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Inversion of Parametric Representations of Geometrical Objects: A General Method for Solving the Canonical Nonuniqueness Problem

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SUMMARY

High-quality geometrical modeling of geological structures requires geometrical objects such as surfaces, volumes or foliations, which need parametric representations to be dealt with in practice. However, there is no one-to-one correspondance between these objects and their representations. As a result, an inversion problem that could be well-posed in geometrical terms is always ill-posed in terms of parametric representations because of their multiplicity for one geometrical object. To overcome this difficulty, we propose a method based on an unphysical least-squares criterion. The criterion first smooths out the mesh associated with parametric representations. In a second step, it is modified in such a way that it does not alter the physical problem considered. The method is designed to work harmoniously with Gauss-Newton optimization algorithms. Numerical results involving one foliation demonstrate the effectiveness of the method.

INTRODUCTION

Efficient intensive oil exploration requires accurate determination of the geological structure's geometry, which can be achieved via multicriteria optimization methods. However, before constraining a structure with all possible data sets, correct mathematical description of unknown structures is necessary.

Models made up of several layers are widely used, since almost all oil prospects systematically consist of layered rocks. Mathematical description of layer boundaries is usually carried out using a function f such that $z = f(x, y)$ where x and y are horizontal coordinates and z elevation or depth. This method is very popular, a lot of software and published papers on traveltime inversion use it.

Other methods related to CAD techniques have appeared recently and they have proven to be more powerful for describing complex structures. Pereyra (1988), Mallet (1989), Léger et al. (1991a) and Virieux & Farra (1991) illustrate this tendency. What these implicit methods have in common is that surfaces are defined using grids of points that can move in three directions instead of one as for explicit representations.

In this paper, we address the key problem that some perturbations in a grid may have no effect on the surface it represents, and we call them *unphysical* perturbations. This is obvious in the case of a plane which remains the same if the gridpoints are moved on it. As a result, inversion techniques will fail to determine the surface via the grid, since only the surface itself has physical significance and several grids may be used for its discretization. This is also true for curves, volumes or foliations, i.e., volumes covered by disjointed surfaces.

We propose a general method for solving the problem so as to obtain a problem as well-posed in terms of grids (parametric representations) as in terms of geometrical objects such as surfaces, volumes or foliations.

First, we briefly discuss the respective advantages and drawbacks of explicit and implicit representations. Second, we identify which parameters are physical or unphysical in the case of foliation parametric representations. Next, we propose a method that works in two steps: mesh smoothing using a least-squares criterion, and modifying the criterion so that it becomes strictly unphysical. Last, numerical results demonstrate that the second step is necessary.

WHY PARAMETRIC REPRESENTATIONS ?

Good geometrical and physical modeling of geological structures seems to be a prerequisite for successfully determining them. Because of the systematically layered structure of oil prospects, we chose the geometrical concept of foliation to represent sediments locally (Léger et al., 1991a). A foliation is simply a volume which is covered by disjointed surfaces. Dealing with faults and unconformities requires considering a geological structure as a set of jointed foliations (Figure 1). Physical properties such as velocity can be defined at any point.

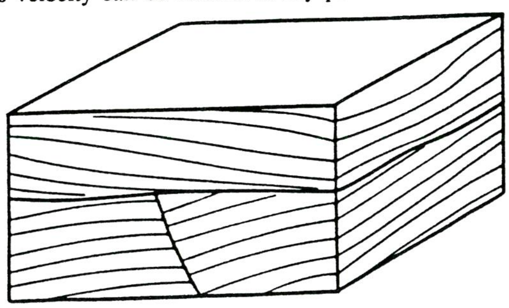

Figure 1. *A geological structure may be considered as a set of jointed foliations, which are volumes covered by disjointed surfaces, called leaves, that represent sedimentation isochrons. Joints between foliations may be faults or unconformities.*

Practical use of geometrical objects such as foliations requires them to be represented by functions. The representation may be explicit:

$$\mathcal{F} = \{(x, y, z)\ ;\ (x, y) \in D\ ;\ z = f(x, y, w)\}$$

where x and y are geographical coordinates in domain D and w is an index that numbers the leaves of foliation $\mathcal{F}$. This method has two basic advantages: it is simple and the correspondance between the foliation and the function f is one-to-one, as long as the first derivative of f with respect to w is continuous and nowhere zero. On the other hand it has two main drawbacks. First, recumbent folds or salt overhangs are difficult to represent. Second, domain D is fixed a priori, which requires that exact fault locations are known. If domain D is considered to be variable, then the discretization of function f will change in an inversion process, which means that the size of the discrete parameter space is not defined.

In our opinion, this behaviour seems incompatible with quality standards normally used when implementing inversion algorithms.

For this reason, we prefer to describe foliations with parametric representations. Once a set $\vec{u} = (u^1, u^2, u^3)$ of curvilinear coordinates and a set $\vec{x} = (x^1, x^2, x^3)$ of Cartesian coordinates are chosen in physical space, a foliation $\mathcal{F}$ can be defined by a parametric representation Φ as follows:

$$\vec{u} \in U \subset R^3 \longrightarrow \vec{x} = \Phi(\vec{u}) = \begin{pmatrix} x^1(u^1, u^2, u^3) \\ x^2(u^1, u^2, u^3) \\ x^3(u^1, u^2, u^3) \end{pmatrix} \in \mathcal{F} \subset R^3$$

The coordinate u^3 numbers the leaves of the foliation. Note that Φ needs to be a diffeomorphism so that the leaves of $\mathcal{F}$ do not cross each other. This property is always assumed in what follows. This method is more complicated as we have three functions instead of one. Its crucial advantage is that the edges of a foliation $\mathcal{F}$ can be considered as unknowns, which is the case in practice, even if domain U remains fixed. However, we lose the very nice one-to-one property of explicit representations, i.e., the same foliation can be described by an infinite number of parametric representations (Figure 2). Indeed, any function Ψ

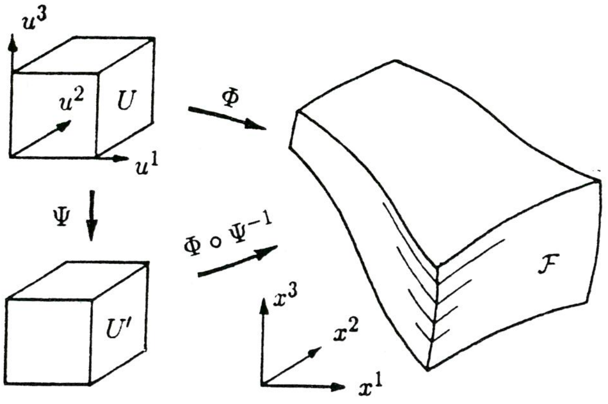

Figure 2. *There are functions Ψ that map U globally onto itself, and are not identity. Consequently, for any parametric representation Φ of a foliation $\mathcal{F}$, there is another one, $\Phi \circ \Psi^{-1}$, that describes exactly the same foliation.*

that maps U globally onto itself, such as:

$$\Psi : \ (u^1, u^2, u^3) \in [0,1]^3 \rightarrow \begin{pmatrix} v^1 = \left[u^1 + (u^1)^2\right]/2 \\ v^2 = \left[u^2 + (u^2)^2\right]/2 \\ v^3 = \left[u^3 + (u^3)^2\right]/2 \end{pmatrix} \in [0,1]^3$$

for instance, defines another parametric representation $\Phi \circ \Psi^{-1}$ of exactly the same foliation as Φ.

In a linearized sense, a parametric representation can be perturbed in many ways such that the foliation it describes remains unchanged to the first order. These perturbations are unphysical.

PHYSICAL AND UNPHYSICAL PARAMETERS

We now discuss which parameters or perturbations are physical or not around a given parametric representation.

What happens if a point inside a foliation is moved in continuity with its neighbours, in such a way that it remains on the same leaf (or its tangent plane T, as a first order approximation), along with its neighbours? Clearly, the answer is that the foliation is unchanged. Therefore, we consider this kind of perturbation to be unphysical. However, if we consider a parametric representation of the foliation, it does change, as suggested by the dashed lines in Figure 3.

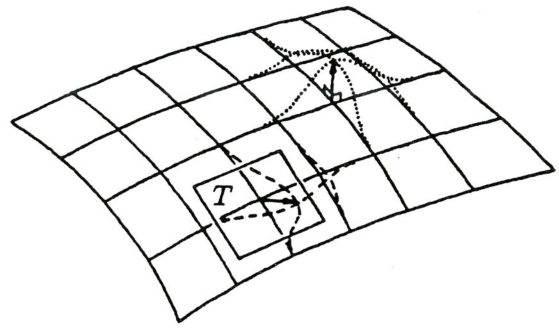

Figure 3. *Physical and unphysical perturbations. If a point is moved but remains on the same leaf of a foliation, this foliation is unchanged despite its modified parametric representation (dashed lines). Purely physical perturbations of a point are orthogonal to the local leaf (dotted lines).*

By virtue of the standard scalar product in physical space, we define purely physical perturbations as orthogonal to the local leaf.

The situation at a point on the boundary may be somewhat different (Figure 4). The general idea is that unphysical perturbations are tangent to all physically significant surfaces at any point of a foliation's closure. At A, two linearly independent perturbations are tangent to the local leaf, hence they are unphysical, like at B, since the local leaf is also a piece of the foliation boundary. At C, the local leaf differs from the boundary so that the only unphysical perturbation is tangent to the intersection curve. At D, the same conclusion holds true since the second piece of the boundary is also a foliation leaf. Points E and F belong to three different physically significant surfaces, and therefore all perturbations are physical.

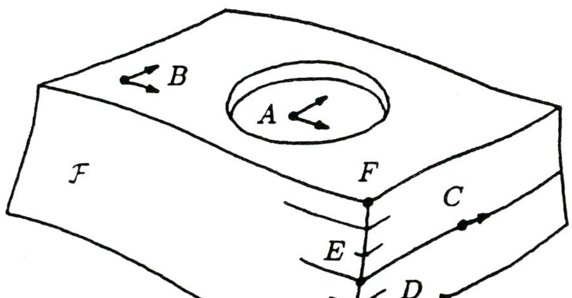

Figure 4. *Unphysical perturbations are tangent to all physically significant surfaces. At points A, B, C and D, the arrows denote unphysical perturbations. At points E and F, all perturbations are physical.*

Surprisingly, some combinations of physical perturbations may be unphysical. If all the points of a leaf S inside a foliation are moved to another leaf S' of that foliation so that the edges of S are moved to the edges of S', then the foliation remains the same. This corresponds to a change of variable Ψ in the curvilinear coordinate domain: $(u^1, u^2, u^3) \rightarrow (u^1, u^2, \Psi(u^3))$. More simply, the leaves of $\mathcal{F}$ are numbered in a different way (Figure 5).

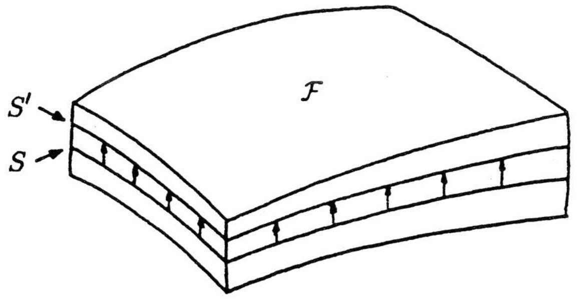

Figure 5. *Some combinations of physical perturbations may be unphysical since they preserve the foliation. They correspond to a change of variable on the curvilinear coordinate that numbers the foliation leaves.*

The problem is that, by definition, unphysical perturbations cannot be determined by any physical (geophysical or geological) information. Therefore, we will now try to constrain these perturbations using an unphysical criterion.

THE METHOD

Let us call $Q_\varphi(\Phi)$ the global physical objective function that derives from geological or geophysical information such as traveltimes, for instance. $Q_\varphi(\Phi)$ is expressed in terms of parametric representation Φ.

The basic idea of the method is to define an additional criterion $Q_\alpha(\Phi)$ that should constrain all unphysical perturbations, and to add it to $Q_\varphi(\Phi)$, to obtain a global criterion: $Q(\Phi) = Q_\varphi(\Phi) + Q_\alpha(\Phi)$. To do this, we define synthetic data as follows:

$$\alpha^{ijk}(\vec{u}) = \frac{\partial^2 \Phi^i}{\partial u^j \partial u^k}(\vec{u})$$

where Φ^i is the i^{th} Cartesian coordinate of a point, whose curvilinear coordinates are $\vec{u} = (u^1, u^2, u^3)$. For each $\vec{u}$, the $\alpha^{ijk}(\vec{u})$ build an eighteen-component vector $\vec{\alpha}(\vec{u})$. We also define "observed" data as $\alpha_0^{ijk}(\vec{u}) = 0$ for any $\vec{u}$. Choosing a L^2 norm on the vector field $(\vec{\alpha} - \vec{\alpha_0}) \circ \Phi = \vec{\alpha} \circ \Phi$ yields the additional criterion $Q_\alpha(\Phi)$.

If zeroed, the $Q_\alpha(\Phi)$ criterion makes Jacobian matrix $\partial\Phi^i/\partial u^j$ a constant, and the foliation leaves are consequently plane. Hence, $Q_\alpha(\Phi)$ has some physical significance and the foliation that minimizes $Q_\varphi(\Phi)$ could be different from the foliation that minimizes $Q_\varphi(\Phi)+Q_\alpha(\Phi)$. In other words, the physical problem associated with $Q_\varphi(\Phi)$ could be disturbed by $Q_\alpha(\Phi)$. To avoid this, we modify $Q_\alpha(\Phi)$ inside the Gauss-Newton procedure we use to optimize $Q_\varphi(\Phi)$.

The standard Gauss-Newton optimization method consists of solving the linearized problem iteratively:

$$J_\varphi^t J_\varphi \vec{\delta p} = -J_\varphi^t \vec{\delta d_\varphi}$$

where vector $\vec{\delta p}$ is the model modification, vector $\vec{\delta d_\varphi}$ is the physical data misfit and J_φ the Jacobian matrix of $\vec{d_\varphi}(\vec{p})$ around the current model, ^{t} denoting transpose. If vector $\vec{p}$ represents a discretized foliation parametric representation, we know that this equation cannot be solved. If we add $Q_\alpha(\Phi)$ to $Q_\varphi(\Phi)$, the above equation becomes:

$$(J_\varphi^t J_\varphi + J_\alpha^t J_\alpha)\vec{\delta p} = -\left(J_\varphi^t \vec{\delta d_\varphi} + J_\alpha^t \vec{\delta d_\alpha}\right)$$

where $\vec{\delta d_\alpha}$ is vector $\vec{\alpha}$ and J_α is the Jacobian matrix of $\vec{d_\alpha}(\vec{p})$.

The key idea of our method is to modify J_α so that the physical problem remains unchanged.

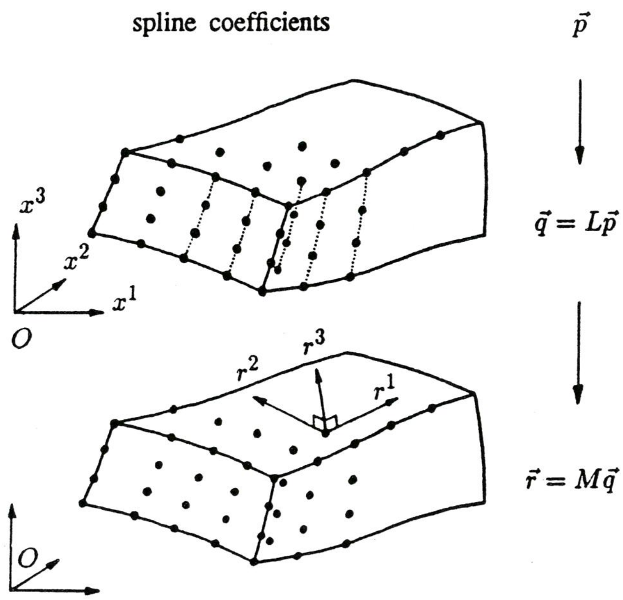

Figure 6. *Starting from the B-spline coefficient discretized parameter space, two successive changes of variable allow easy modification of the additional criterion that keeps it from having any physical signification.*

To do this, we use the above discussion about physical and unphysical perturbations after two changes of variable in the parameter space (Figure 6). The first one, $\vec{q} = L\vec{p}$, consists in choosing a grid of points, whose Cartesian coordinates build a vector $\vec{q}$. The second one, $\vec{r} = M\vec{q}$, consists in replacing the frame in physical space to which these coordinates are related, by a specific frame for each of the gridpoints. This frame is such that two basis vectors lie in the tangent plane to the local leaf and the third one is orthogonal to it (Figure 6). The origin O remaining the same, we call r the vector of all the gridpoints coordinates in their respective frame. Hence, the Jacobian matrix J_α becomes: $J_\alpha^r = J_\alpha L^{-1} M^{-1}$.

Now the additional criterion can be easily modified in such a way that it becomes strictly unphysical by zeroing the columns in J_α^r that correspond to physical perturbations of the gridpoints. If we call P the matrix associated to this projection in parameter space, then the modified Jacobian matrix J_α expressed in the original parameter space simply becomes:

$$J_\alpha^p = J_\alpha L^{-1} M^{-1} P M L$$

so that the modified linearized problem becomes:

$$\left(J_\varphi^t J_\varphi + (J_\alpha^p)^t J_\alpha^p\right)\vec{\delta p} = -\left(J_\varphi^t \delta\vec{d}_\varphi + (J_\alpha^p)^t \delta\vec{d}_\alpha\right)$$

Besides its simplicity, an important advantage of this method is that it is designed and can be implemented completely independently of the physical criteria. Moreover, the possibility of discussing which perturbations are physical or not at the edges of a foliation in detail suggests that this method will still work for complicated geological structures made up of several foliations.

Numerical experiments show that the discrete problem is well-posed using our method. We are currently studying a mathematical proof related to the continuous problem.

NUMERICAL RESULTS

The results that we present in a joint paper (Léger et al., 1991a) were obtained using this method and illustrate it.

Nevertheless, we present two numerical results here that demonstrate the need for the Jacobian matrix modification step. Figure 7 represents a foliation which is optimized with respect to a parallelism criterion using our method. The top surface is given and the foliation is known along a well trajectory. The result corresponds to our expectations.

Figure 8 displays the solution of the same problem but we have "forgotten" to modify the Jacobian matrix J_α. This result should obviously be rejected.

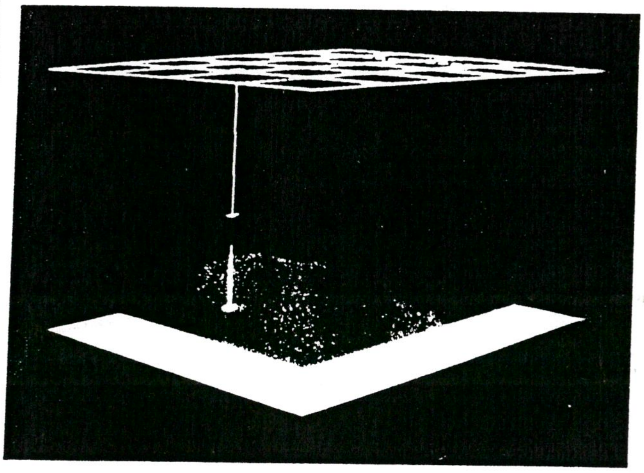

Figure 7. *Using our method, the optimization of a foliation with respect to a parallelism criterion yields the correct expected result. The top surface and the well are fixed.*

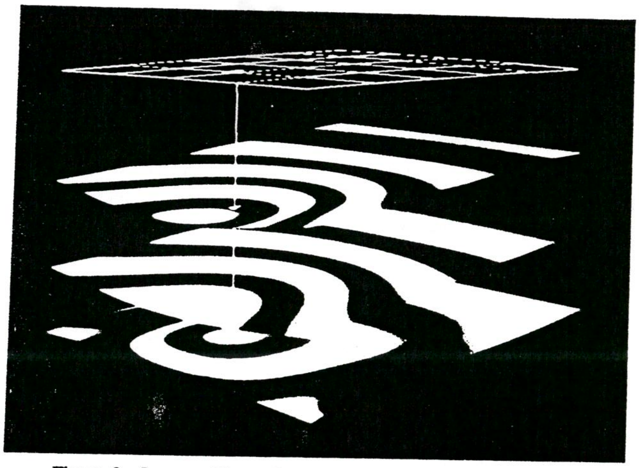

Figure 8. *Same as Figure 7, but without modifying the Jacobian matrix J_α. This result clearly is incorrect since the the foliation is not parallel. Gray intensity is a sine function of depth.*

CONCLUSIONS

High-quality inversion of complex geological structures involves geometrical objects such as surfaces, volumes or foliations and requires them to be dealt with via parametric representations. Even if sufficient data are available for the inversion problem to be well-posed in terms of geometrical objects, it is ill-posed in terms of parametric representations because of their multiplicity for one geometrical object. This means that degrees of freedom of parametric representations may be physical or unphysical.

To overcome this difficulty, we propose a general method that works in two steps. First, a least-squares criterion smooths out the mesh associated with parametric representation. Second, it is modified so that the solution of the physical inverse problem is unchanged.

This method can be implemented in harmony with the standard Gauss-Newton optimization procedure. Numerical results involving one foliation (a geological structure without fault or unconformity) demonstrate its effectiveness. We believe that this method could also work for more complicated and realistic geological structures involving several foliations.

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