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EXTERNAL CURVATURES AND INTERNAL TORSION OF A RIEMANNIAN SUBMANIFOLD

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0. Introduction

The geometrical idea of this work is quite natural. Following the construction of the torsion of a submanifold given by Otsuki [16], and using the principal normal spaces introduced by Allendorfer, we define the "external curvatures" of a submanifold to be entities which, in a certain sense, measure the distance between the submanifold and osculator spaces. Roughly speaking, the second external curvature (or external torsion), for example, measures the rate of which the E_1-sections leave E_1 after parallel displacement; E_1 is the first principal normal space, i.e., the space spanned by the image of the second fundamental form (cf. for example [17], [4]).

The study of the case where $\dim E_1 > 1$ leads us to introduce the notion of internal torsion $\theta^{(M)}$. In analogy with the external torsion, $\theta^{(M)}$ describes the rate of parallel displacement of E_1-section which stay in E_1.

Using these quantities, we give a description of the submanifolds of a space form in the case where $\dim E_1$ is constant and $\leqslant 2$.

1. Preliminaries

Note. When we want to indicate that the dimension of a manifold M is n, we write M^n.

Let (M^n, g) and $(\tilde{M}^{n+p}, \tilde{g})$ be two Riemannian manifolds, and $f\colon M \to \tilde{M}$ be an isometric immersion. We use the following notation: TM and $T\tilde{M}$ are the tangent spaces of M and $\tilde{M}$, ∇ and $\tilde{\nabla}$ are the Levi-Civita connexions on M and $\tilde{M}$, R and $\tilde{R}$ are the curvature tensor of M and $\tilde{M}$, $T^\perp M$ is the normal bundle, $\nabla^\perp$ is the Riemannian connexion induced by $\tilde{\nabla}$ on $T^\perp M$, σ is the second fundamental form of M and K the associated tensor defined by

$$g(K(X,\xi),Y) = \tilde{g}(\sigma(X,Y),\xi),$$

where $X, Y \in TM$, and $\xi \in T^\perp M$.

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We have

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \forall X, Y \in TM$$
$$\tilde{\nabla}_X \xi = \nabla^{\perp}_X \xi - K(X, \xi), \quad \forall X \in TM, \forall \xi \in T^{\perp} M$$

and the following Gauss-Codazzi and Codazzi-Ricci equations:

$$\tilde{R}(X, Y)Z = R(X, Y)Z + K(X, \sigma(Y, Z)) - K(Y, \sigma(X, Z)) + (\overline{\nabla}_X \sigma)(Y, Z) - (\overline{\nabla}_Y \sigma)(X, Z), \tag{1}$$

$$\tilde{R}(X, Y)\xi = R^{\perp}(X, Y)\xi + \sigma(X, K(Y, \xi)) - \sigma(Y, K(X, \xi)) - (\overline{\nabla}_X K)(Y, \xi) + (\overline{\nabla}_Y K)(X, \xi) \tag{2}$$

$\forall X, Y, Z \in TM, \forall \xi \in T^{\perp} M$, where $R^{\perp}$ is the curvature tensor on $T^{\perp} M$, and

$$(\overline{\nabla}_X \sigma)(Y, Z) = \nabla^{\perp}_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z),$$
$$(\overline{\nabla}_X K)(Y, \xi) = \nabla_X(K(Y, \xi)) - K(\nabla_X Y, \xi) - K(Y, \nabla^{\perp}_X \xi).$$

By $[A]$ we denote the vector space spanned by the subspace A of a vector space.

2. External curvatures and internal torsion of a Riemannian submanifold

Let $f = M^n \to \tilde{M}^{n+p}$ be an isometric immersion.

Lemma 1. *Let $\mathfrak{D}$ be a distribution on $T^{\perp} M$. If $\xi \in \mathfrak{D}$ and $X \in T_m M$, then $\mathrm{pr}_{\mathfrak{D}^{\perp}} \nabla^{\perp}_X \xi$ depends only on ξ_m.*

The proof is obvious.

This lemma allows us to give the following definitions.

Definition 1 (cf. [17] *for instance*). Let $m \in M$; we define $(E_0)_m = T_m M$ and

$$(E_1)_m = [\mathrm{Im}\, \sigma_m],$$

(i.e., the space spanned by the image of σ_m). If $\dim E_1$ is constant on a neighborhood of m, we define

$$L_2 \colon T_m M \times (E_1)_m \to T^{\perp}_m M$$
$$(X, \xi) \mapsto \mathrm{pr}_{E_1^{\perp}} \nabla^{\perp}_X \xi$$

and $(E_2)_m = [\mathrm{Im}\, L_2]$. By induction if $\dim(E_{i-1})_m$ is constant on a neighborhood of m, we define

$$L_i \colon T_m M \times (E_{i-1})_m \to T^{\perp}_m M$$
$$(X, \xi) \mapsto \mathrm{pr}_{(\bigoplus_{j<i} E_j)^{\perp}} \nabla^{\perp}_X \xi,$$

and $(E_i)_m = [\mathrm{Im}\, L_i]$, and call E_i the ith principal normal space.

Definition 2. A submanifold M of $\tilde{M}$ is said to be E_j-nicely curved if E_i is a subbundle of $T^{\perp} M$, $\forall i \leqslant j$.

Definition 3. Let $m \in M$. If $(E_1)_m, \ldots, (E_i)_m$ are defined, we call the norm of the bilinear map L_j (with $L_1 = \sigma$), i.e.,

$$\left(k_j^{(M)}\right)_m = \sup_{\substack{X \in T_m M, \|X\|=1 \\ \xi \in (E_{j-1})_m \|\xi\|=1}} \|L_j(X, \xi)\|$$

the jth-external curvature (or jth-Frenet curvature) at m.

The principal normal space gives a decomposition of the normal space $T^{\perp} M$. In order to study submanifolds such that $\dim E_1 > 1$ we introduce a decomposition of E_1. Let $F_1 = \{\eta \in E_1 \mid L_2(X, \eta) = 0 \ \forall X \in TM\}$, and give the map

$$\Theta\colon TM \times F_1 \to E_1$$
$$(X, \eta) \mapsto \mathrm{pr}_{F_1^{\perp}} \nabla_X^{\perp} \eta.$$

We define

$$F_2 = [\operatorname{Im} \Theta] \quad \text{and} \quad (\theta^{(M)})_m = \sup_{\substack{X \in T_m M, \|X\|=1 \\ \eta \in (F_1)_m, \|\eta\|=1}} \|\Theta(X, \eta)\|.$$

If $(F_1)_m = \{0\}$, we say that $\theta_m^{(M)} = -\infty$.

Definition 4. $\theta^{(M)}$ is called the *internal torsion* of M.

Remarks on these definitions.

1. $(E_i)_m = 0 \Leftrightarrow (k_i^{(M)})_m = 0$.

2. A point $m \in M^n$ such that $(k_1^{(M)})_m, \cdots, (k_s^{(M)})_m$ are defined and nonzero will be said to be s-regular.

3. If M is a curve, then $k_i^{(M)}$ coincides with the ith Frenet curvature of the curve. In this case, $\theta^{(M)}$ is finite only if the curve is plane, and $\theta^{(M)} = 0$.

4. Clearly, if $\dim E_1 = 1$ at every point, then $\theta^{(M)} = 0$ or $-\infty$.

5. It can be more interesting (cf. [5]) to take the tensorial norm of the maps L_i to define $k_i^{(M)}$.

Using the work of Burstin, Mayer, Allendoerfer (cf. M. Spivak [17, Vol. IV, Chap. 7, p. 241]), we can immediately deduce the following result.

Theorem 1. *Let M^n be a connected, simply connected submanifold of a space form $\tilde{M}^{n+p}(c)$ (of constant curvature c). Suppose that the principal normal space $E_1 \cdots E_p$ of M satisfy the following conditions:*

M^n is E_p niced-curved, $\dim E_1 \oplus \cdots \oplus \dim E_p = r = \text{const.}$, $k_{p+1}^{(M)} \equiv 0$. Then M^n is a submanifold of $\tilde{M}^{n+p}(c)$ with substantial codimension r (i.e., there exists a totally geodesic submanifold of dimension $n + r$ in $\tilde{M}^{n+p}(c)$ which contains M^n).

Examples.

(a) The unit sphere S^n in the euclidean space $\mathbf{E}^{n+p}$. We have $\dim E_1 = 1$, $k_1^{(S^n)} = 1$, $\dim E_j = 0$ for $j > 1$.

(b) A *cylinder*, i.e., a submanifold M^n in $\mathbf{E}^{n+p}$ such that $M^n = C \times \mathbf{E}^{n-1}$, where C is a curve. The second fundamental form of M^n has the following expression:

$$\sigma(X, Y) = \alpha\langle X, T\rangle\langle Y, T\rangle\xi_1,$$

where T is the unit vector tangent to the curve C, $|\alpha|$ is the curvature of C, and ξ_1 is the first principal normal vector of C. We have

$$\begin{aligned} \nabla_X^{\perp}\xi_1 &= k_2^{(C)}\langle X, T\rangle\xi_2, \\ \nabla_X^{\perp}\xi_{i-1} &= k_i^{(C)}\langle X, T\rangle\xi_i - k_{i-2}^{(C)}\langle X, T\rangle\xi_{i-2}, \\ \nabla_X^{\perp}\xi_i &= -k_{i-1}^{(C)}\langle X, T\rangle\xi_{i-1}, \end{aligned}$$

where $k_j^{(C)}$, $1 \leqslant j \leqslant i$, are the Frenet curvatures of C in $\mathbf{E}^{n+p}$ when these curvatures are defined. We can deduce that if $k_{i-1}^{(C)} \neq 0$ on an open set U, and $k_i^{(C)} = 0$ on U, then

$$\begin{aligned} &\dim E_j = 1 && \text{if} \quad 1 \leqslant j \leqslant i, \\ &\dim E_j = 0 && \text{if} \quad j > 1, \\ &k_j^{(M^n)} = k_j^{(C)} && \text{if} \quad 1 \leqslant j \leqslant p. \end{aligned}$$

(c) The product of two curves C_1, C_2: $M^2 = C_1 \times C_2$, where C_1 and C_2 are two closed curves in $\mathbf{E}^3$, the torsion of which is never zero (cf. [18]). In this case, $\dim E_1 = 2$, $\dim E_2 = 2$. This is an example of a compact submanifold of Euclidean space such that $\dim E_2 \neq 0$ at each point.

(d) A nonextrinsic sphere M^n of a Hermitian symmetric space of compact type, [3], is an example of submanifold such that $\dim E_1 = 1$, $\dim E_2 = n$.

(e) In [10] N. Kuiper proved that any substantial tight compact submanifold M in Euclidean space satisfies $(E_1^{\perp})_m = 0 \ \forall m \in M$.

3. Submanifolds in spaces of constant curvature such that $\dim E_1 \leqslant 1$

Let us consider a submanifold of a Riemannian manifold. Generally, if we suppose that its first principal normal space has dimension 1, we cannot deduce any strong restriction on the second principal normal space (see Example (c), §2). However, we shall show that, if the ambient space has constant curvature, and $\dim E_1 = 1$, then the submanifold is cylindrical (in the sense of B. Y. Chen [2]), and $\dim E_i = 1$ or 0. This will allow us to give a classification of submanifolds such that $\dim E_1 \leqslant 1$.

We shall prove the two following theorems.

Theorem 2. *Let $\tilde{M}^{n+p}(c)$ be a $(n+p)$-dimensional manifold of constant curvature c, and $f\colon M^n \hookrightarrow \tilde{M}^{n+p}(c)$ be an isometric immersion of a connected Riemannian manifold in $\tilde{M}^{n+p}(c)$. Suppose that the first principal normal space E_1 of M satisfy the condition:*

$$\dim E_1 \leqslant 1 \text{ at every point.}$$

Then there exists a dense open set M' of M such that $M' = M_1 \cup M_2$ with $M_1 \cap M_2 = \varnothing$, where M_1 and M_2 are two open sets such that:

(a) *The connected components of M_1 are submanifolds with substantial codimension* 1 *in $\tilde{M}^{n+p}(c)$.*

(b) *M_2 is foliated by hypersurfaces which are totally geodesic in $\tilde{M}^{n+p}(c)$.*

Theorem 3. *Let $\tilde{M}^{n+p}(c)$ be a $(n+p)$-dimensional manifold of constant curvature c, and $f\colon M^n \to \tilde{M}^{n+p}(c)$ be an isometric immersion of a Riemannian manifold in $\tilde{M}^{n+p}(c)$. Suppose that*

(α) *M is connected, complete, and E_s-nicely curved, $s \geqslant 1$,*

(β) $\dim E_1 = 1$ *at every point,*

(γ) $k_2^{(M)} \neq 0$ *at every point (i.e., each point is biregular),*

(δ) $\exists i \in \{1,\cdots,s\}$ *such that* $k_i^{(M)} = \text{const.} \neq 0$.

Then:

(1) $c = 0$,

(2) *M is flat,*

(3) *$M = C \times M_1$, where M_1 is totally geodesic in $\tilde{M}^{n+p}(c)$ and C is a curve of $\tilde{M}^{n+p}(c)$ such that $k_j^{(M)} = k_j^{(C)}$, $j = 1,\cdots,p$, $k_j^{(C)}$ being the classical Frenet curvatures of C in $\tilde{M}^{n+p}(c)$.*

Remark. If $\tilde{M}^{n+p}(c) = \mathbf{E}^{n+p}$, and M^n satisfies only (α), (β), (γ), using a theorem of O'Neill [15] we can conclude that $M = C \times \mathbf{E}^{n-1}$, where C is a curve in $\mathbf{E}^{n+p}$.

In order to prove this theorem, we need the following propositions.

Proposition 1. *Let $f\colon M_1^n \to \tilde{M}^{n+p}(c)$ be an isometric immersion of a connected manifold in a space $\tilde{M}^{n+p}(c)$ of constant curvature c. Suppose that the first principal normal space E_1 of M_1 has dimension* 1 *at every point of M_1, and that the second external curvature $k_2^{(M_1)}$ of M_1 is null everywhere. Then M_1 is a submanifold of substantial codimension equal to* 1 *in $\tilde{M}^{n+p}(c)$.*

Proof of Proposition 1. Use Theorem 1.

Proposition 2. *Let $f\colon M_2^n \to \tilde{M}^{n+p}(c)$ be an isometric immersion of a connected n-dimensional ($n \geqslant 2$) manifold in a space $\tilde{M}^{n+p}(c)$ of constant curvature c. Suppose that the first principal normal space E_1 of M_2 has dimension* 1 *at every point of M_2, and that every point of M_2 is biregular. Then for every s-regular ($2 \leqslant s \leqslant p$) $m \in M_2$ there exists a unique, except for the sign, unit vector system*

$\{\xi_1,\cdots,\xi_s\}$ orthogonal to M_2, and defined on a neighborhood of m, and $s-1$ nonnull linear forms $\tau_2,\cdots,\tau_s$, which are closed, proportional, and defined on a neighborhood of m, such that:

$$\|\tau_i\| = k_i^{(M_2)}, \quad \text{and} \quad \forall X \in T_m M_2,$$

$$\nabla^{\perp}_X \xi_1 = \tau_2(X)\xi_2,$$
$$\nabla^{\perp}_X \xi_2 = \tau_3(X)\xi_3 - \tau_2(X)\xi_1,$$
$$\cdots\cdots\cdots$$
$$\nabla^{\perp}_X \xi_{s-1} = \tau_s(X)\xi_s - \tau_{s-1}(X)\xi_{s-2}, \quad \text{for } s \leq p-1,$$
$$\nabla^{\perp}_X \xi_p = -\tau_p(X)\xi_{p-1}.$$

To prove this proposition we need the following.

Lemma 2. *Let h: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be a symmetric bilinear form, and L: $\mathbf{R}^n \to \mathbf{R}^p$ a linear application. If $L \neq 0$, $h \neq 0$ and*

$h(Y,Z)\,L(X) - h(X,Z)\,L(Y) = 0 \;\; \forall X, Y, Z \in \mathbf{R}^n$,

then rg h = rg $L = 1$ *and* Ker h = Ker L.

The proof of Lemma 2 is obvious.

Proof of Proposition 2. Since the curvature of $\tilde{M}^{n+p}$ is a constant c, the normal component of $\tilde{R}'(X,Y)Z$ is null $\forall X, Y, Z \in TM_2$. Consequently, (1) $\Rightarrow (\overline{\nabla}_X\sigma)(Y,Z) = (\overline{\nabla}_Y\sigma)(X,Z)$. Since dim $E_1 = 1$, we can write $\sigma(X,Y) = h(X,Y)\xi_1$, where ξ_1 is unique except for the sign. (1) $\Leftrightarrow$

$$(1)' \qquad (\nabla_X h)(Y,Z)\xi_1 - (\nabla_Y h)(X,Z)\xi_1 + h(Y,Z)\nabla^{\perp}_X \xi_1 - h(X,Z)\nabla^{\perp}_Y \xi_1 = 0.$$

The projection of (1′) on ξ_1 gives

$$h(Y,Z)\,L(X) - h(X,Z)\,L(Y) = 0, \quad \forall X, Y, Z \in TM,$$

where $L(X) = \nabla^{\perp}_X \xi_1$.

Since dim $E_1 = 1$, $h \neq 0$; since m is s-regular with $s \geq 2$, $L \neq 0$. Then applying Lemma 2 we obtain rg h = rg $L = 1$ at m and consequently on a neighborhood of m. Let $L(X) = \tau_2(X)\xi_2$, where ξ_2 is a unit vector field of Im L on a neighborhood of m.

Since the curvature of $\tilde{M}^{n+p}$ is a constant c, (2) gives $\tilde{R}(X,Y)\xi_1 = 0$. Then the normal componant of $\tilde{R}(X,Y)\xi_1$ is null:

$$(2)' \qquad R^{\perp}(X,Y)\xi_1 = \sigma(X, K(Y,\xi_1)) - \sigma(Y, K(X,\xi_1)).$$

The projection of (2′) on ξ_2 gives $d\tau_2 = 0$.

Now assume that m is s-regular, $s \geq 3$. The projection of (2′) on ξ_1 gives

$$\mathrm{pr}_{\xi_1^{\perp}}[\tau_2(Y)\nabla^{\perp}_X \xi_2 - \tau_2(X)\nabla^{\perp}_Y \xi_2] = 0.$$

Let $M(X) = \mathrm{pr}_{\xi_1^\perp}(\nabla^\perp_X \xi_2)$. $M \neq 0$ because $s \geqslant 3$. (2′) gives

$$(2'') \qquad \tau_2(Y)M(X) - \tau_2(X)M(Y) = 0 \quad \forall X, Y \in TM_2.$$

Since $\tau_2 \not\equiv 0$ and $M \not\equiv 0$, we deduce that $\operatorname{Ker}\tau_2 = \operatorname{Ker} M$. Hence $\operatorname{rg} M = 1$, and there exist a unit vector ξ_3 and a linear form τ_3 such that $M(X) = \tau_3(X)\xi_3$. Moreover by (2″) we have

$$\tau_2(Y)\tau_3(X) - \tau_2(X)\tau_3(Y) = 0,$$

i.e., $\tau_2 \wedge \tau_3 = 0$. Finally, $\nabla^\perp_X \xi_2 = \tau_3(X)\xi_3 - \tau_2(X)\xi_1$.

We proceed in such a way, studying the projection of $\tilde{R}(X, Y)\xi_i$ on ξ_{i+1} and ξ_{i+2}, $1 \leqslant i \leqslant s$. Now we can evaluate the external curvatures of M_2:

$$\left(k_2^{(M_2)}\right)_m = \sup_{\substack{\eta \in E_{1_m} \\ \|\eta\|=1}} \sup_{\substack{X \in T_m M_2 \\ \|X\|=1}} \|\mathrm{pr}_{E_{1m}^\perp} \nabla^\perp_X \eta\|$$

$$= \sup_{\substack{X \in T_m M_2 \\ \|X\|=1}} \|\tau_2(X)\xi_2\| = \|\tau_2\|_m,$$

and, since $E_1 = [\xi_1]$, $E_2 = [\xi_2], \cdots, E_i = [\xi_i], \cdots$,

$$\left(k_i^{(M)}\right)_m = \sup_{\substack{\eta \in (E_{i-1})_m \\ \|\eta\|=1}} \sup_{\substack{X \in T_m M_2 \\ \|X\|=1}} \left\|\mathrm{pr}_{\underset{j<i}{\oplus}(E_j)_m^\perp} \nabla^\perp_X \eta\right\|$$

$$= \sup_{\substack{X \in T_m M_2 \\ \|X\|=1}} \|\tau_i(X)\xi_i\| = \|\tau_i\|_m.$$

Proposition 3. *Let f: $M_2^n \to \tilde{M}^{n+p}(c)$ be an isometric immersion of an n-dimensional manifold M_2 in an $(n+p)$-dimensional manifold of constant curvature c, $(n \geqslant 2)$, such that* $\dim E_1 = 1$. *If $k_2^{(M_2)} \neq 0$ at every point of M_2, then M_2 is foliated by totally geodesic $(n-1)$-submanifolds of $\tilde{M}^{n+p}$.*

Proof of Proposition 3. Since every point of M_2 is 2-regular, the form $\tau_2 = \|\nabla^\perp \xi_1\|$ is defined (except for the sign) on M_2. Let T_2 be the vector field ($\neq 0$ for $k_2^{(M)} \neq 0$) associated with τ_2 in the duality defined by the metric, and let $T = T_2/\|T_2\|$.

$$(1') \qquad \Leftrightarrow h(Y, Z)\langle T, X\rangle = h(X, Z)\langle T, Y\rangle.$$

Thus $h(X, Y) = \beta\langle X, T\rangle\langle Y, T\rangle$ with $\beta = h(T, T) \neq 0$. Consequently, the relative nullity index is constant $(= n-1)$ on M_2. Hence applying a result of [1] we conclude that M_2 is foliated by totally geodesic $(n-1)$-dimensional submanifolds of $\tilde{M}^{n+p}$.

We shall now prove Theorem 2 and Theorem 3.

Proof of Theorem 2. Let $m \in M$. One of the following three possibilities can happen.

A. $\exists U_1$, an open neighborhood of m, such that $\dim E_1|_{U_1} \equiv 0$. In this case, U_1 is totally geodesic, and of course, foliated by hypersurfaces which are totally geodesic in $\tilde{M}^{n+p}(c)$.

B. $\exists U_2$, an open neighborhood of m, such that $\dim E_1|_{U_2} = 1$ and $k^{(M)}_{2|U_2} \equiv 0$. In this case, using Proposition 1 we can conclude that locally the substantial codimension of U_2 is one.

C. $\exists U_3$, an open neighborhood of m, such that $\dim E_1|_{U_3} = 1$ and $k_2^{(M)} \neq 0$. Then using Proposition 2 we can conclude that U_3 is foliated by hypersurfaces which are totally geodesic in $\tilde{M}^{n+p}(c)$.

Finally, it is clear that there exists a dense open set M' of M on which one of these three possibilities happens. Hence Theorem 2 is proved.

Proof of Theorem 3. We can suppose that M is simply connected. The general result is obtained by passing to the universal covering of M. The proof consists in building a parallel vector field on M. Then we apply the De Rham decomposition theorem (cf. [9]). We need the following lemmas.

Lemma 3. $k_i^{(M)} = |\tau_i(T)|$ *if* $i \geqslant 2$.

This is a consequence of Proposition 2.

Lemma 4. *Let* ω *be the form associated to* T *in the duality defined by the metric. Then* $d(\beta\,\omega) = 0$.

Proof of Lemma 4. Since $\tilde{M}^{n+p}$ is of constant curvature, the normal componant of $\tilde{R}(X, Y)T$ is null $\forall X, Y \in TM$.

$$(1) \Leftrightarrow (\nabla_X \sigma)(Y, T) = (\nabla_Y \sigma)(X, T).$$

Projecting this equality on ξ_1, we obtain $d(\beta\,\omega) = 0$.

Lemma 5. *If there exists* $i \in [1 \cdots p]$ *such that* $k_i^{(M)} = \text{const.} \neq 0$, *then* $X(\beta) = 0$, $\forall X \perp T$.

Proof of Lemma 5. If $i = 1$, then $k_1^{(M)} = \operatorname{Sup}\|\sigma(X, Y)\| = |h(T, T)| = |\beta|$. Thus $\beta = \text{const}$. Hence $X(\beta) = 0$, $\forall X \perp T$.

If $i \geqslant 2$, since $\omega = \tau_i/\|\tau_i\|$, by Lemma 4 we have $d(\beta\tau_i/\|\tau_i\|) = 0$. $\|\tau_i\| = k_i^{(M)} = \text{const.} \Rightarrow d(\beta\,\tau_i) = 0 \Rightarrow d\beta \wedge \tau_i = 0$ since $d\,\tau_i = 0$, (by Proposition 2) $\Rightarrow X(\beta) = 0$, $\forall X \perp T$.

Lemma 6. *If there exists* $i \in [1 \cdots p]$ *such that* $k_i^{(M)} = \text{const.} \neq 0$, *then* T *is parallel.*

Proof of Lemma 6. From (2) we deduce

$$(2''') \qquad (\nabla_X K)(T, \xi_1) = (\nabla_T K)(X, \xi_1).$$

Let $X \perp T$, $X \in TM$. Since $K(Y, \xi_1) = \beta\langle Y, T\rangle T$, $\forall Y \in TM$, we have $K(X, \xi_1) = 0$. Hence (2''') $\Leftrightarrow X(\beta)T + \beta\,\nabla_X T = \beta\langle X, \nabla_T T\rangle T$. Since $X \perp T$, $X(\beta) = 0$. Therefore $\beta\,\nabla_X T = \beta\langle X, \nabla_T T\rangle T$. Since $\beta \neq 0$ and $\nabla_X T \perp T$, we deduce $\nabla_X T = 0$ if $X \perp T$, and $\nabla_T^{\perp} T = 0$. Consequently T is parallel.

Now we return to the proof of Theorem 3. Since $\tilde{M}^{n+p}$ is of constant curvature c,

$$\begin{aligned}\tilde{R}(X,Y)Z &= c\{\langle X,Z\rangle Y - \langle Y,Z\rangle X\}\\ &= R(X,Y)Z - K(X,\beta\langle Y,T\rangle\langle Z,T\rangle\xi_1)\\ &\quad + K(Y,\beta\langle X,T\rangle\langle Z,T\rangle\xi_1)\\ &= R(X,Y)Z.\end{aligned}$$

Hence the curvature of M is c, and M possesses a parallel field. It follows that $c = 0$ so that M and $\tilde{M}^{n+p}$ are flat.

On the other hand, the distributions Δ_1 and Δ_2 defined by T and $T^{\perp}$ are parallel and differentiable. Hence M is the product of $C \times M_1$ where C and M_1 are maximal integral submanifolds of Δ_1 and Δ_2. It is easy to see that M_1 is totally geodesic in $\tilde{M}^{n+p}$.

Now we can estimate the Frenet curvatures of C in $\tilde{M}^{n+p}$:

$$\tilde{\nabla}_T T = \nabla_T T + \beta\xi_1 = \beta\xi_1, \quad k_1^{(C)} = |\beta| = k_1^{(M)};$$

$$\tilde{\nabla}_T \xi_1 = \nabla_T^{\perp}\xi_1 - K(T,\xi_1) = \tau_2(T)\xi_2 - \beta T, \quad k_2^{(C)} = |\tau_2(T)| = k_2^{(M^n)};$$

$$\tilde{\nabla}_T \xi_i = \nabla_T^{\perp}\xi_i - K(T,\xi_i) = \tau_{i+1}(T)\xi_{i+1} - \tau_i(T)\xi_{i-1},$$

$$k_{i+1}^{(C)} = |\tau_{i+1}(T)| = k_{i+1}^{(M)}.$$

Therefore $k_i^{(C)} = k_i^{(M^n)}$, $\forall i \in [1 \cdots p]$.

4. Submanifolds such that $\dim E_1 = 2$

Let us now consider a submanifold M of a space of constant curvature, such that $\dim E_1 = 2$. We shall show that it is possible to describe M with the external curvatures and the internal torsion. We shall prove the following theorems.

Theorem 4. *Let $f\colon M^n \mapsto \tilde{M}^{n+p}(c)$ be an isometric immersion of an n-dimensional manifold M^n in the space form $\tilde{M}^{n+p}(c)$, $n \geqslant 3$, $p \geqslant 2$. Suppose that*

$$\dim E_1 = 2 \text{ at every point } m \in M.$$

Then M contains a dense open set M' such that

$$M' = M_1 \cup M_2 \cup M_3, \quad (M_i \cap M_j = \varnothing,\ i \neq j),$$

where M_1, M_2, M_3 are three open sets such that:

(a) *The connected componants of M_1 are submanifolds of $\tilde{M}^{n+p}(c)$ which have a substantial codimension equal to 2,*

(b) M_2 *is foliated by hypersurfaces of substantial codimension equal to* 2 *in* $\tilde{M}^{n+p}(c)$,

(c) M_3 *is foliated by* $(n-2)$*-dimensional totally geodesic submanifolds of* $\tilde{M}^{n+p}(c)$.

Theorem 5. *Let* $f\colon M^n \to \tilde{M}^{n+p}(c)$ *be an isometric immersion of an n-dimensional manifold* M^n *in the space form* $\tilde{M}^{n+p}(c)$, $n \geqslant 3$, $p \geqslant 2$, *such that*

(i) $\dim E_1 = 2$ *at every point* $m \in M$,

(ii) *every point of M is s-regular*, $s \geqslant 2$,

(iii) *the internal torsion* $\theta^{(M)}$ *is constant.*

Then each of the following holds:

(A) *If the internal torsion* $\theta^{(M)} = 0$, *and* $\exists i \in \{2, \cdots, s\}$ *such that* $k_i^{(M)} = \text{const.} \neq 0$ *and M is complete, connected, then* $M = C \times M_1$, *where C is a curve, and* M_1 *a submanifold with substantial codimension* 1. *Moreover, if* $c = 0$, *we have* $k_j^{(C)} = k_j^{(M)}$, $\forall j \geqslant 2$; *if* $c \neq 0$, *then* M_1 *is an open set of an "n-sphere".*

(B) *If the internal torsion* $\theta^{(M)} = \text{const} \neq 0$, *and* $\exists i \in \{2, \cdots, s\}$ *such that* $k_i^{(M)} = \text{const} \neq 0$, *then M is foliated by* $(n-1)$*-dimensional submanifolds* M_2 *with substantial codimension* 2. *In particular, if* $c \neq 0$, *then* M_2 *is included in an "n-sphere".*

(C) *If the internal torsion* $\theta^{(M)} = -\infty$, *then M is foliated by* $(n-2)$*-dimensional submanifolds which are totally geodesic in* $\tilde{M}^{n+p}$.

In order to prove these theorems, we need to study the biregular submanifolds such that $\dim E_1 = 2$. This will be done in §§4.1, 4.2, 4.3. The proof of the theorems are in §§4.4 and 4.5.

4.1. Biregular submanifolds such that $\dim E_1 = 2$

Proposition 4. *Let* $f\colon M^n \to \tilde{M}^{n+p}(c)$ *be an isometric immersion of an n-dimensional manifold* M^n *in an* $(n+p)$*-dimensional* ($n \geqslant 3$, $p \geqslant 2$) *manifold* $\tilde{M}^{n+p}(c)$ *of constant curvature c such that* $\dim E_1 = 2$ *at every point and such that every point is* 2*-regular. Then each of the following holds*:

(i) *If* $\theta^{(M)} \neq -\infty$ *at every point of M, there exists a global, except for the sign, frame* (ξ, η) *of* E_1 *such that* $L_\xi \neq 0$ *and* $L_\eta = 0$, *where* $L_\xi(x) = \text{pr}_{E_1^\perp} \nabla_x^\perp \xi$. *Moreover*, $\dim E_2 = 1$ *at every point of M.*

(ii) *If* $\theta^{(M)} = -\infty$ *at every point, then the index of relative nullity of M is* $n-2$ *at every point of M. Moreover*, $\dim E_2 \leqslant 2$.

Proof of Proposition 4. (i) Since $k_2^{(M)} \neq 0$ at every point $m \in M$, then $\dim F_{1_m} < \dim E_{1_m}$ at every point (F_1 is defined in §2). Since $\dim E_1 = 2$, $\dim F_{1_m} < 2$.

On the other hand, since $\theta_m^{(M)} \neq -\infty$ at every point m, $\dim F_{1_m} > 0$ at every point m. Consequently $\dim F_1 \equiv 1$, and F_1 is a subbundle of $T^{\perp} M$, with fibers of dimension 1.

Let η be the global section (except for the sign), which spans F_1. We have $L_\xi = 0$ at every point m. If ξ is a section of E_1 such that $\langle \eta, \xi \rangle = 0$ and $\|\xi\| = 1$, it is clear that $L_\xi \neq 0$ at every point.

(ii) Let ν be the index of relative nullity of M. ($\nu(m) = \dim N_m$, where $N_m = \{X \in T_m M / \sigma(X, Y) = 0, \ \forall Y \in TM\}$). We have $\nu(m) \leqslant n - 2$ for every $m \in M$. In fact, if $\nu(m) = n$, m is a flat point; this is impossible for $(k_2^{(M)})_m \neq 0$. If $\nu(m) = n - 1$, then $\dim (E_1)_m = 1$, which is excluded.

In order to show that $\nu(m) = n - 2$, and that $\dim E_2 \leqslant 2$, we need the following two lemmas.

Lemma 7. *Let $m \in M$ such that there exists an orthonormal frame (ξ, η) of $(E_1)_m$ such that L_ξ and L_η are not proportional. Then $\nu(m) = n - 2$ (and $\dim (E_2)_m \leqslant 2$).*

Lemma 8. *Let $\theta^{(M)} = -\infty$ at every point of M. Then, for every $m \in M$, every neighborhood of m and every orthonormal frame (ξ, η) of E_1 on U, there exists a neighborhood $V \subset U$ such that L_ξ and L_η are not proportional on V.*

Combining these two lemmas we obtain

(*) $\forall m \in M$, $\forall U$, neighborhood of m, $\exists v$, open, $V \subset U$, such that $\nu|_V = n - 2$.

Now assume that there exists $m \in M$ such that $\nu(m) < n - 2$. Since ν is upper semicontinuous, there exists a neighborhood U of m such that $\nu|_U < n - 2$. But this is impossible because of (*). Thus $\nu_m = n - 2$ at every point $x \in M$.

Proof of Lemmas 7 *and* 8. The proof of Lemma 7 results from the following algebraic lemma.

Lemma. *Let L, M: $\mathbf{R}^n \to \mathbf{R}^p$ be two linear maps. If there exist α, β: $\mathbf{R}^n \to \mathbf{R}$ not simultaneously null such that*

$$\alpha(X)L(X) + \beta(X)M(X) = 0 \quad \forall X \in \mathbf{R}^n,$$

Then L and M are proportional or rg $L \leqslant 1$ *and* rg $M \leqslant 1$.

Proof. Let rg $L = k$, and let $v_1, \cdots, v_p$ be a basis of $\mathbf{R}^p$ such that

$$L(X) = \omega_1(X)v_1 + \cdots + \omega_k(X)v_k,$$
$$M(X) = \pi_1(X)v_1 + \cdots + \pi_p(X)v_p,$$

where $\omega_1, \cdots, \omega_k$ are independent linear forms.

(a) If $\exists l > k$ such that $\pi_l \neq 0$, there exists X_0 such that $\pi_l(X_0) \neq 0$. Thus $\beta_l(X_0)\pi_l(X_0) = 0$. Consequently $\beta_l(X_0) = 0$ and therefore $\alpha_l(X_0) \neq 0$, from which it follows that $L(X_0) = 0$. But the set of the X_0 such that $\pi_l(X_0) \neq 0$ is dense, and L continuous, so $L = 0$. (In particular L and M are proportional.)

(b) Suppose $L \neq 0$ and $M \neq 0$. By the argument of (a) we see that rg $L =$ rg M. If rg $L = 1$, the lemma is proved.

Suppose rg $L = k > 1$, and let, for example,

$$L(X) = \omega_1(X)v_1 + \cdots + \omega_k(X)v_k,$$
$$M(X) = \pi_1(X)v_1 + \cdots + \pi_k(X)v_k.$$

We have

$$\alpha(X)[\omega_1(X)v_1 + \cdots + \omega_k(X)v_k] + \beta(X)[\pi_1(X)v_1 + \cdots + \pi_k(X)v_k] = 0.$$

Let X_0 be an element of Ker ω_k. Then $\beta(X_0)\pi_k(X_0) = 0$. If $\beta(X_0) = 0$, we have $\alpha(X_0) \neq 0$. Thus

$$\omega_1(X_0)v_1 + \cdots + \omega_{k-1}(X_0)v_{k-1} = 0,$$

so that $X_0 \in$ Ker L; therefore rg $L \leqslant 1$ which is excluded. Hence $\beta(X_0) \neq 0$ and $\pi_k(X_0) = 0$.

Then Ker $\omega_k \subset$ Ker π_k so that

$$\pi_k = \lambda_k \omega_k \quad (\lambda_k \in \mathbf{R}).$$

Thus

$$L(X) = \omega_1(X)v_1 + \cdots + \omega_k(X)v_k,$$
$$M(X) = \lambda_1\omega_1(X)v_1 + \cdots + \lambda_k\omega_k(X)v_k.$$

We deduce

$$\alpha(X)\omega_1(X) + \beta(X)\lambda_1\omega(X) = 0,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\alpha(X)\omega_k(X) + \beta(X)\lambda_k\omega_k(X) = 0.$$

By choosing an X_0 such that $\omega_1(X_0) = 1$ and $\omega_2(X_0) = 1$, we obtain

$$\alpha(X_0) + \lambda_1\beta(X_0) = 0,$$
$$\alpha(X_0) + \lambda_2\beta(X_0) = 0,$$

from which it follows that $\lambda_1 = \lambda_2$ since $\alpha(X_0)$ and $\beta(X_0)$ are not both zero.

In the same way one can prove that $\lambda_2 = \lambda_3$, etc. So L is proportional to M.

Lemma 9. *Let h and k be two nonnull and nonproportional bilinear symmetric forms on $\mathbf{R}^n$ ($n \geqslant 3$), and L, M two linear maps from $\mathbf{R}^n$ into $\mathbf{R}^p$ such that*

$$(**)\quad h(Y,Z)\,L(X) + k(Y,Z)\,M(X) = h(X,Z)\,M(Y) + k(Y,Z)\,M(Y), \quad \forall X, Y, Z \in \mathbf{R}^n.$$

Then

(1) $\operatorname{Ker} h \cap \operatorname{Ker} k = \operatorname{Ker} L \cap \operatorname{Ker} M$,

(2) $\dim(\operatorname{Ker} h \cap \operatorname{Ker} k) = n - 2$,

(3) $\dim[\operatorname{Im} L \cup \operatorname{Im} M] \leqslant 2$.

The fact that $\operatorname{Ker} h \cap \operatorname{Ker} k = \operatorname{Ker} L \cap \operatorname{Ker} M$ is a straightforward exercise.

On the other hand, $\dim \operatorname{Ker}(h \cap \operatorname{Ker} k) \leqslant n - 2$ because h and k are nonproportional and nonnull. We prove that $\dim(\operatorname{Ker} h \cap \operatorname{Ker} k) \geqslant n - 2$.

Suppose that $\dim(\operatorname{Ker} h \cap \operatorname{Ker} k) \leqslant n - 3$, and let $F = (\operatorname{Ker} h \cap \operatorname{Ker} k)^{\perp}$, $\dim F \geqslant 3$. For $X_0 \in F$, let $G_1 = \{Y \in F \mid h(Y, X_0) = 0\}$ and $G_2 = \{Y \in F \mid k(Y, X_0) = 0\}$. We have

$$\dim G_1 \cap G_2 \geqslant \dim F - 2 \geqslant 1.$$

Therefore there exists $Z_0 \in F$ such that $h(X_0, Z_0) = 0$ and $k(X_0, Z_0) = 0$. Thus $\forall X_0 \in F$, $\exists Z_0 \in F$ such that $h(Y, Z_0)\,L(X_0) + k(Y, Z_0)\,M(X_0) = 0$, $\forall Y \in \mathbf{R}^n$. Since $Z_0 \notin \operatorname{Ker} h \cap \operatorname{Ker} k$, there exists $Y_0 \in \mathbf{R}^n$ such that $\alpha = h(Y_0, Z_0)$ and $\beta = k(Y_0, Z_0)$ are not simultaneously null (α and β depend on X_0). Hence $\forall X_0 \in F$, $\exists \alpha_{X_0}, \beta_{X_0} \in \mathbf{R}$ not both zero such that

$$\alpha_{X_0} L(X_0) + \beta_{X_0} M(X_0) = 0.$$

Going back to the problem, if $\bar{L} = L|_F$ and $\bar{M} = M|_F$, then $\bar{L}$ and $\bar{M}$ are proportional or $\operatorname{rg} \bar{L} \leqslant 1$ and $\operatorname{rg} \bar{M} = 1$. Since $F = (\operatorname{Ker} L \cap \operatorname{Ker} M)^{\perp}$, L and M are proportional or $\operatorname{rg} L \leqslant 1$ and $\operatorname{rg} M \leqslant 1$. Hence these two cases are excluded respectively by the hypothesis and the assumption that $\dim(\operatorname{Ker} h \cap \operatorname{Ker} k) < n - 2$.

For the proof of the last part (3), see [14].

Proof of Lemma 8. Let (ξ, η) be an orthonormal frame of E_1 on U. Then $(L_\xi)_\eta = 0$ and $(L_\eta)_m = 0$ is impossible for $(k_2^{(M)})_m \neq 0$.

Suppose that $(L_\xi)_m \neq 0$ and $(L_\eta)_m = 0$. Let $W \subset U$ be a neighborhood of m on which $L_{\xi|W} \neq 0$. On W there exists a point p such that $(L_\eta)_p \neq 0$ (for if $L_{\eta|W} = 0$, then $\theta_p^{(M^n)} \neq -\infty$). If there exists a neighborhood W' of p such that $L_\xi = \alpha L_\eta$ on W', then $L_{\xi'|W} = 0$ where $\xi' = (-\xi + \alpha\eta)(1 + \alpha^2)^{-1/2}$. But this is impossible because $\theta_p^{(M^n)} = -\infty$. Therefore $\forall W$ neighborhood of p, there exists $p' \in W'$ such that at p', $L_\xi \neq 0$ and $L_\eta \neq 0$, and L_ξ, L_η are not proportional. Since L_ξ and L_η are continuous, there exists a neighborhood V of p' such that these properties are satisfied.

Finally, if $(L_\xi)_m \neq 0$ and $(L_\eta)_m \neq 0$, we can take $p = m$.

4.2. The case $\theta^{(M)} \neq -\infty$ and a Frenet frame over $T^{\perp} M$

Proposition 5. *Let $\tilde{M}^{n+p}$ be an $(n+p)$-dimensional $(n \geqslant 3, p \geqslant 2)$ manifold of constant curvature, and M^n be an n-dimensional isometric submanifold of $\tilde{M}^{n+p}$ such that*

(i) $\dim E_1 = 2$ *at every point,*

(ii) *every point of M is s-regular* $(s \geqslant 2)$,

(iii) $\theta^{(M)} \neq -\infty$ *at every point.*

Let (ξ, η) be the orthonormal frame of E_1 (defined in Proposition 4), and $\sigma = (h \otimes \xi + k \otimes \eta)$ be the second fundamental form of M^n. Then each of the following holds:

(1) *There exist s nonnull and nonproportional scalar forms $\tau_2, \cdots, \tau_s, \theta$ on M everywhere such that*

$$d\tau_i = 0, \quad \|\tau_i\| = k_i^{(M)} \, (i \geqslant 2), \quad \|\theta\| = \theta^{(M)}.$$

(2) *There exist $s-1$ normal orthonormal global (except for the sign) sections $\xi_2, \cdots, \xi_s$ such that*

$$E_2 = [\xi_2], \cdots, [E_s] = \xi_s,$$
$$\tilde{\nabla}_X \xi = -\theta(X)\,\eta + \tau_2(X)\,\xi_2, \quad \nabla^{\perp}_X \eta = \theta(X)\,\xi,$$
$$\tilde{\nabla}_X \xi_2 = -\tau_2(X)\,\xi + \tau_3(X)\,\xi_3,$$
$$\tilde{\nabla}_X \xi_3 = -\tau_3(X)\,\xi_2 + \tau_4(X)\,\xi_4,$$
$$\cdots\cdots\cdots$$
$$\tilde{\nabla}_X \xi_{s-1} = -\tau_{s-1}(X)\xi_{s-2} + \tau_s(X)\xi_s.$$

(3) $h(X, Y) = \beta\langle X, T\rangle\langle Y, T\rangle$ *where* $\beta = h(T, T)$, *and T is the global (except for the sign) vector field on M, which is associated to $\tau_2/\|\tau_2\|$ in the duality defined by the metric.*

(4) $d\theta = \beta[k(X, T)\langle Y, T\rangle - k(Y, T)\langle X, T\rangle]$.

(5) *The distribution on M, defined by $T^{\perp}$, is involutive.*

Proof. We know that ξ and η satisfy $L_\xi \neq 0$ and $L_\eta = 0$. Using the Gauss-Codazzi equation

$$(\overline{\nabla}_X \sigma)(Y, Z) = (\overline{\nabla}_Y \sigma)(X, Z)$$

and projecting on $E_1^{\perp}$, we find

$$h(Y, Z)\, L_\xi(X) - h(XZ)\, L_\xi(Y) = 0.$$

Therefore by Lemma 2 we deduce that $\operatorname{rg} h = \operatorname{rg} L_\xi = 1$, so that there exist a scalar 1-form τ_2 and a vector field ξ_2 such that

$$h(X, Y) = h(T, T)\langle X, T\rangle\langle Y, T\rangle,$$
$$\operatorname{pr}_{E_1^{\perp}} \nabla^{\perp}_X \xi = \tau_2(X)\, \text{-}\xi_3, \quad \tau_2 \neq 0,$$

where T is the vector field associated to $\tau_2/\|\tau_2\|$ in the duality defined by the metric.

On the other hand, since $\dim E_1 = \dim[\operatorname{Im}\sigma] = 2$ and $\langle \xi, \eta \rangle = 0$, we can find a scalar form θ such that

$$\mathrm{pr}_{E_1} \nabla^{\perp}_X \eta = \theta(X)\,\xi.$$

Consequently, we have

$$\nabla^{\perp}_X \xi = -\theta(X)\,\eta + \tau_2(X)\,\xi_2,$$
$$\nabla^{\perp}_X \eta = \theta(X)\,\xi,$$

from which we deduce that $E_2 = [\xi_2]$.

By Gauss-Codazzi equations we have that $\tilde{R}(X, Y)\eta = 0\ \forall X, Y \in TM$, so that

(2) $$R^{\perp}(X, Y)\,\eta - \sigma(X, K(Y, \eta)) + \sigma(Y, K(X, \eta)) = 0.$$

Projecting (2) on $E_1^{\perp}$ gives $\theta \wedge \tau_2 = 0$.

In the same way, we have

(3) $$\tilde{R}(X, Y)\,\xi = 0, \quad \forall X, Y \in TM.$$

Projecting (3) on ξ_2 we find $d\,\tau_2 = 0$.

Finally

$$k_{2_m}^{(M)} = \sup_{X \in T_m M,\, \|X\| = 1} \|\mathrm{pr}_{E_1^{\perp}} \nabla_X \xi\|_m = \|\tau_2\|_m,$$

and $k_2^{(m)} = \|\tau_2\|$.

We conclude by induction. Since $d\,\tau_2 = 0$, $T^{\perp}$ is involutive. Thus

$$\|\theta\|_m = \sup_{X \in T_m M,\, \|X\| = 1} \|\mathrm{pr}_{E_1} \nabla^{\perp}_X \eta\|_m.$$

Since η is the only section of F_1, we deduce immediately that $\|\theta\| = \theta^M$.

Finally projecting on η the equation $\tilde{R}(X, Y)\,\xi = 0$ yields readily

$$d\,\theta(X, Y) = \beta[\langle Y, T\rangle k(X, T) - \langle X, T\rangle k(Y, T)].$$

4.3. The case where $\exists i$ such that $k_i^{(M)} = \text{const.}$ and $\theta^{(M)} = \text{const.}$

Proposition 6. *With the same hypotheses as in Proposition 5, if* $\exists i \in \{2, \cdots, s\}$ *such that* $k_i^{(M)} = \text{const.} \neq 0$, $\theta^{(M)} = \text{const.} \neq -\infty$, *then*

(1°) $d\,\theta = 0$,

(2°) $k(X, T) = k(T, T)\langle X, T\rangle$,

(3°) $\theta^{(M)} k(X, Y) = \theta^{(M)} k(T, T)\langle X, T\rangle\langle Y, T\rangle + \beta\langle \nabla_X T, Y\rangle$,

(4°) $\nabla_T T = 0$.

Proof. (1°) We have $k_i = \|\tau_i\| = \text{const.}$ and $d\tau_i = 0$. If $\pi = \tau_2/\|\tau_2\|$, $\forall i \in [2 \cdots s]$, then $d\pi = 0$ since $\pi = \tau_i/\|\tau_i\|$. Thus $\theta = \theta^{(M)}\pi$ (cf. Proposition 5 (1)), and consequently $d\theta = 0$, because $\theta^{(M)} = \text{const}$.

(2°) is a consequence of Proposition 5 (4).

(3°) The Gauss-Codazzi equations give $(\overline{\nabla}_X \sigma)(X, Z) = (\overline{\nabla}_Y \sigma)(X, Z)$. Projecting this equation on ξ and η we obtain

(i) $(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) - k(X, Z)\theta(Y) + k(Y, Z)\theta(X) = 0$,

(ii) $(\nabla_X k)(Y, Z) - (\nabla_Y k)(X, Z) = 0$.

Since $h = \beta\pi \otimes \pi$, from (i) it follows that

$$m(X, Z)\langle Y, T\rangle = m(Y, Z)\langle X, T\rangle,$$

where

$$m(X, Y) = \beta\langle \nabla_X T, Z\rangle - \theta^{(M)}k(X, Z).$$

Hence

$$m(X, Y) = m(T, T)\langle X, T\rangle\langle Y, T\rangle,$$

i.e.,

$$\beta\langle \nabla_X, T, Y\rangle - \theta^{(M)}k(X, Y) = -\theta^{(M)}k(T, T)\langle X, T\rangle\langle Y, T\rangle.$$

(4°) is an immediate consequence of (3°) with $X = T$.

4.4. Proof of Theorem 4

We shall use Propositions 4 and 5.

Let M_1 be the interior of the set of the points $m \in M$ such that $(k_2^{(M)})_m = 0$. Let $\tilde{M}_2$ be the interior of the set of the points $m \in M$ such that $(k_2^{(M)})_m \neq 0$ and $\theta_m^{(M)} \neq -\infty$. Let M_3 be the interior of the set of the points $m \in M$ such that $(k_2^{(M)})_m \neq 0$ and $\theta_m^{(M)} = -\infty$. We shall study M_1, $\tilde{M}_2$ and M_3.

Since $\dim E_1 = 2$, M_1 is an open set, the connected components of which are submanifolds with substantial codimension 2 (cf. Theorem 1). In order to study $\tilde{M}_2$, we shall use Proposition 5. Since on $\tilde{M}_2$ the distribution $T^\perp$ is involutive, $\tilde{M}_2$ is foliated by hypersurfaces $\overline{M_2}$ such that $\sigma(X, Y) = k(X, Y)\eta$, $\forall X, Y \in T\overline{M_2}$. If $\bar{\sigma}^2$ denotes the second fundamental form of $\overline{M_2}$ in $\tilde{M}^{n+p}$, we have

$$\bar{\sigma}^2(X, Y) = k(X, Y)\eta + \langle \nabla_X Y, T\rangle T.$$

Thus $\dim E_1^{\overline{M_2}} = 2$. Consequently, we can find two open sets N_1 and N_2 such that $N_1 \cup N_2$ is dense in M_2, and N_1 and N_2 satisfy

$$\dim E_1^{\overline{M_2}}|_{N_1} = 1, \quad \dim E_1^{\overline{M_1}}|_{N_2} = 2.$$

On N_1, $\dim E_2^{\overline{M_2}} \leqslant 1$, and it is clear that $\dim E_3^{\overline{M_2}} = 0$ on a dense open set of N_1. On N_2, $\dim E_2^{\overline{M_2}} = 0$ since $L_\eta = 0$.

Using Theorem 1 we conclude that $\tilde{M}_2$ contains a dense open set M_2 which is foliated by hypersurfaces with substantial codimension 2 in $\tilde{M}^{n+p}$.

In order to study M_3, we shall use Proposition 4. On M_3, the index of relative nullity is equal to $n-2$. Using a well-known theorem (cf. [1] for instance), we conclude that M_3 is foliated by totally geodesic submanifolds of dimension $n-2$.

Theorem 4 is proved.

4.5. Proof of Theorem 5

(A) *Let* $\theta^{(M)} = 0$.

(1°) From Proposition 6 (3), we obtain $\beta\langle \nabla_X T, Y\rangle = 0, \forall X, Y \in TM$. Since $\beta \neq 0$, T is parallel. If M is complete, connected, and simply connected, from De Rham theorem, we have $M = C \times M_1$, where C and M_1 are maximal integral submanifolds of T and $T^\perp$ at a point $p \in M$. The general result is obtained by passing to the universal covering of M.

(2°) We have $\dim E_1^{(M_1)} = 1$ and $k_2^{(M_1)} = 0$. In fact, let σ^{M_1} be the second fundamental form associated with the restriction of the immersion to M_1. We have $TM = TM_1 \oplus T$. Hence $\forall X, Y \in TM_1$, $\sigma^{M_1}(X, Y) = \sigma(X, Y) + \langle \tilde{\nabla}_X Y, T\rangle T = k(X, Y)\eta$. Consequently, $\dim E_1^{(M_1)} \leqslant 1$. If, at a point $m \in M$, $k_m(X, Y) = 0$ $\forall X, Y \in T_m T_1$, then $\dim \operatorname{Ker} k_m = n-1$, and therefore $k_m(X, Y) = \gamma\langle X, T\rangle\langle Y, T\rangle$, which implies that h_m and k_m are proportional; this is excluded. Hence $\dim E_1^{(M_1)} = 1$.

Let $\nabla^{\perp^{M_1}}$ be the normal connexion on M_1. Then $\forall X \in TM_1$ we have $\nabla_X^{\perp^{M_1}} \eta = k(X, T)T = 0$ since $X \perp T$, and thus $(k_2^{(M_1)})_m = 0$, $\forall m \in M_1$.

(3°) On the other hand, since T is parallel, $R(X, T)T = 0$, $\forall X \in TM$. From Gauss-Codazzi equations we have

$$\tilde{R}(X, T)T = K(X, \sigma(T, T)) - K(T, \sigma(X, T)).$$

If c is the curvature of $\tilde{M}^{n+p}$, then

$$c(\langle X, Y\rangle - \langle X, T\rangle\langle Y, T\rangle) = k(T, T)[k(X, Y) - k(T, T)\langle X, T\rangle\langle Y, T\rangle].$$

If $c \neq 0$, we have $k_m(T, T) \neq 0$, $\forall m \in M$, since the equality does not hold for every X, Y. Thus

$$k(X, Y) = \frac{c}{k(T, T)}\langle X, Y\rangle, \quad \forall X, Y \in TM_1.$$

Consequently, if $c \neq 0$, the submanifold M_1 is totally umbilical and is contained in an "hypersphere".

If $c = 0$, we have $k(T, T) = 0$. In fact, if at a point $m \in M$, $k_m(T, T)_m \neq 0$, then $k_m(X, Y) = k_m(T, T)_m \langle X, T\rangle\langle Y, T\rangle$ which is impossible because h_m is not proportional to k_m.

Computing the Frenet curvatures of C, we find:

$$\tilde{\nabla}_T T = \nabla_T T + \sigma(T, T) = \sigma(T, T) = \beta\,\xi \Rightarrow k_1^{(C)} = \beta,$$
$$\tilde{\nabla}_T \xi = \nabla_T^{\perp}\xi - K(T, \xi) = \tau_2(T)\,\xi_2 - \beta T \Rightarrow k_2^{(C)} = |\tau_2(T)| = k_2^{(M)},$$
$$\cdots\cdots\cdots$$
$$\tilde{\nabla}_T \xi_i = \nabla_T^{\perp}\xi_i = \tau_{i+1}(T)\,\xi_{i+1} - \tau_i(T)\,\xi_{i-1}$$
$$\Rightarrow k_{i+1}^{(C)} = |\tau_{i+1}(T)| = \|\tau_{i+1}\| = k_{i+1}^{(M^n)}.$$

Hence

$$k_i^{(C)} = k_i^{(M)}, \quad \forall i \in [2 \cdots s].$$

(B) Let $\theta^{(M)} = \text{const.} \neq 0$. From Proposition 6(3), we have

$$(*)\quad k(X, Y) = k(T, T)\langle X, T\rangle\langle Y, T\rangle + \frac{\beta}{\theta^{(M^n)}}\langle \nabla_X T, Y\rangle, \quad \forall X, Y \in TM.$$

Let M_2 be a maximal integral submanifold of the distribution $T^{\perp}$, and σ^{M_2} the second fundamental form associated to M_2. Then we have

$$\sigma^{M_2}(X, Y) = \sigma(X, Y) + \langle \nabla_X Y, T\rangle T$$
$$= \langle \nabla_X T, Y\rangle\left(\frac{\beta}{\theta^{(M^n)}}\eta - T\right).$$

Thus $\dim E_1^{(M_2)} \leqslant 1$.

On the other hand, $\nabla T \neq 0$ at every point. In fact, if $(\nabla T)_m = 0$ at $m \in M$, k_m is proportional to h_m, and $\dim(E_1)_m = 1$. Consequently, $\dim E_1^{(M_2)} = 1$.

Finally, let $\nabla^{\perp M_2}$ be the normal connexion on M_2 induced by $\nabla^{\perp}$. If $x \in TM_2$, then

$$\nabla_X^{\perp M_2}\left(\frac{\beta}{\theta^{(M)}}\eta - T\right) = \mathrm{pr}_{TM^{\perp}\oplus T}\tilde{\nabla}_X\left(\frac{\beta}{\theta^{(M)}}\eta - T\right)$$
$$= \nabla_X \frac{\beta}{\theta^{(M)}}\eta + k(X, T)T - \sigma(X, T) - \langle \nabla_X T, T\rangle T$$
$$= 0,$$

since $X \perp T$. Consequently $k_2^{(M_2)} = 0$.

Now we shall study the case where $\tilde{M} = R^{n+p}(c)$, $c \neq 0$. We shall need the following lemmas.

Lemma 10. *With the notations of Proposition* 6, $\forall X \in TM$ *we have*

$$c\{\langle X, T\rangle T - X\} = -\left[\frac{\theta^{(M)}}{\beta} k(T,T) + \frac{T(\beta)}{\beta} + k(T,T)\frac{\beta}{\theta^{(M'')}}\right]\nabla_X T.$$

Proof of Lemma 10. From Gauss-Codazzi equations, we have

$$\begin{aligned} \tilde{R}(X,Y)T &= R(X,Y)T - K(X,\sigma(Y,T)) + K(Y,\sigma(X,T)), \\ (**) \qquad &= R(X,T)T - k(T,T)\frac{\beta}{\theta^{(M'')}}\nabla_X T. \end{aligned}$$

Let us compute $R(X,T)T$. From the proof of Proposition 6 (3) (ii) we have

$$(\nabla_X k)(Y,Z) = (\nabla_Y k)(X,Z).$$

Replacing k by its expression (Proposition 3.3) gives

$$\begin{aligned} &\frac{\beta}{\theta^{(M)}}\langle R(X,Y)T, Z\rangle + d(k(T,T)\pi)(X,Y)\langle Z,T\rangle \\ &\qquad + \frac{1}{\theta^{(M)}}X(\beta)\langle \nabla_Y T, Z\rangle - \frac{1}{\theta^{(M)}}Y(\beta)\langle \nabla_X T, Z\rangle \\ &\qquad + k(T,T)\langle Y,T\rangle\langle Z,\nabla_X T\rangle - k(T,T)\langle Z,\nabla_Y T\rangle = 0. \end{aligned}$$

Thus we deduce

$$\begin{aligned} R(X,Y)T = \frac{\theta^{(M)}}{\beta}\Bigg[&\left\{k(T,T)\langle X,T\rangle - \frac{X(\beta)}{\theta^{(M)}}\right\}\nabla_Y T \\ &- \left\{k(T,T)\langle Y,T\rangle - \frac{Y(\beta)}{\theta^{(M)}}\right\}\nabla_X T\Bigg]. \end{aligned}$$

From $(**)$ and $\nabla_T T = 0$ it follows that

$$\tilde{R}(X,T)T = -\left[\frac{\theta^{(M)}}{\beta}k(T,T) + \frac{T(\beta)}{\beta} + k(T,T)\frac{\beta}{\theta^{(M)}}\right]\nabla_X T.$$

Since the curvature of $\tilde{M}^{n+p}$ is constant $(= c)$, we have

$$c\{\langle X,T\rangle T - X\} = \gamma \nabla_X T,$$

with

$$\gamma = -\left[k(T,T)\frac{\beta}{\theta^{(M)}} + \frac{T(\beta)}{\beta} + k(T,T)\frac{\theta^{(M)}}{\beta}\right].$$

Lemma 11. *If* $c \neq 0$, *the direction* η *is quasiumbilical.*

Proof of Lemma 11. At first we recall that a directiion $\nu \in TM^{\perp}$ is quasiumbilical if $\exists f_1$ and $\exists f_2 \in C^\infty(M)$ such that

$$\langle K(X,\nu), Y\rangle = f_1\langle X,U\rangle\langle Y,U\rangle + f_2\langle X,T\rangle,$$

where $U \in TM$. Let $Y \perp T$. From Lemma 10 it follows that $-cY = \gamma \nabla_Y T$. Since $c \neq 0$, we deduce that $\gamma \neq 0$ at every point of M. Consequently, $\nabla_X T = c/\gamma\{\langle X, T\rangle T - X\}$. Thus from Proposition 6 (3) we deduce

$$k(X, Y) = f_1\langle X, T\rangle\langle Y, T\rangle + f_2\langle X, T\rangle$$

with

$$f_1 = k(T, T) - \frac{\beta c}{\theta^{(M)}\gamma}, \quad f_2 = -\frac{\beta c}{\theta^{(M)}\gamma}.$$

Hence η is quasiumbilical.

We can now proceed to prove (B). To this end, let M_2 be a maximal integral submanifold of T, and σ^{M_2} the second fundamental form associated to M_2. Then

$$\sigma^{M_2}(X, Y) = k(X, Y)\,\eta + \langle \nabla_X Y, T\rangle T.$$

Since $c \neq 0$, we deduce

$$\sigma^{M_2}(X, Y) = f_2\langle X, T\rangle \eta + \frac{c}{\gamma}\langle X, T\rangle T.$$

Thus $\sigma^{M_2}(X, Y) = \langle X, T\rangle(f_2\eta + c/\gamma T)$, which shows that M_2 is totally umbilical and contained in $(n-1)$-dimensional hypersphere. Hence M^n is foliated by $(n-1)$-dimensional hyperspheres, when $c \neq 0$, and (B) is proved.

(C) *Let* $\theta^{(M)} = -\infty$. In this case, we know that the index of relative nullity of M is equal to $(n-2)$ at every point m of M. Consequently, M is foliated by totally geometric submanifolds of dimension $n-2$.

Hence Theorem 5 is completely proved.

Remarks. Some of the results in this paper are summarized in [6], [7], [8]. The topological properties of the principal normal spaces are exposed in [13] and summarized in [11] and [12]. The existence of immersions with prescribed external curvatures has been studied in [5]. These papers are a part of the second author's thesis [14].

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