Least-Squares Optimization of Thread Surfaces

M. Léger, J.-M. Morvan, and M. Thibaut

Abstract. In geology, faults may sometimes be considered as slipping surfaces between two approximately rigid blocks. Geometrically, we define these surfaces as follows: they are preserved by at least one single-parameter family of moves, and we call them *threads* by analogy with a screw and a nut. Besides, we need to optimize our subsurface models for oil exploration purposes. Therefore, we propose a method that optimizes a surface with respect to several least-squares criteria related to curvature, proximity to known points, and the "thread property".

§1. Introduction

Geophysics aims to determine and image a subsurface by using seismic data and geologic information. Inversion [5] is a popular technique to achieve this goal, but it requires a least-squares formulation of geological knowledge [2]. We focus our attention on the shape of faults. Sometimes, faults separate blocks which may be considered rigid. We formulate in least-squares terms this geological knowledge: "a fault surface separates two approximately rigid blocks" and we optimize the surface with respect to that criterion. We also constrain the surface to be smooth and to be close to given points.

In the first section, we derive the geometrical consequences of the rigid block hypothesis which leads us to introduce the concept of *thread* surface. We briefly review inversion in the second section. Next, we describe the physical objective functions related to following data: the *thread* property, the proximity to given points and the smoothness of the surface. In the fourth section, we add three nonphysical criteria to make the problem well-posed. Finally, we present numerical results and conclusions.

§2. Rigid Blocks and Thread Surfaces

We consider a fault as a surface, *i.e.*, we neglect the thickness of the possible gauge zone. We represent the surface using a parameterization $\Phi(u, v)$ (u and v are curvilinear coordinates) because many faults are almost vertical and therefore $z(x, y)$ representations are awkward. We discretize the parameterization by using B-spline tensor products.

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2.1. The rigid block approximation

In some geologic circumstances, strains in the blocks separated by a fault are weak. In these cases, the rigid block approximation is valid. Besides, blocks usually remain in contact during faulting, and consequently the fault surface remains the same during the move. We call *threads* the surfaces that have this property, by analogy with a screw and a nut.

Definition 1. *A surface S is a thread if and only if there exists a single parameter family of moves that preserves surface S.*

Surfaces of revolution and cylinders are typical examples of threads. On the contrary, an egg-box is not a thread.

2.2. A characteristic property of threads

We review the definition of a twistor.

Definition 2. *A twistor $\mathcal{T}$ is a vector field such that, for any M,*

$$\mathcal{T} = \begin{bmatrix} T(\vec{O}) \\ \vec{\Omega} \end{bmatrix}_O = \begin{bmatrix} T(\vec{O}) + \vec{\Omega} \wedge O\vec{M} \\ \vec{\Omega} \end{bmatrix}_M .$$

According to an idea of C.-M. Marle, Definition 1 is equivalent to

Definition 3. *A surface S is a thread if and only if there exists a nonzero twistor $\mathcal{T}$ such that vector $T(P)$ belongs to the tangent plane $T_P(S)$ to S at any point P of S.*

In the general case (Figure 1), the field lines of a twistor are helices with the same axis and the same pitch. If $\Omega = 0$, the vector field is constant, the thread is a cylinder and the relative move between the blocks is a translation. If $T(M)$ is everywhere orthogonal to Ω, the thread is a surface of revolution, and the relative move is a rotation. A thread may be considered as a single parameter family of field lines of a nonzero twistor. Geologists call *striae* the lines made on one block by the bumps of the other. We call *computed striae* the *field lines* of the (projection of the) twistor on a (quasi-) thread surface.

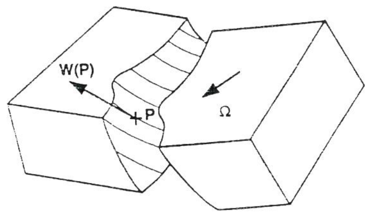

Fig. 1. Two rigid blocks.

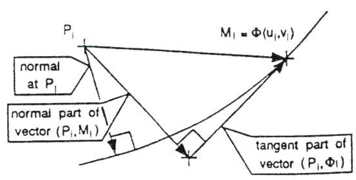

Figure 2. Proximity to a given point.

§3. Inversion

We have chosen the inversion approach because it can take various kinds of data into account, which is our problem since we wish to constrain a surface to be smooth, to be close to given points and to approach a thread.

3.1. The direct and inverse problems

The laws of physics, represented by a map f, enable us to compute predicted observations d as a function of a known model m, $f(m) = d$. This is called the *direct problem.* In geophysics, the situation differs since we wish to determine an earth model m from known observed data d, $f^{-1}(d) = m$. This is called the *inverse problem.* Unfortunately, the map f is generally not invertible, and the model parameters cannot be computed by using that equation. Therefore, we look for a model $\tilde{m}$ such that computed data $\tilde{d} = f(\tilde{m})$ are as close as possible to observed data d.

3.2. The objective function

We need to make clear what "as close as possible" means. To do that, we define an objective function Q which measures the discrepancy between computed and observed data. If we assume that the data space is a vector space, a norm on the residual vector $d - \tilde{d}$ will define objective function Q. We choose an L_2-norm for convenience $Q(m) = \frac{1}{2}\|d - \tilde{d}\|^2$. The Gauss-Newton method gives the solution model by minimization of objective function Q.

§4. The Physical Data

In our problem, the data space is the product of three data spaces, one for each kind of information. We use geometric terms to translate the geological problem and we obtain a geometric objective function depending on surface S and twistor $\mathcal{T}$. Next, according to the functional viewpoint, we introduce a parameterization Φ by a change of variable.

4.1. The thread criterion

According to the above definition, threads are the only surfaces that zero the geometric objective function

$$Q_T(S, \mathcal{T}) = \frac{1}{2} \int_S \langle T(M), N(M) \rangle^2 dS,$$

where $N(M)$ is the vector normal to the surface at M and $T(M)$ is the value of twistor $\mathcal{T}$ at M. Here, the data space is the space of functions $\langle T(M), N(M) \rangle$ defined on S. In this case, the "observed" dataset is the zero function in this space. Note that the twistor is *a priori* unknown, therefore it is an auxiliary unknown.

4.2. The given points

Sometimes, wells cross faults and hence we know that intersection points P_i belong to the thread surface. Since well trajectories are not perfectly known, the location of points P_i is not exactly known. We introduce the geometric objective function $Q_P(S) = \sum_i \mathcal{D}(P_i, S)$, where $\mathcal{D}(P_i, S)$ is the Euclidean distance between surface S and point P_i.

This distance is a complicated function of parameterization Φ. Therefore, we approximate it by the normal component of vector $\vec{M_iP_i}$ where $M_i = \Phi(m_i)$ is a point of S specified by its curvilinear coordinates $m_i = (u_i, v_i) \in U$ (Figure 2). This approximation is satisfactory if the modulus of $\vec{M_iP_i}$ is much smaller than the principal radii of curvature of the surface. Then, the objective function is $\tilde{Q}_{\frac{1}{P}}(\Phi) = \frac{1}{2}\sum_i \|\langle P_i\vec{M}_i, N(M)\rangle\|^2$.

4.3. The curvature criterion

We smooth the surface by minimizing the geometric objective function

$$Q_C(S) = \frac{1}{2}\int_S (\|h\|^2\, dS) = \frac{1}{2}\int_U (\lambda_1^2 + \lambda_2^2) \circ \Phi\ |d\Phi|\ dU,$$

where h is the norm of the second fundamental form of a surface. Note that $h^2 = \lambda_1^2 + \lambda_2^2$, where λ_1 and λ_2 are the principal curvatures. As in the case of the thread criterion, the "observed" data are zero. A curvature-based criterion is more satisfactory than a second derivative-based criterion because the former is intrinsic, *i.e.*, insensitive to the parameterization, whereas the latter is not. The "smoothing effect" of a second derivative-based criterion is sensitive to the first derivative of the parameterization, and then may be anisotropic or heterogeneous.

4.4. The physical objective function

Finally, we state our physical problem as the minimization of overall geometric objective function Q_φ which is the weighted sum of the above objective functions $Q_\varphi(S, \mathcal{T}) = W_T.Q_T + W_P.Q_{\frac{1}{P}} + W_C.Q_C$, where W_T, W_P, W_C are weights that represent the confidence we have in each kind of data (the higher the confidence, the higher the weights).

From a functional viewpoint, the physical objective function is written

$$\tilde{Q}_\varphi(\Phi, \mathcal{T}) = W_T.\tilde{Q}_T + W_P.\tilde{Q}_{\frac{1}{P}} + W_C.\tilde{Q}_C.$$

§5. The Three Causes for Indetermination

Since there is an infinity of parameterizations describing the same surface, the minimization of $\tilde{Q}_\varphi(\Phi, \mathcal{T})$ is always an ill-posed problem, even if the minimization of $Q_\varphi(S, \mathcal{T})$ is a well-posed one. We call that the *canonical indetermination*.

Besides, the null twistor zeroes Q_T for any given surface, hence, the surface that optimizes $W_P.Q_P^{\perp} + W_C.Q_C$ will always optimize Q_φ. This is not acceptable because every surface would be a thread in such a case. To avoid this situation, and to match the definition of a thread, we normalize the twistor.

Moreover, the given point criterion controls only the normal component of vectors $P_i\vec{M}_i$, and thus we need to introduce the tangential component of these vectors.

5.1. The additional criterion

In order to solve the canonical indetermination problem, we introduce an additional criterion $\tilde{Q}_A$, which automatically selects one particular parameterization in the set of those that describe the optimal surface. This criterion which is added to $\tilde{Q}_\varphi$ should not modify the optimal surface (condition 1) and should be sufficiently constraining to make the problem well-posed (condition 2).

Let us consider the space $\mathcal{P}$ of parameterizations, *i.e.*, the set of $\mathcal{C}^2$-diffeomorphisms from the curvilinear coordinate domain U to $\mathbb{R}^3$. We define $\mathcal{S}$ as the set of surfaces described by parameterizations in $\mathcal{P}$. We define a map s from $\mathcal{P}$ to $\mathcal{S}$ which associates the surface S with a parameterization Φ describing S. We define an equivalence relationship $\sim$ on $\mathcal{P}$ as follows $\Phi_1 \sim \Phi_2 \iff s(\Phi_1) = s(\Phi_2)$. Hence, we have $\mathcal{S} = \mathcal{P}/\sim$.

Now, we introduce an additional objective function $\tilde{Q}_a$ which meets condition 2, and we will later modify it to make it compatible with condition 1. Generally, there is no intersection between the set of the critical points of $\tilde{Q}_a$ and the set of the critical points of $\tilde{Q}_\varphi$. In other words, the physical solution changes if we optimize $\tilde{Q}_a + \tilde{Q}_\varphi$. We need to modify $\tilde{Q}_a$ to meet condition 1. Therefore, at any point Φ of $\mathcal{P}$, we project the gradient of $\tilde{Q}_a$ on the space $T_\Phi\mathcal{S}$ tangent to the local leaf $S = s(\Phi)$. It can be shown [3] that there exists, locally, a function $\tilde{Q}_A$ on $\mathcal{P}$ whose gradient is this projected gradient.

In practice, we choose to define $\tilde{Q}_a$ as a second derivative L_2-seminorm by analogy with the curvature criterion.

5.2. The twistor normalization

To normalize the twistor, we define the objective function

$$\tilde{Q}_N(\Phi, \mathcal{T}) = \frac{1}{2}\| \int_U \|\mathcal{T}(M)\|^2 \circ \Phi |d\Phi| \, dU - \int_U |d\Phi| \, dU\|^2.$$

Clearly, optimizing this objective function makes the RMS value of twistor $\mathcal{T}$ on surface S close to 1.

5.3. The "tangential distance"

To control tangential perturbations of the parameterizations, we define the objective function

$$\tilde{Q}^{\|}_P(\Phi) = \sum_i \|\vec{M_iP_i}\|^2 - \langle \vec{M_iP_i}, N(M_i)\rangle^2.$$

5.4. The overall objective function

The overall objective function $\tilde{Q}$ is the sum of physical objective function $\tilde{Q}_\varphi$ and nonphysical objective function $W_A\tilde{Q}_A + W_N\tilde{Q}_N + W_P\tilde{Q}^{\|}_P$.

§6. Numerical Examples

We now describe several numerical experiments. First, we solve an approximation problem, *i.e.*, we optimize the curvature and proximity to data point criteria, but not the thread criterion. Then, we optimize the surface with respect to a known twistor. Finally, we solve the inverse problem with both the twistor and the surface unknown.

6.1. Approximation problem

Figure 3 gives information about 25 given points P_i defined by their Cartesian coordinates (x_i, y_i, z_i). They are displayed by crosses if $z_i = 0$ and diamonds if $z_i = 1$. Each point P_i is associated with a point $M_i = \Phi(m_i)$ lying on the surface, the curvilinear coordinates of which are $m_i = (u_i, v_i)$. We used these data points in all numerical experiments.

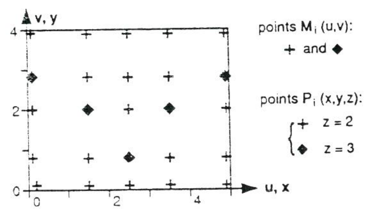

Fig. 3. Given points.

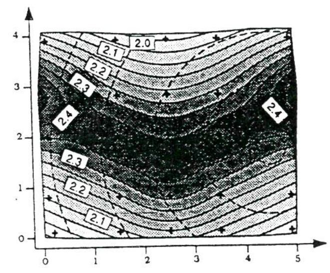

Figure 4. Approximation surface.

To solve the approximation problem, we simply choose a piece of the plane as the initial model. Figure 4 shows the result. This surface is smooth, and as close as possible to data points. Indeed, the RMS distance between data points and the surface is 0.008 instead of 0.2 in the initial model. Note that it is clearly not a thread because of the two bumps.

6.2. Unknown surface and fixed twistor

Figure 5 shows the result of the optimization of the surface with respect to all the criteria (except the normalization criterion) under the constraint of a fixed twistor. We used a translation twistor such that $T(M) = (1, 0, 0)$ everywhere. The surface becomes almost a cylinder, and the computed striae (dashed lines in Figures 4 – 6) are almost straight lines. This surface is smooth, and almost as close to the given points as the surface displayed in Figure 4 which was the initial surface. Dashed lines in Figure 4 suggest the initial twistor.

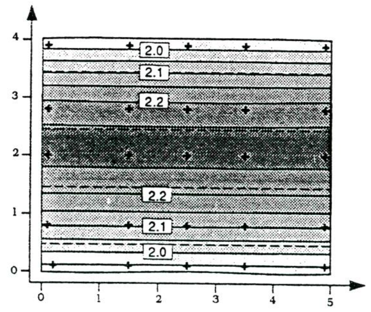

Fig. 5. Unknown surface and fixed twistor.

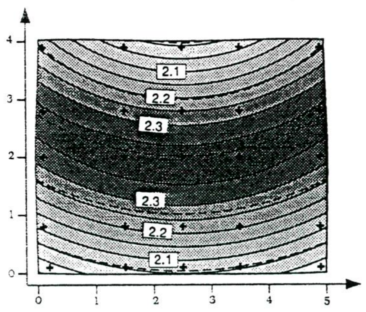

Fig. 6. Unknown surface and twistor.

6.3. Unknown surface and twistor

Figure 6 shows the result of the complete inverse problem, *i.e.*, the twistor and the surface are both unknown. After 12 iterations, convergence is achieved (the gradient is divided by 500,000). This surface is clearly a surface of revolution (and hence a thread). For this twistor, the reduction elements are $T(O) = (1.32, -0.50, 0.0091)$ and $\Omega = (-0.0003, 0.0033, 0.20)$, and the axis Δ is defined parametrically by $\Delta = \{M/\ \vec{OM} = \vec{OA} + \mu\vec{V}/\ \mu \in \mathcal{R}\}$ with $A = (-2.20, -6.6, -0.104)$ and $\vec{V} = (-0.0079, 0.082, 5.0)$. The pitch of the thread is 0.023 (the theoretical value is 0 for surfaces of revolution) and the thread turns to the right. The RMS value of the (twistor-tangent plane) angle is 0.15 degree, and 3.5 degrees for Figure 4. We conclude that we found an optimal twistor which is almost tangent to the surface.

To sum up, a slight downgrade of the proximity and curvature criteria is compatible with a great improvement of the thread criterion.

§7. Conclusions

Our results demonstrate that it is possible to constrain the shape of faults with more elaborate geological knowledge than only smoothness and proximity to data points. The least-squares "thread criterion", which derives from the rigid block approximation, improves predictions about the shape of faults, especially if only a few wells cross it, because a simple approximation may give inaccurate results in this case. The "thread criterion" yields the direction of striae as a by-product. The method is flexible with respect to the available knowledge about the twistor which represents the relative move between the two blocks. Moreover, if a surface is represented parametrically, its smoothing by the minimization of principal curvatures leads to more satisfying results than by the minimization of second derivatives because curvature is an intrinsic quantity. Besides, the "additional cirterion method" solves the canonical indermination problem due to the multiplicity of the parametrizations for one surface. In the future, we plan to test our method with field data. Moreover, striae measurements could be used to formulate another criterion, if available.

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Michel Léger
Institut Français du Pétrole
1-4 avenue de Bois Préau
92500 Rueil Malmaison, FRANCE
legerm@irsun1.ifp.fr