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Reprinted from
JAPANESE JOURNAL OF MATHEMATICS
Vol. 21, No. 1, pp. 189–205
June 1995

Japan. J. Math.
Vol. 21, No. 1, 1995

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(Received December 7, 1992)
(Revised April 19, 1994)

§1. Introduction

Let $\widetilde{M}$ be a Kaehler surface endowed with almost complex structure J and inner product $\langle\ ,\ \rangle$ and M an oriented surface immersed in $\widetilde{M}$. We denote the tangent space and the normal space of M in $\widetilde{M}$ at p by T_pM and $T_p^{\perp}M$, respectively.

For a given positive orthonormal basis $\{e_1, e_2\}$ of T_pM, we put

$$\theta(T_pM) = \arccos(\langle Je_1, e_2\rangle). \tag{1.1}$$

Then $\theta(T_pM) \in [0, \pi]$ is independent of the choice of the positive orthonormal basis $\{e_1, e_2\}$ and θ is a continuous function on M. $\theta(T_pM)$ is called the *Wirtinger angle* of M at p (or the *holomorphic angle* by A. Lichnerowicz [7]). If $\theta(T_pM) = 0$ (respectively, $\theta(T_pM) = \pi$, or $\theta(T_pM) = \frac{\pi}{2}$), then the point p is called a *complex* (respectively, *anti-complex, or Lagrangian*) point of M in $\widetilde{M}$. A surface M in $\widetilde{M}$ is said to be *holomorphic* (respectively, *anti-holomorphic* or *Lagrangian*) if $\theta(T_pM) = 0$ (respectively, $\theta(T_pM) = \pi$ or $\theta(T_pM) = \frac{\pi}{2}$), for any $p \in M$. Furthermore, a surface M in $\widetilde{M}$ is called *slant* if its Wirtinger angle is constant.

Throughout this article, by *totally real surface* in $\widetilde{M}$ we mean a surface without complex points and anti-complex points. (Hence, the meaning of totally real surfaces in this article is different from the one given in [5]). Since the Wirtinger function θ fails to be differentiable only at complex and anti-complex points, θ is a differentiable function on a totally real surface.

Recall that a *regular homotopy* is a family of immersions π_t, $t \in [0,1]$ from a manifold into another such that π_t and all its derivatives depend continuously on the parameter t. Immersions π_0 and π_1 are *regularly homotopic* if there exists a regular homotopy connecting π_0 to π_1. It is known that every totally real immersion of a surface in $\mathbb{C}^2$ is regularly homotopic to a Lagrangian immersion (Gromov [6]). Moreover, all totally real immersions of a surface in $\mathbb{C}^2$ are regularly homotopic to each other. In particular, this result implies that there exist no regularly homotopic invariants for totally real surfaces in $\mathbb{C}^2$ through regular homotopy.

In this article, we define a differentiable 1-form on a totally real surface in a Kaehler surface $\widetilde{M}$. (In fact, this 1-form can be defined on any surface in $\widetilde{M}$ by removing all complex and anti-complex points on the surface.) When $\widetilde{M}$ is flat, this 1-form is closed and hence it defines a cohomology class on M. In §4 we prove that this cohomology class is indeed an integral class when the ambient space $\widetilde{M}$ is $\mathbb{C}^2$ which is nothing but an extension of the Maslov class. As a consequence, this integral cohomology class gives rise to a homotopic invariant for totally real immersions of a surface in $\mathbb{C}^2$ through totally real regular homotopy.

As applications we derive some isoperimetric inequalities for totally real surfaces in $\mathbb{C}^2$.

§2. Preliminaries

Let M be an oriented surface immersed in a Kaehler surface $\widetilde{M}$. We denote by A, D, h, and H the Weingarten map, the normal connection, the second fundamental form, and the mean curvature vector of M in $\widetilde{M}$, respectively. Let $\widetilde{\nabla}$ and ∇ denote the Levi-Civita connections of $\widetilde{M}$ and of M with the induced metric, respectively.

For a tangent vector $X \in T_pM$, we put

$$JX = PX + FX \tag{2.1}$$

where PX and FX are the tangential and the normal components of JX, respectively. Consequently, we have an endomorphism $P : TM \to TM$ and a linear map $F : TM \to T^{\perp}M$.

For a normal vector $\xi \in T_p^{\perp}M$, we put

$$J\xi = t\xi + f\xi, \tag{2.2}$$

where $t\xi$ and $f\xi$ are the tangential and the normal components of $J\xi$, respectively.

In the following we assume that M is an orientable totally real surface in $\widetilde{M}$ unless mentioned otherwise.

For a given positively oriented local orthonormal frame field $\{e_1, e_2\}$ on M, we choose a local orthonormal frame field $\{e_3, e_4\}$ such that

$$e_3 = (\csc\theta)Fe_1, \quad e_4 = (\csc\theta)Fe_2. \tag{2.3}$$

Then we have $Pe_1 = (\cos\theta)e_2$, $Pe_2 = -(\cos\theta)e_1$. We call such a local frame field $\{e_1, e_2, e_3, e_4\}$ an *adapted frame field*. From (2.3) we have

$$te_3 = -\sin\theta e_1, \quad te_4 = -\sin\theta e_2, \quad fe_3 = -\cos\theta e_4, \quad fe_4 = \cos\theta e_3. \tag{2.4}$$

Let $\{e_1, e_2, e_3, e_4\}$ be a local orthonormal frame field on $\widetilde{M}$ such that, restricted to M, it is an adapted frame field. Let $\{\omega^1, \omega^2, \omega^3, \omega^4\}$ be the field of dual frames. The structure equations of $\widetilde{M}$ are given by

$$d\omega^A = -\sum_B \omega^A_B \wedge \omega^B, \quad \omega^A_B + \omega^B_A = 0, \quad 1 \le A, B, C, D \le 4, \tag{2.5}$$

and

$$d\omega^A_B = -\sum_C \omega^A_C \wedge \omega^C_B + \widetilde{\Omega}^A_B, \quad \widetilde{\Omega}^A_B = \frac{1}{2}\sum_{C,D} \widetilde{R}^A_{BCD}\omega^C \wedge \omega^D. \tag{2.6}$$

If we restrict these forms to M, then $\omega^r = 0$, $r = 3, 4$. Since

$$0 = d\omega^r = -\sum_i \omega^r_i \wedge \omega^i, \quad 1 \le i, j, k, \ell \le 2, \tag{2.7}$$

Cartan's lemma yields

$$\omega^r_i = \sum_j h^r_{ij}\omega^j, \quad h^r_{ij} = h^r_{ji}, \tag{2.8}$$

where

$$h^r_{ij} = \omega^r_i(e_j) = \langle A_{e_r}e_i, e_j\rangle = \langle h(e_i, e_j), e_r\rangle. \tag{2.9}$$

From these formulas we obtain

$$d\omega^i = -\sum_j \omega^i_j \wedge \omega^j, \quad \omega^i_j + \omega^j_i = 0, \tag{2.10}$$

$$d\omega^i_j = -\sum_k \omega^i_k \wedge \omega^k_j + \Omega^i_j, \quad \Omega^i_j = \frac{1}{2}\sum_{k,\ell} R^i_{jk\ell}\omega^k \wedge \omega^\ell, \tag{2.11}$$

$$R^i_{jk\ell} = \widetilde{R}^i_{jk\ell} + \sum_r (h^r_{ik}h^r_{j\ell} - h^r_{i\ell}h^r_{jk}), \tag{2.12}$$

$$d\omega^r_i = -\sum_j \omega^r_j \wedge \omega^j_i - \sum_s \omega^r_s \wedge \omega^s_i + \widetilde{\Omega}^r_i. \tag{2.13}$$

We need the following two lemmas from [2].

LEMMA 2.1. *Let M be a totally real surface immersed in a Kaehler surface $\widetilde{M}$. Then for any vector fields X, Y tangent to M, we have*

$$(\nabla_X P)Y = th(X,Y) + A_{FY}X,$$
$$(\nabla_X F)Y = fh(X,Y) - h(X,PY),$$

where $(\nabla_X P)Y = \nabla_X(PY) - P(\nabla_X Y)$ *and* $(\nabla_X F)Y = D_X(FY) - F(\nabla_X Y)$.

LEMMA 2.2. *Let M be a totally real surface immersed in a Kaehler surface. Then, with respect to an adapted frame $\{e_1, e_2, e_3, e_4\}$ we have*

$$(\nabla_X P)e_1 = X(\cos\theta)e_2, \quad (\nabla_X P)e_2 = -X(\cos\theta)e_1.$$

PROOF. By the definition of ∇P, we have

$$\begin{aligned}(\nabla_X P)e_1 &= \nabla_X(Pe_1) - P(\nabla_X e_1)\\ &= \nabla_X((\cos\theta)e_2) - P(\omega_1^2(X)e_2) = X(\cos\theta)e_2.\end{aligned}$$

Similarly we obtain the corresponding formula for $(\nabla_X P)e_2$. □

We also need the following.

LEMMA 2.3. *Let M be an oriented totally real surface immersed in a Kaehler surface M. Then, with respect to an adapted frame, we have*

(i) $h_{12}^3 = h_{11}^4 - d\theta(e_1), \quad h_{12}^4 = h_{22}^3 + d\theta(e_2)$,

(ii) $\omega_3^4(e_1) - \omega_1^2(e_1) = -\cot\theta(\operatorname{trace} h^3 + d\theta(e_2))$,

(iii) $\omega_3^4(e_2) - \omega_1^2(e_2) = -\cot\theta(\operatorname{trace} h^4 - d\theta(e_1))$.

PROOF. (i) follows easily from Lemma 2.1 and Lemma 2.2.

By using (2.3) and (2.4), we get

$$\begin{aligned}D_{e_1}e_3 &= D_{e_1}(\csc\theta Fe_1) = (\csc\theta)De_{e_1}(Fe_1) - (\cot\theta)(e_1\theta)e_3\\ &= (\csc\theta)\{F(\nabla_{e_1}e_1) + fh(e_1,e_1) - h(e_1,Pe_1)\} - (\cot\theta)(e_1\theta)e_3\\ &= (\csc\theta)\{\omega_1^2(e_1)Fe_2 + h_{11}^3 fe_3 + h_{11}^4 fe_4 - \cos\theta(h_{12}^3e_3 + h_{12}^4e_4)\}\\ &\qquad - (\cot\theta)(e_1\theta)e_3\\ &= \omega_1^2(e_1)e_4 - (\cot\theta)((\operatorname{trace} h^3)e_4 + d\theta(e_2))e_4.\end{aligned}$$

This implies (ii). Similarly, we may obtain (iii). □

§3. 1-form Φ

For an oriented Riemannian surface M with volume form $*1$, there exists a canonical endomorphism $j : TM \to TM$ defined by

$$\langle jX, Y\rangle = 2(*1)(X,Y), \quad X, Y \in TM. \tag{3.1}$$

This endomorphism j is the canonical almost complex structure of the Riemannian surface M. In particular, if e_1, e_2 is a positive orthonormal frame field of M, we have $je_1 = e_2$, $je_2 = -e_1$.

We recall that a Kaehler surface $\widetilde{M}$ admits a canonical symplectic structure Ω given by

$$\Omega(X,Y) = \langle X, JY\rangle, \quad X, Y \in T\widetilde{M}. \tag{3.2}$$

For an oriented totally real surface M immersed in a Kaehler surface $\widetilde{M}$, we introduce a 1-form Φ on M defined by

$$\Phi(X) = \frac{1}{2\pi \sin^2\theta}\{2\Omega(H,X) + \sin\theta(d\theta\circ j)(X)\}, \quad X \in TM, \tag{3.3}$$

where θ is the Wirtinger function on M and $H = \frac{1}{2}\operatorname{trace} h$ is the mean curvature vector field.

In this section we shall prove the following.

THEOREM 3.1. *Let M be an oriented totally real surface in a Kaehler surface $\widetilde{M}$. With respect to an adapted frame, we have*

$$d\Phi = \frac{1}{2\pi\sin\theta}\left(\widetilde{\Omega}_1^3 + \widetilde{\Omega}_2^4\right). \tag{3.4}$$

In particular, if $\widetilde{M}$ is flat, then $d\Phi = 0$; and hence Φ defines a cohomology class: $[\Phi] \in H^1(M;\mathbb{R})$.

PROOF. Let e_1, e_2, e_3, e_4 be an adapted frame field on the oriented totally real surface M in $\widetilde{M}$ and let $X = X^1e_1 + X^2e_2$ be a tangent vector field of M. Then, from (2.3), (3.1), and (3.2), we have

$$2\Omega(H,X) = \sin\theta\left((h_{11}^3 + h_{22}^3)X^1 + (h_{11}^4 + h_{22}^4)X^2\right), \tag{3.5}$$

$$(d\theta\circ j)(X) = X^1 d\theta(e_2) - X^2 d\theta(e_1). \tag{3.6}$$

Thus, by (3.3), (3.5), and (3.6), we get

$$\Phi(X) = \frac{1}{2\pi\sin\theta}\{(h_{11}^3 + h_{22}^3)X^1 + (h_{11}^4 + h_{22}^4)X^2 + X^1 e_2\theta - X^2 e_1\theta\}.$$

By combining this with (i) of Lemma 2.3, we may obtain

$$\Phi(X) = \frac{1}{2\pi \sin\theta}(h^3_{11}X^1 + h^3_{12}X^2 + h^4_{21}X^1 + h^4_{22}X^2).$$

This implies

$$\Phi = \frac{1}{2\pi \sin\theta}(\omega^3_1 + \omega^4_2). \tag{3.7}$$

By (2.13), we find

$$d(\omega^3_1 + \omega^4_2) = (\omega^2_1 - \omega^4_3) \wedge (\omega^3_2 - \omega^4_1) + \widetilde{\Omega}^3_1 + \widetilde{\Omega}^4_2. \tag{3.8}$$

Therefore, by (3.7) and (3.8), we get

$$2\pi d\Phi = d(\csc\theta) \wedge (\omega^3_1 + \omega^4_2) + \csc\theta\{(\omega^2_1 - \omega^4_3) \wedge (\omega^3_2 - \omega^4_1) + \widetilde{\Omega}^3_1 + \widetilde{\Omega}^4_2\}.$$

This yields

$$\begin{aligned} 2\pi d\Phi(e_1, e_2) = &-\frac{\cos\theta}{2\sin^2\theta}\{d\theta(e_1)(h^3_{12} + h^4_{22}) - d\theta(e_2)(h^3_{11} + h^4_{12})\} \\ &+ \csc\theta\{(\omega^2_1 - \omega^4_3) \wedge (\omega^3_2 - \omega^4_1) + \widetilde{\Omega}^3_1 + \widetilde{\Omega}^4_2\}(e_1, e_2). \end{aligned}$$

Combining this formula with Lemma 2.3 imply (3.4). □

If L is a Lagrangian surface in $\mathbb{C}^2$, then L has no complex points and anti-complex points. Since $\theta = \frac{\pi}{2}$ in this case, (3.3) reduces to

$$\Phi = \frac{1}{\pi}\langle JH, \quad \rangle. \tag{3.9}$$

Since the Maslov class $m(L)$ of a Lagrangian surface L in $\mathbb{C}^2$ is also represented by $\frac{1}{\pi}\langle JH, \quad \rangle$ (cf. [8]), we have the following.

THEOREM 3.2. *If L is a Lagrangian surface in $\mathbb{C}^2$, then the cohomology class $[\Phi]$ of L is equal to the Maslov class $m(L)$ of L.*

As an application of Theorem 3.1 we have the following.

THEOREM 3.3. *Let M be a totally real minimal surface in a flat Kaehler surface $\widetilde{M}$. If M is compact, then M is a slant surface (in the sense of [2]). Moreover, we have $\Phi = 0$.*

PROOF. Since M is a totally real minimal surface in $\widetilde{M}$, (3.3) gives

$$\Phi = \frac{1}{2\pi}\left(d\left(\ln\left(\tan\frac{\theta}{2}\right)\right) \circ j\right). \tag{3.10}$$

Because $\widetilde{M}$ is flat, $d\Phi = 0$ by Theorem 3.1. Hence, by the Poincaré lemma, Φ is locally exact. By passing, if necessary, to the two-fold covering surface, we may assume M is orientable. It is then possible to choose a system of isothermal coordinates $\{x, y\}$ covering M with the metric tensor g given by $g = E(dx^2 + dy^2)$. Because Φ is locally exact, there exists a local function f such that

$$d\left(\ln\left(\tan\frac{\theta}{2}\right)\right) \circ j = df. \tag{3.11}$$

From (3.11), we see that $\ln\left(\tan\frac{\theta}{2}\right)$ and f are harmonic conjugates, *i.e.*,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial y}\left(\ln\left(\tan\frac{\theta}{2}\right)\right), \quad \frac{\partial f}{\partial y} = -\frac{\partial}{\partial x}\left(\ln\left(\tan\frac{\theta}{2}\right)\right). \tag{3.12}$$

Since the Laplacian of M is given by $\Delta = -\frac{1}{E}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, (3.12) implies that $\ln\left(\tan\frac{\theta}{2}\right)$ is a harmonic function on M. Since M is compact, θ is constant. Hence M is a slant surface in $\widetilde{M}$. In particular, from (3.3), we get $\Phi = 0$. □

REMARK 3.1. Although there exist no compact minimal surfaces in $\mathbb{C}^2$, there do exist compact minimal slant surfaces in flat Kaehler surfaces with an arbitrarily prescribed slant angle.

§4. The cohomology class of totally real surfaces in $\mathbb{C}^2$

The main purpose of this section is to prove the following two theorems.

THEOREM 4.1. *Let M be an oriented totally real surface in $\mathbb{C}^2$. Then the cohomology class $[\Phi]$ of M is an integral class, i.e., $[\Phi] \in H^1(M; \mathbb{Z})$.*

THEOREM 4.2. *Let M be an oriented totally real surface in $\mathbb{C}^2$. Then the cohomology class $[\Phi]$ is an invariant through totally real regular homotopy.*

In order to prove these two results, we need to derive a precise expression of the canonical 1-form Φ in terms of the Gauss map. Thus we need to recall some basic facts concerning the geometry of the Grassmannian $G(2,4)$ which consists of all oriented 2-planes in the Euclidean 4-space E^4 (see, for instances [2, 3], for more details).

Assume E^4 is oriented by its canonical orthonormal frame:

$$\epsilon_1 = (1,0,0,0), \quad \epsilon_2 = (0,1,0,0), \quad \epsilon_3 = (0,0,1,0), \quad \epsilon_4 = (0,0,0,1).$$

Let $\langle\ ,\ \rangle$ be the canonical inner product on E^4. Denote by $\{\theta^1, \theta^2, \theta^3, \theta^4\}$ the dual frame of $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$. Let $\wedge^2 E^4$ be the space of 2-vectors of E^4 and $D_1(2,4)$ the set consisting of all unit decomposable 2-vectors in $\wedge^2 E^4$. $\wedge^2 E^4$ is a 6-dimensional real vector space with its canonical inner product given by

$$\langle X_1 \wedge X_2, Y_1 \wedge Y_2 \rangle = \det(\langle X_i, Y_j \rangle). \tag{4.1}$$

It is known that the Grassmannian $G(2,4)$ can be identified with the set $D_1(2,4)$ via the map $\phi : G(2,4) \to D_1(2.4)$ defined by $\phi(V) = e_1 \wedge e_2$, for a positive orthonormal basis $\{e_1, e_2\}$ of $V \in G(2,4)$.

The Hodge star operator $* : \wedge^2 E^4 \to \wedge^2 E^4$ is defined by

$$\langle *\alpha, \beta \rangle \Psi = \alpha \wedge \beta, \tag{4.2}$$

for any $\alpha, \beta \in \wedge^2 E^4$, where Ψ denotes the volume element of E^4. If we regard an oriented 2-plane $V \in G(2,4)$ as an element in $D_1(2,4)$ via ϕ, then $*V = V^\perp$, where $V^\perp$ is the oriented orthogonal complementary subspace of the oriented 2-plane V in E^4.

Since $*^2 = 1$ and $*$ is a self-adjoint endomorphism of $\wedge^2 E^4$, we have the following orthogonal decomposition:

$$\wedge^2 E^4 = \wedge^2_+ E^4 \oplus \wedge^2_- E^4 \tag{4.3}$$

of eigenspaces of $*$ with eigenvalues 1 and -1, respectively. Let π_+ and π_- denote the natural projections: $\pi_\pm : \wedge^2 E^4 \to \wedge^2_\pm E^4$, respectively.

If $\alpha \in D_1(2,4)$, we have

$$\pi_+(\alpha) = \frac{1}{2}(\alpha + *\alpha), \quad \pi_-(\alpha) = \frac{1}{2}(\alpha - *\alpha).$$

If S^2_+ (respectively, S^2_-) denotes the 2-sphere of $\wedge^2_+ E^4$ (respectively, of $\wedge^2_- E^4$) centered at the origin and with radius $\frac{1}{\sqrt{2}}$, then $\pi = (\pi_+, \pi_-)$ gives rise to the following indentification of $G(2,4) \cong D_1(2,4)$ with $S^2_+ \times S^2_-$:

$$\pi : G(2,4) \cong D_1(2,4) \to S^2_+ \times S^2_-; \quad \alpha \mapsto \left(\frac{1}{2}(\alpha + *\alpha), \frac{1}{2}(\alpha - *\alpha)\right).$$

Associated with the canonical frame field of E^4, we have a canonical orthonormal frame field of $\wedge^2 E^4$ given by

$$\eta_1 = \frac{1}{\sqrt{2}}(\epsilon_1 \wedge \epsilon_2 + \epsilon_3 \wedge \epsilon_4), \quad \eta_2 = \frac{1}{\sqrt{2}}(\epsilon_1 \wedge \epsilon_3 - \epsilon_2 \wedge \epsilon_4),$$
$$\eta_3 = \frac{1}{\sqrt{2}}(\epsilon_1 \wedge \epsilon_4 + \epsilon_2 \wedge \epsilon_3), \quad \eta_4 = \frac{1}{\sqrt{2}}(\epsilon_1 \wedge \epsilon_2 - \epsilon_3 \wedge \epsilon_4),$$
$$\eta_5 = \frac{1}{\sqrt{2}}(\epsilon_1 \wedge \epsilon_3 + \epsilon_2 \wedge \epsilon_4), \quad \eta_6 = \frac{1}{\sqrt{2}}(\epsilon_1 \wedge \epsilon_4 - \epsilon_2 \wedge \epsilon_3).$$

$\{\eta_1, \eta_2, \eta_3\}$ and $\{\eta_4, \eta_5, \eta_6\}$ form orthonormal bases of $\wedge^2_+ E^4$ and $\wedge^2_- E^4$, respectively. We shall orient the spaces $\wedge^2_+ E^4$ and $\wedge^2_- E^4$ in such way that these two bases are positive, *i.e.*, they give positive orientations for $\wedge^2_+ E^4$ and $\wedge^2_- E^4$, respectively.

In general, if $\{e_1, e_2, e_3, e_4\}$ is a positive orthonormal basis of E^4, we can build an orthonormal basis for $\wedge^2 E^4$ by

$$\gamma_1 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4), \quad \gamma_2 = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 - e_2 \wedge e_4),$$

$$\gamma_3 = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 + e_2 \wedge e_3), \quad \gamma_4 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 - e_3 \wedge e_4),$$

$$\gamma_5 = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 + e_2 \wedge e_4), \quad \gamma_6 = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 - e_2 \wedge e_3).$$

It is clear that $\{\gamma_1, \gamma_2, \gamma_3\}$ and $\{\gamma_4, \gamma_5, \gamma_6\}$ also form orthonormal bases of $\wedge^2_+ E^4$ and $\wedge^2_- E^4$, respectively.

Put

$$\widetilde{\gamma}_i = \frac{1}{\sqrt{2}}\gamma_i; \quad 1 \leq i \leq 6.$$

Then $\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3$ belong to S^2_+ and $\widetilde{\gamma}_4, \widetilde{\gamma}_5, \widetilde{\gamma}_6$ belong to S^2_-.

Let $\mathbb{C}^2$ be the complex Euclidean 2-plane endowed with the canonical almost complex structure J. For a given Lagrangian plane L in $\mathbb{C}^2$, we identify $\mathbb{C}^2$ with the real 4-space $E^4 = L \oplus J(L)$. We orient $\mathbb{C}^2$ via this identification, *i.e.*, if $\{\epsilon_1, \epsilon_2\}$ is an orthonormal basis of L, then $\epsilon_1 \wedge \epsilon_2 \wedge J\epsilon_1 \wedge J\epsilon_2$ gives the positive orientation of $\mathbb{C}^2$.

Let M be an oriented surface in $\mathbb{C}^2$. We denote by

$$\nu : M \to G(2, 4)$$

the Gauss map of M in $\mathbb{C}^2$ which is defined by

$$\nu(p) = (e_1 \wedge e_2)_p$$

where $\{e_1, e_2\}$ is a positive orthonormal basis of T_pM. We put

$$\nu_+ = \pi_+ \circ \nu, \quad \nu_- = \pi_- \circ \nu.$$

Then $\nu_+ : M \to S^2_+ \subset \wedge^2_+ E^4$ and $\nu_- : M \to S^2_- \subset \wedge^2_- E^4$.

We need the following.

Lemma 4.3. *Let M be an oriented totally real surface in $\mathbb{C}^2$. Then, with respect to an adapted frame field $\{e_1, e_2, e_3, e_4\}$, we have*

$$(\nu_+)_*(X) = \frac{1}{\sqrt{2}}\{(\omega_1^4 + \omega_2^3)(X)\gamma_2 + (-\omega_1^3 + \omega_2^4)(X)\gamma_3\},$$

and

$$(\nu_-)_*(X) = \frac{1}{\sqrt{2}}\{(-\omega_1^4 + \omega_2^3)(X)\gamma_5 + (\omega_1^3 + \omega_2^4)(X)\gamma_6\}$$

for any X tangent to M.

PROOF. We have

$$\begin{aligned}\nu_*(X) &= (\widetilde{\nabla}_X e_1) \wedge e_2 + e_1 \wedge (\widetilde{\nabla}_X e_2)\\ &= \omega_1^3(X)e_3 \wedge e_2 + \omega_1^4(X)e_4 \wedge e_2 + e_1 \wedge \omega_2^3(X)e_3 + e_1 \wedge \omega_2^4(X)e_4\\ &= \frac{1}{\sqrt{2}}\{(\omega_1^4 + \omega_2^3)(X)\gamma_2 + (-\omega_1^3 + \omega_2^4)(X)\gamma_3\\ &\qquad + (-\omega_1^4 + \omega_2^3)(X)\gamma_5 + (\omega_1^3 + \omega_2^4)(X)\gamma_6\},\end{aligned}$$

from which we obtain the lemma. □

The canonical almost complex structure J on $\mathbb{C}^2$ gives a unique 2-vector $\zeta_J \in \wedge^2 E^4$, defined by

$$\langle \zeta_J, X \wedge Y \rangle = \langle JX, Y \rangle \tag{4.4}$$

for X, Y tangent to $\mathbb{C}^2$. It is easy to check that

$$\zeta_J = \epsilon_1 \wedge \epsilon_3 + \epsilon_2 \wedge \epsilon_4 = \epsilon_1 \wedge J\epsilon_1 + \epsilon_2 \wedge J\epsilon_2 = \sqrt{2}\eta_5.$$

Moreover, from the definition of S^2_-, we see that $\widetilde{\zeta}_J = \frac{1}{\sqrt{2}}\eta_5$ is an element in S^2_-.

Let $\widetilde{\gamma} \in \overline{S^2_-} = S^2_- - \{\pm\widetilde{\zeta}_J\}$ and let α be the angle between $\widetilde{\gamma}$ and $\widetilde{\zeta}_J$. Denote by $S^1(\alpha)$ the intersection of S^2_- with the 2-plane in $\wedge^2_- E^4 = E^3$ containing the endpoint of $\widetilde{\gamma}$ and which is orthogonal to ζ_J. So $S^1(\alpha)$ is a circle of radius $\frac{1}{\sqrt{2}}\sin\alpha$. If $\alpha = \frac{\pi}{2}$, the circle is an equator of S^2_-.

For each $\widetilde{\gamma} \in \overline{S^2_-}$, we define a tangent vector of S^2_- at $\widetilde{\gamma}$ by

$$T(\widetilde{\gamma}) = \left(\frac{\sqrt{2}}{\pi \sin^2 \alpha}\right) \widetilde{\gamma} \times \widetilde{\zeta}_J \tag{4.5}$$

where $\times$ denotes the cross-product on $\wedge^2_- E^4 \cong E^3$. It is clear that $T(\widetilde{\gamma})$ is tangent to the circle $S^1(\alpha)$ at $\widetilde{\gamma}$. Let ω be the dual 1-form of T on $\overline{S^2_-}$, *i.e.*,

$$\omega(Z) = \langle T, Z \rangle, \quad \forall Z \in T\overline{S^2_-}. \tag{4.6}$$

By direct computation, we have

$$\int_{S^1(\alpha)} \omega = 1 \tag{4.7}$$

for any fixed $\alpha \in (0, \pi)$.

LEMMA 4.4. *For an oriented totally real surface M in $\mathbb{C}^2$, the 1-form Φ satisfies*

$$\Phi(X_p) = \frac{\sqrt{2}}{\pi \sin^2 \theta_p} \det(\nu_-(p), \widetilde{\zeta}_J, (\nu_-)_*(X_p)) \tag{4.8}$$

for any $X_p \in T_pM$, where the determinant is computed with respect to the orientation of $\wedge^2_- E^4 \cong E^3$ given by $\{\eta_4, \eta_5, \eta_6\}$. Moreover, the 1-form Φ is the pullback of the 1-form ω by ν_-, i.e., $\Phi = (\nu_-)^*\omega$, where ω is the 1-form on $\overline{S^2_-}$ defined by (4.6).

PROOF. For each point $p \in M$, we have

$$\nu_-(p) = \frac{1}{2}(e_1 \wedge e_2 - e_3 \wedge e_4) = \frac{1}{\sqrt{2}}\gamma_4. \tag{4.9}$$

Thus, $|\nu_-(p)| = \frac{1}{\sqrt{2}}$. Since $|\zeta_J| = \sqrt{2}$, (4.4) yields

$$\cos\alpha = \langle \nu_-(p), \zeta_J \rangle = \frac{1}{2}\langle e_1 \wedge e_2 - e_3 \wedge e_4, \zeta_J \rangle = \cos\theta(p)$$

where $\theta(p)$ is the Wirtinger angle of M at p and α is the angle between $\nu_-(p)$ and ζ_J.

From (4.5) and (4.6), we have

$$\begin{aligned} ((\nu_-)^*\omega)(X_p) &= \omega((\nu_-)_*(X_p)) = \langle T(\nu_-(p)), (\nu_-)_*(X_p) \rangle \\ &= \frac{\sqrt{2}}{\pi \sin^2\theta} \langle \nu_-(p) \times \zeta_J, (\nu_-)_*(X_p) \rangle. \end{aligned} \tag{4.10}$$

On the other hand, we have

$$\langle \zeta_J, \gamma_6 \rangle = \frac{1}{\sqrt{2}} \langle \zeta_J, e_1 \wedge e_4 - e_2 \wedge e_3 \rangle = 0,$$

therefore, γ_6 is perpendicular to both ζ_J and $\nu_-(p)$. Hence, by (4.5), we may obtain

$$T(\nu_-(p)) = \left(\frac{1}{\sqrt{2}\pi \sin\theta(p)} \right) \gamma_6. \tag{4.11}$$

Here we remark that the orientation of γ_6 and $\widetilde{\gamma} \times \widetilde{\zeta}_J$ are the same. By applying (3.7), (4.10) and Lemma 4.3, we get

$$\begin{aligned} ((\nu_-)^*\omega)(X_p) &= \langle T(\nu_-(p)), (\nu_-)_*(X_p) \rangle = \frac{1}{\sqrt{2}\pi \sin\theta} \langle \gamma_6, (\nu_-)_*(X_p) \rangle \\ &= \frac{1}{2\pi \sin\theta} (\omega_1^3 + \omega_2^4)(X_p) = \Phi(X_p). \end{aligned}$$

These imply the lemma. □

PROOF OF THEOREMS 4.1 AND 4.2. To show that the cohomology class $[\Phi]$ is an integral class, *i.e.*, $[\Phi] \in N^1(M;\mathbb{Z})$, we choose a Cartesian coordinate system (x,y,z) on $\wedge^2_- E^4$ such that $\widetilde{\zeta}_J$ and $-\widetilde{\zeta}_J$ are given by $\left(0,0,\frac{1}{\sqrt{2}}\right)$ and $\left(0,0,-\frac{1}{\sqrt{2}}\right)$, respectively. We consider $\widetilde{\zeta}_J$ and $-\widetilde{\zeta}_J$ as the north and south poles of S^2_-. For a point $\widetilde{\gamma} \in \overline{S^2_-}$, we denote by F the half-plane containing $\widetilde{\gamma}$ and the north and south poles. We define β to be the angle measured from the half-plane $\{(x,y,z) : x \geq 0,\, y = 0\}$ to the half-plane F. From (4.5) and (4.6), we can prove by direct computation that $\omega = \frac{1}{2\pi} d\beta$ on $\overline{S^2_-}$.

Now, if Γ is a loop in surface M. Then, by Lemma 4.4, we have

$$\int_\Gamma \Phi = \int_\Gamma (\nu_-)^* \omega = \int_{\nu_-(\Gamma)} \omega = \frac{1}{2\pi} \int_{\nu_-(\Gamma)} d\beta.$$

From this we see that the value of $\int_\Gamma \Phi$ of the loop Γ in M is equal to the turning number of Γ around the north-south axis of S^2_- under the map ν_- which is an integer. Hence, $[\Phi]$ is an integral cohomology class.

Theorem 4.2 follows easily from Theorem 4.1, since integers are isolated in the real line $\mathbb{R}$. □

EXAMPLE 1. Let $\mathbb{C}^2$ be the complex plane given by E^4 endowed with the canonical complex structure J defined by $J(x,y,z,w) = (-z,-w,x,y)$ and $U = E^2 - \{(0,0)\}$. And let ϕ and ψ be the two totally real imbeddings of U in $\mathbb{C}^2$ defined by

$$\phi(u,v) = (u,u,v,0), \tag{4.12}$$

$$\psi(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, 0\right), \tag{4.13}$$

respectively. The imbedding ψ is obtained from the stereographic projection of the xz-plane into the unit sphere in E^3 centered at the origin.

By direct straight-forward computation, we can verify that the cohomology class $[\Phi]_\phi$ of ϕ is trivial and the cohomology class $[\Phi]_\psi$ of ψ is non-trivial. Thus, by applying Theorem 4.2, we conclude that the two totally real imbeddings ϕ and ψ are not regularly homotopic through totally real regular homotopy, although ϕ and ψ are regularly homotopic through regular homotopy.

EXAMPLE 2. (Enneper's Minimal Surface) Consider the following surface in $\mathbb{C}^2$ defined by

$$\phi(u,v) = \left(u - \frac{u^3}{3} + uv^2, u^2 - v^2, v - \frac{v^3}{3} + vu^2, 0\right), \tag{4.14}$$

$$(u,v) \in U = E^2 - (0,0),$$

$$\frac{\partial}{\partial u} = (1 - u^2 + v^2, 2u, 2uv, 0), \tag{4.15}$$

$$\frac{\partial}{\partial v} = (2uv, -2v, 1 - v^2 + u^2, 0). \tag{4.16}$$

From (4.15) and (4.16) it follows that (u, v) is an isothermal coordinate system and ϕ is a totally real minimal immersion whose Wirtinger angle θ is given by

$$\theta = \cos^{-1}\left(\frac{1 - u^2 - v^2}{1 + u^2 + v^2}\right). \tag{4.17}$$

Let (r, α) be the polar coordinates on U with $u = r\cos\alpha$, $v = r\sin\alpha$. By (3.3), (4.17) and straight-forward computation, we may obtain

$$\Phi = -\frac{1}{2\pi}d\alpha. \tag{4.18}$$

This implies $[\Phi] \neq 0$. This example shows that not all minimal totally real surfaces in $\mathbb{C}^2$ have trivial canonical homology class $[\Phi]$, although all Lagrangian minimal surfaces in $\mathbb{C}^2$ have trivial Maslov class [8].

Example 3. (Catenoid) Consider the catenoid defined by

$$\psi(u, v) = (\cosh u \cos v, u, \cosh u \sin v, 0), \quad (u, v) \in E^2. \tag{4.19}$$

It is easy to verify that (u, v) is an isothermal coordinate system and ψ is a totally real minimal immersion whose Wirtinger angle is given by

$$\theta = \cos^{-1}(\tanh u). \tag{4.20}$$

From these together with (3.3) we may obtain

$$\Phi = -\frac{1}{2\pi}dv.$$

Thus $[\Phi] = 0$.

§5. Applications and remarks

5.1: Complex Curves in $\mathbb{C}^2$. If M is a complex curve in $\mathbb{C}^2$, then the 1-form Φ is obviously undefined. However, we can modify this situation slightly as follows: First, according to [3], we know that there exists an almost complex structure J_1 on $\mathbb{C}^2$ such that M becomes a minimal Lagrangian surface with respect to the new almost complex structure. Since M is minimal and θ is constant, (3.3) shows that the 1-form Φ associated with M in (E^4, J_1) is trivial.

5.2: Slant Surfaces. A totally real surface M in $\mathbb{C}^2$ is called a proper *slant surface* if its Wirtinger function θ is constant (cf. [2]). In this case formula (3.3) reduces to

$$\Phi = \frac{1}{\pi \sin^2 \theta} \langle JH, \quad \rangle. \tag{5.1}$$

From (5.1) it follows that the integral cohomology class $[\Phi]$ is trivial if the slant surface is a minimal surface in $\mathbb{C}^2$.

Although the cohomology class $[\Phi]$ is not necessary trivial for an arbitrary minimal totally real surface in $\mathbb{C}^2$, however, by applying Theorem 4.2, we have the following result.

COROLLARY 5.1. *An oriented totally real surface in $\mathbb{C}^2$ has trivial cohomology class $[\Phi]$ if it is regularly homotopic to a minimal proper slant surface in $\mathbb{C}^2$ through totally real homotopy.*

5.3: Φ-index. Let M be an oriented totally real surface in $\mathbb{C}^2$. For an oriented loop γ in M, we put

$$i_\Phi(\gamma) = \int_\gamma \Phi. \tag{5.2}$$

We call the integer $i_\Phi(\gamma)$, the *Φ-index of the loop* γ. If $\overline{\gamma}$ is another oriented loop in M such that γ is homotopic to $\overline{\gamma}$ through orientation-preserving homotopy in M, then $i_\Phi(\gamma) = i_\Phi(\overline{\gamma})$ by $d\Phi = 0$. Thus, the Φ-index of loops is a homotopic invariant.

Take an arbitrary surface M in $\mathbb{C}^2$, the surface may contain complex and anti-complex points. According to Thom's transversality theorem, generically, the complex and anti-complex points are isolated. In particular, if M is compact without boundary, the number of such points is, generically, finite. Let $\{p_1, \dots, p_n\}$ be the set of complex and anti-complex points on M. Then $\overline{M} = M - \{p_1, \dots, p_n\}$ is a totally real surface in $\mathbb{C}^2$ and hence we have the canonical closed 1-form Φ defined on $\overline{M}$. For each $i \in \{1, \dots, n\}$, let γ_{r_i} be a small circle of radius r_i around p_i and D_i the disk centered at p_i enclosed by γ_{r_i}. By using the orientation of M, each γ_{r_i} is endowed with a canonical orientation. We define the *Φ-index* of M by $i_\Phi(M) = \sum_{i=1}^n i_\Phi(\gamma_{r_i})$. It is easy to see that $i_\Phi(M)$ is well-defined.

By Stokes' theorem we have

$$i_\Phi(M) = \sum_{i=1}^n \int_{\gamma_{r_i}} \Phi = \int_{M - \bigcup_{i=1}^n D_i} d\Phi = 0. \tag{5.3}$$

This implies that the Φ-index $i_\Phi(M)$ of M in $\mathbb{C}^2$ vanishes when M is compact and it has only a finite number of complex and anti-complex points.

5.4: Isoperimetric inequalities. Let M be an oriented totally real surface in $\mathbb{C}^2$. Then, with respect to an adapted frame field, we have

$$\|\omega_1^3 + \omega_2^4\| \le \sqrt{2}\|h\|. \tag{5.4}$$

Therefore, by (3.7), (5.2) and (5.4), we may obtain the following

COROLLARY 5.2. *Let M be an oriented totally real surface in $\mathbb{C}^2$. Then, for any loop γ in M, the length of γ satisfies*

$$L(\gamma) \ge \left(\frac{\sqrt{2}\pi \sin\theta_0}{\sup\|h\|_\gamma}\right)|i_\Phi(\gamma)|, \tag{5.5}$$

where

$$\sin\theta_0 = \min_{p\in\gamma} \sin\theta(p), \quad \sup\|h\|_\gamma = \sup_{p\in\gamma}\|h\|(p).$$

In particular, this implies the following

COROLLARY 5.3. *Let M be a totally real surface in $\mathbb{C}^2$ with non-trivial cohomology class $[\Phi]$ and α a positive real number with $\sin\theta \ge \sin\alpha$ on M. Then there exists a homology class in $H_1(M;R)$ such that the length of each loop γ in the homology class satisfies*

$$L(\gamma) \ge \left(\frac{\sqrt{2}\pi \sin\alpha}{\sup\|h\|}\right). \tag{5.6}$$

If M is a minimal totally real surface in $\mathbb{C}^2$, then (5.4) can be sharpened to

$$\|\omega_1^3 + \omega_2^4\| \le \|h\|.$$

Thus we obtain the following

COROLLARY 5.4. *We have*

$$L(\gamma) \ge \left(\frac{2\pi \sin\theta_0}{\sup\|h\|_\gamma}\right)|i_\Phi(\gamma)|, \tag{5.7}$$

for any loop γ in a totally real minimal surface M in $\mathbb{C}^2$.

COROLLARY 5.5 *Let M be a totally real, minimal surface in $\mathbb{C}^2$ with non-trivial cohomology class $[\Phi]$ and α a positive real number with $\sin\theta \ge \sin\alpha$ on M. Then there exists a homology class in $H_1(M;\mathbb{R})$ such that the length of each loop γ in the homology class satisfies*

$$L(\gamma) \ge \left(\frac{2\pi \sin\alpha}{\sup\|h\|}\right). \tag{5.8}$$

If T^2 is a Lagrangian imbedded torus in $\mathbb{C}^2$, it is known that the Maslov class $m(T^2)$ of T^2 is even and there exists a simple loop γ on T^2 such that (Viterbo [9])

$$\int_\gamma \Psi = 2$$

where $\Psi = \frac{1}{\pi}\langle JH, \quad \rangle$ is the Maslov 1-form on T^2. Therefore, by Theorem 3.2 and Corollary 5.2 we obtain the following

COROLLARY 5.6. *Let M be a totally real torus in $\mathbb{C}^2$. If M is regularly homotopic to a Lagrangian torus through totally real homotopy, then there exists a homology class in $H_1(M;\mathbb{R})$ such that the length of each loop in the homology class satisfies*

$$L(\gamma) \geq \frac{2\sqrt{2}\pi \sin\theta_0}{\sup\|h\|}. \tag{5.9}$$

ACKNOWLEDGEMENT. The authors would like to take this opportunity to express their thanks to Prof. C. Viterbo for the valuable discussions.

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