MINIMIZATION OF GEOMETRICAL CRITERIA DEFINED ON A SURFACE

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0. Introduction

0.1. Petroleum geosciences and differential geometry

a- Oil

Oil derives from the organic matter contained in some sedimentary rocks called source rocks if these rocks are sufficently heated, (100 to 1500) by burying in the first kilometers of the earth's crust. Since some water impregnates all rocks, and since oil is lighter than water, it usually percolates upwards and degrades at the ground surface. Sometimes, however, this migration is deviated by impermeable layers and oil may concentrate in a reservoir below a high of the overburden, which makes an oil trap. Since oil has no particular physical property which coulde make possible its direct detection by geophysical methods, oil exploration mainly consists in finding these traps in which oil could be found by drilling.

b- Sismic exploration

Prospecting for oil involves geologic and geophysical methods. Whereas physics aim at predicting the effects of known actions on a known system, the purpose of geophysics is to determine the sytem, knowing the actions and measuring the effects. The sismic method is almost the only geophysical method which is used in oil exploration because it yields detailed images of the subsurface in which traps can be recognized. Sismic exploration consists in sending acoustic waves in the subsurface from many shot-points, and in recording the echoes a several receiver-points for each shot (for redundancy and signal-to-noise ratio improvement). A source sends in a see acoustic waves which propagate in the rocks below and reflect on the layer discontinuities. The echoes are recorded by hydrophones, (100 or mores) in a streamer which is about 3km long. Typical shot interval is 25 m. usual maximum depth of investigation is 5 km. The frequencies of these seismic signals range from 10 to about 100 Hz. In case of a 3D acquisition, the distance between parallel lines is about 100 m. Seismic velocities range from 1.5 km/s for water to 6 km/s for very compact rocks. These echoes are due to the discontinuities of mechanical properties located at the interface between two layers. Since the beginning, shot locations were organized in seismic lines from which 2D images, or cross sections, were obtained by various processing techniques. About 15 years ago, oil companies began to to acquire 3D seismic datasets in order to image exhaustively the studied structures. This 2D to 3D change was necessary because companies explore oil fields which are smaller and smaller, or more and more complicated, most giant oil fields being already discovered. This change was also possible because of the dramatic increase of the available computiong power. Seismic images are often strongly distored because the seismic wave propagation velocity field is complicated and poorly known. For this reason, the accurate determination of the shape of the structure is a difficult problem.

c- Structural geology

Structural geology consists in studying the present shape of the layers from outcrop observation, well data or seismic image interpretation, and in reconstructing the history of the structure. After deposition, rocks may be compacted, heated chemically altered, eroded, folded or fractured. Structural geology specifically studies the two last transformations, that is, the geometrical aspects of geology. The history of rocks deformations need difficult and very expensive strain measurements. For this reason, one prefers to extrapolate general rules from particular detailed studies, which leads sometimes to the conclusion that deformations are small. However, this "rigid-block" approximation is often coarse, and the "constant bend- length" approximation is more realistic. This means that, during folding, hard layers are bent almost isometrically, whereas weaker layers allow them to slip on each other. For short, sedimentary rocks behave like a paperbound book. In the context of oil exploration, this knowledge about

deformations is used to constrain geological structures, as they derive from seismic imaging, to be more likely.

d- Modeling geological structures

Quantitative managing of geological structure geometries requires their mathematical modeling. Differential geometry yields nice basic concepts such as surface or foliations to describe the shape of geological objects and to state the problems related to them. Thereafter, funcional tools of applied mathematics, i.e. parametrisations in our case, may be used to prepare the final discretization and implementation.

e- Inversion

As far as structure determination is concerned, these model parameters are the unknowns. Data are seismic wave traveltimes, geologic knowledge about structure geometry and velocity field, and observations made in boreholes, if any. Since these data may be somewhat contradictory, direct methods (operating the data to obtain the result) are not suitable. In the opposite, we hope that inversion methods will achieve this synthesis. Basically, they consist in finding a model such that the least-squares criteria related to the various datasets are optimized. In tomography, or traveltime tomography, the main dataset contains the traveltimes picked on seismic records.

0-2 The geometric approach

The purpose of this article is to study a particular aspect of the previous problem. We give a general frame-work to the problem of determining a parametric surface which minimizes geometric criteria. By a surface, we mean a 2-D submanifold of a 3-D vector space, usually the physical space in applications. By geometric criteria, we mean criteria which describe the shape of the surface, and which are independent of the parametrisation. We adress the problem of minimizing, under equality (or penalty) conditions, the norm of the second fundamental form of a surface, which is closely related to total and mean curvatures. The first version of this prolem is probably due to S. Germain, about two centuries ago. The same kind of problems were studied more recently, essentially related with elasticity of membranes. All of them are related to the classical Hooke's Law, that is the equation:

$$(*) \quad E(S)= \int_S (a+bH^2+cG)dS,$$

where H is the mean curvature and K is the total curvature of the surface S, which are symmetric functions of the second fundamental form.

It is well known that the theorical problem of determining the existence, unicity and the value of the minimum of (*) is very difficult. It is obviously related to the famous theorem of Gauss-Bonnet, (when a=b=0, c=1), and to Willmore conjecture (when a=c=0, b=1). Our approach is much more practical: We would like to determine a particular parametrisation, whose range approaches, (at least), the expected surface. Minimizing the norm of the second fundamental form is *a priori* an ill-posed problem, since many parametrisations have the same surface as range. Therefore, we need to adapt the formulation of our problem, and the good tool, which generalises the second fundamental form of a surface, is the second fundamental form of a map, that is, its second derivatives. Its decomposition in tangent part and normal part allows to get interesting results. Practically, minimizing this tangent part, and the second fundamental form of the surface, yields automatically a parametrisation that solves the original geometrical problem. Besides, we present partial theoretical results about the existence and unicity of maps solving this kind of problems.

The plan of this article is the following:

- In the first paragraph, we set the general minimization problem, and give typical examples of such a situation.

- In the second paragraph, we give a general method to find critical points of functionals defined on a quotient of manifolds, which is the case which appears in the first paragraph.

- In the third paragraph, we describe the main tool used here: that is, the second fundamental form of a map.

- In the fourth paragraph, we give a fundamental inequality, which relates the second fundamental form of a map, and the second fundamental form of its range.

- In the fifth paragraph, we expose the problem which we deal with, in terms of second fundamental form.

- In the sixth paragraph, we restrict our attention to surfaces in E^3.

- In the seventh paragraph, we modify the problem into a simplier one, solve it, and look at the relationship between their solutions. We also remark that we can get some restrictions on the type of singularities of the limit of a sequence of immersions.

- Finally, we present numerical experiments which illustrate the effect of minimizing the second fundamental form h of a map, that is, its second derivatives, and the second fundamental form of its range, plus the tangent part of h.

Figure 1   Offshore seismics. A source sends in the sea acoustic waves which propagate in the rocks below and reflect on the layer discontinuities. The echoes are recorded by hydrophones (100 or more) in a about 3km long streamer. Typical shot interval is 25m. Usual maximum depth of investigation is 5 to 10km. The frequencies of these seismic signals range from 10 to about 100Hz. In case of a 3D acquisition, the distance between parallel lines is about 100m. Seismic velocities range from 1.5km/s for water to about 6km/s for very compact rocks.
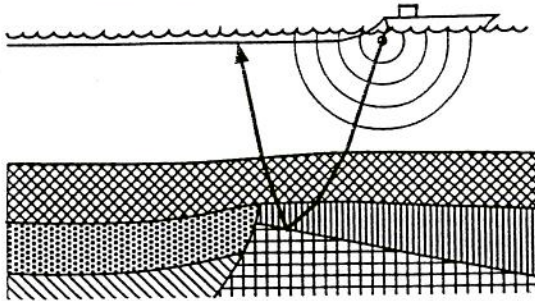
Figure 2   Structural geology. After deposition, rocks may be compacted, heated, chemically altered, eroded, folded or fractured. Structural geology specifically studies the two last transformations.
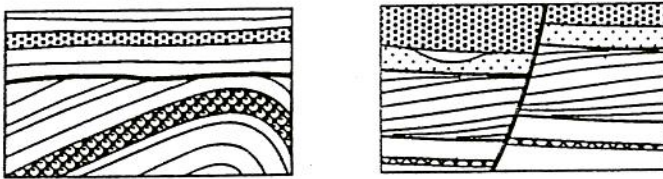
1. The general problem

We deal with the following problem : Let (M,g) be a n-dimensional Riemannian manifold. Let $\{m_1,....,m_k\}$ be k points of $\mathbb{E}^{n+p}$. Let $\mathfrak{Im}$ be the set of all $\mathfrak{C}^2$-immersions ψ of M into $\mathbb{E}^{n+p}$ endowed with the following relation $\sim$:

$$\psi_1 \sim \psi_2 \Leftrightarrow \psi_1(M) = \psi_2(M)$$

Let F be an operator defined on $\mathfrak{Im}/_\sim$.

1.1. The Problem ($\mathfrak{P}$)

Does there exist an unique class of $\mathfrak{C}^2$-immersions ϕ in $\mathfrak{Im}/_\sim$, such that

$$F(\phi) = \inf_{\psi \in \mathfrak{Im}/_\sim} F(\psi),$$

and

ϕ satisfies the constraint (C) : $m_j \in \phi(M), \forall j \in \{1,..,k\}$.

Of course, this problem is very difficult in general. It can have no or many solutions, depending on the dimensions of M and M', F, k, and the positions of the points $m_1,....,m_k$.

1.2. Examples of problems

a) $M = [0,1]$; $p = 1$; $k = 2$.

$$F(\psi) = \int_M |K|(\psi)ds ,$$

where $K(\psi)$ is the curvature of the curve $\psi([0,1])$. Then ($\mathfrak{P}$) has an infinity of solutions : $\phi([0,1])$ can be any segment which contains m_1 and m_2 .

b) $M = [0,1]$; $p = 1$; $k = 3$.

$$F(\psi) = \int_M |K|(\psi)ds,$$

where $K(\psi)$ is the curvature of the curve $\psi([0,1])$. We assume here that $\{m_1, m_2, m_3\}$ are not in same straight line. This problem has no $\mathfrak{C}^2$-solution.

1.3. Examples of operators F

Since we are interested in the shape of the submanifold $\psi(M)$, we can use the second fundamental form h of $\psi(M)$:

a) We can take $F(\psi) = \int_M \|h\|^2 dv$, or more generally,

$$F(\psi) = \int_M (\|h\|^2 + \|\nabla h\|^2 + \dots + \|\nabla^{(p)}h\|^2)^q \, dv \; .$$

(See the following paragraph, for details on the definitions of h and $\nabla^{(p)}$).

b) We can also specify particular geometrical objects :

Let ξ be a normal vector field over M, and h_ξ be defined by

$$h_\xi \,(x,y) = <h(x,y),\xi>.$$

Consider the polynomial function of variable t, defined by

$$P(t) = \det(\mathrm{Id} + t.h_\xi).$$

We can write

$$P(t) = 1 + k_1 t + k_2 t^2 + \dots + k_n t^n \; ,$$

where k_1, k_2..., k_n are symetric functions of the eigenvalues of h_ξ. In particular,

$$k_1 = n< H, \, \xi >; \;\; k_n = \det h_\xi.$$

where H is the mean curvature vector field of $\psi(M)$. Now we can put

$$F(\psi) = \int_M (\alpha|k_1|^{p_1} + \beta|k_2|^{p_2} + \dots + \omega|k_n|^{p_3})^q \, dv$$

where $\{\alpha, \beta..., \omega\}$ are constants, and $\{p_1,...,p_n,q\} \subset \mathbb{Q}$.

c) A simple example of the situation b) is the case of surfaces S in $\mathbb{E}^3$,

$$\psi : S \to \mathbb{E}^3.$$

Then,

$$F(\psi) = \int_S (\alpha|H|^2 + \beta|K|) \, da$$

where H is the mean curvature vector field of $\psi(S)$ and K is the Gauss curvature of $\psi(S)$, endowed with the induced metric.

d) Remark that for a closed surface S in $\mathbb{E}^3$, the number

$$F(\psi) = \int_S K \, ds = \chi(S)$$

is a topological invariant , independant of ψ. In this case F is constant.

2. A method of studying critical points of F

The problem of finding a solution to $\mathcal{P}$ in the quotient $\mathfrak{Im}/\sim$ being too difficult, we shall restrict it to the local problem of finding critical points of F. Theorically, it is well known that the quotient $\mathfrak{Im}/\sim$ has a complicate topological and geometrical structure. However, the space $\mathfrak{Im}$ has a simple structure of infinite dimensional manifold. On the other hand, practically, in order to exhibate a surface, we are often lead to construct a parametrisation of it. That is why, instead of working in $\mathfrak{Im}/\sim$, we prefer to work in $\mathfrak{Im}$. Of course, we need to show that the solution found in $\mathfrak{Im}$, projected in $\mathfrak{Im}/\sim$, gives a solution in $\mathfrak{Im}/\sim$. This is the purpose of this paragraph.

Working with differential technics, we need to assume some assumptions on the regularity of $\mathfrak{Im}/\sim$ and the surjection between $\mathfrak{Im}$ and $\mathfrak{Im}/\sim$. Since our problem is local, we assume that we work on open sets which have a differential structure, and such that the surjection is a (smooth) submersion . So, this geometric frame leads to set the following two lemmas. The first one claims that the projection of a critical point in $\mathfrak{Im}$ gives a critical point in $\mathfrak{Im}/\sim$. This is enough if it is easy to find a critical point in $\mathfrak{Im}$. However, to exhibate a particular critical point in $\mathfrak{Im}$, it may be interesting to modify the function F. We show that, by suitable modifications of the function defined on $\mathfrak{Im}$, a critical point found in $\mathfrak{Im}$ for this new problem gives a critical point for the first function in $\mathfrak{Im}$. We state now the two lemmas:

2.1. Lemma

Let M, N be two manifolds, and s be a submersion from M to N. Let F be a $\mathcal{C}^1$ map from N to $\mathbb{R}$. Let $m \in M$ be a critical point of $F \circ s$. Then $s(m)$ is a critical point of F.

2.2. Proposition

Let M, N be two manifolds, s be a submersion from M to N, F be a real $\mathcal{C}^1$ function on N. Let f be a real $\mathcal{C}^1$ function defined on M. We denote by $\mathfrak{M}$ the foliation defined by $Ker(s_*)$, and by $\mathcal{G}$ any complementary distribution of $T\mathfrak{M}$ in TM. Let ω be the differential one-form defined on M by :

$$\omega = df \text{ on } T\mathfrak{M}$$
$$\omega = 0 \text{ on } \mathcal{G}.$$

Let $m \in M$ such that

$$\omega_m + d(F \circ s)_m = 0$$

Then $s(m)$ is a critical point of F .

2.3. Proof

The first lemma is obvious. We only prove the proposition .

$$M \xrightarrow{s} N \xrightarrow{F} \mathbb{R}$$
$$f \downarrow$$
$$\mathbb{R}$$

Let $n = s(m)$. We shall prove that

$$d(F \circ s)_m = 0.$$

Let $y \in T_n N$, and $z \in \mathcal{G}$ such that

$$ds_m(z) = y.$$

We have

$$\omega_m(z) = 0.$$

and then,

$$d(F \circ s)_m(z) = 0.$$

The first lemma implies that $s(m)$ is a critical point of F.

3. The notion of second fundamental form

The operators that we shall study in this article describe the shape of submanifolds. So, they are much related to the second fundamental form of them. However, we have seen in the previous paragraphs that, instead of working with the submanifolds itself, we need to work with the immersions. Then, we need to use a generalisation of the notion of second fundamental form of a submanifold, which is the second fundamental form of a map. This object has been studied by J. Eells, [Ee], in his work on harmonic maps. We give here a brief summary of the basic definitions, and relations between these two notions. (See [No] for a general study).

3.1. Second fundamental form of a submanifold

3.1.1. The general theory

Let

$$i : (M, g) \to (M', g')$$

be an isometric immersion of a n-dimensional Riemannian manifold (M,g) into a Riemannian manifold (M',g'). We denote by ∇,R, (resp. ∇', R'), the Levi-Civita connexion and the curvature tensor on M , (resp. M'). Let h be the second fundamental form of the immersion. h is a symmetric tensor which takes its values into the normal bundle $TM^\perp$:

$$h : TM \times TM \to T^\perp M$$
$$h(x,y) = \nabla'_x y - \nabla_x y , \quad \forall x, y \in TM.$$

For any $\xi \in T^\perp M$, we put

$$\langle A_\xi(x),y \rangle = \langle h(x,y), \xi \rangle , \quad \forall x, y \in TM.$$

(A_ξ is the adjoint of $\langle h(.,.),\xi \rangle$).

$H = \frac{1}{n}$ Trace(h) is the mean curvature vector field. It is well known that $H = 0$ if and only if M is minimal (for the volume).

The Gauss equation relates R, R' and h :

$$<R'(x,y)z,w> = <R(x,y)z,w> - <h(x,z), h(y,w)> + <h(x,w), h(y,z)>$$

3.1.2. The case of submanifolds of $\mathbb{E}^m$.

Suppose that $M' = \mathbb{E}^m$. Let r be the scalar curvature of M :

$$r = \Sigma_{i,j} <R(e_i,e_j)e_j,e_i>)$$

where $\{e_i,e_j\}$ is a local orthonormal frame over M.

We deduce from the Gauss equation:

$$r = H^2 - \|h\|^2.$$

In particular, if M is a surface in $\mathbb{E}^3$, its Gauss curvature K satisfies:

$$K = H^2 - \|h\|^2.$$

where $\|h\|^2 = \Sigma_{i,j} <h(e_i,e_j), \varepsilon_\alpha>^2$

where (e_i) is an g-orthonormal frame on M, and (ε_α) is an orthonormal frame on $\mathbb{E}^m$.

3.2. Second fundamental form of a map

3.2.1. Classical fiber bundles induced by a map

Let M, M' be two differentiable manifolds. Let

$$f : M \to M'$$

be a $\mathbb{C}^k$-map, $(k \geq 2)$.

3.2.1.1. The fiber bundle $f^{-1}(TM')$

We denote by $f^{-1}(TM')$ the pull-back of TM' by the map f :

$$
\begin{array}{ccc}
f^{-1}(TM') & & TM' \\
\downarrow & & \downarrow \\
M & \to & M' \\
& f &
\end{array}
$$

The basis of this bundle is M, and the fiber over a point $m \in$ M, is $T_{f(m)}M'$. A $\mathbb{C}^p$-section of

$$f^{-1}TM',$$

is called a $\mathbb{C}^p$-field along f. In particular, every field x on M spans the field $f_*(x)$ along f, defined by:

$$f_*(x)_m = (f_*)_m(x)_m.$$

Moreover, every field X' over M' induces a field $f_*(x')$ along f, defined by

$$f_*(x')_m = x'_{f(m)}.$$

3.2.1.2. The fiber bundle $f_*(TM)$

We denote by $f_*(TM)$ the fiber subbundle of $f^{-1}(TM')$, whose fiber at the point m is $f_*(T_mM)$.

3.2.2. Canonical connexion on $f^{-1}(TM')$. [Li]

3.2.2.1. Proposition

Let (M, g) et (M', g') be two Riemannian manifolds, endowed with there Levi-Civita connexion ∇, ∇'.
Let

$$f : M \to M'$$

be a $\mathfrak{C}^k$ - map, $(k \geq 2)$.

Then, there exists an unique linear connexion $\overline{\nabla}'$ over $f^{-1}(TM')$ such that, for every $X \in TM$, and every vector-field Y' over M', we have:

$$\overline{\nabla}'_x\xi' = [\nabla'_{f_*(x)}y']_{f(M)},$$
$$(if \ \xi' = f^*(y')).$$

The proof of this proposition can be found in [Li].

3.2.2.2. Definition

The connexion $\overline{\nabla}'$ is called the pull-back of the connexion ∇'.

3.2.2.3. Local expression of $\overline{\nabla}'$

Let $(x^1,..x^i, ...,x^m)$, (resp. $(y^1,..y^\alpha, ...,y^{m'})$) be local coordinates over M, (resp. M'). Let $\partial_i = \dfrac{\partial}{\partial x_i}$, i = 1,...,m, (resp. $\partial'_\alpha = \dfrac{\partial}{\partial y_\alpha}$) be the vector fields associated to these local coordinates over M. (resp. M'), let $(y_i{}^\alpha)_m = (\dfrac{\partial y^\alpha}{\partial x_i})_m$ be the Jacobian matrix of f at m., and $\partial^2{}_{ij}y^\gamma = \dfrac{\partial^2 y^\gamma}{\partial x_i \partial x_j}$ be the Hessian of f at m.

If $x = x^i\partial_i$, $\xi' = \phi^\alpha\partial'_\alpha$, then

$$\overline{\nabla}'_x\xi' = x^i(\partial_i\phi^\gamma \, \partial_i + y_i{}^\alpha\phi^\beta\Gamma^\gamma_{\alpha\beta})\partial'_\gamma,$$

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where $\Gamma^{'\gamma}_{\alpha\beta}$ are the Riemann-Christoffel symbols of ∇'.

In particular, if $\xi' = f_*(y)$, we get:

$$\bar{\nabla}'_x\xi' = x^i y^j (y_{ij}{}^\gamma + y_i{}^\alpha y_j{}^\beta \Gamma^{'\gamma}_{\alpha\beta})\partial'_\gamma + x^i \partial_i y^j \partial_j y^\gamma \partial'_\gamma.$$

3.2.3. Definition of the second fundamental form of a map

In the following, we shall assume that (M, g), (M', g') are two Riemannian manifolds and f is a $\mathfrak{C}^k$-map of constant rank, $(k \geq 2)$:

$$f : (M, g) \rightarrow (M', g').$$

3.2.3.1. The connexion $\bar{\nabla}$

We denote by $\bar{\nabla}$ the connexion which is the direct sum of ∇ and $\bar{\nabla}'$, défined on $TM \oplus f^{-1}(TM)$. We have:

$$(\bar{\nabla}f_*)(x,y) = \bar{\nabla}'_x f_*(y) - f_*(\nabla_x y)$$

3.2.3.2. Definition

The symmetric tensor

$$h : TM \times TM \rightarrow TM'$$

defined by

$$h(x,y) = (\bar{\nabla}f_*)(x,y) , \forall x, y \in TM,$$

is called the second fundamental form of f.

3.2.3.3. Expression in local coordinates

Using the usual notations , we get:

$$h(x,y) = x^i y^j (y_{ij}{}^\gamma + y_i{}^\alpha y_j{}^\beta \Gamma^{'\gamma}_{\alpha\beta} - y_k{}^\gamma \Gamma^k_{ij})\partial'_\gamma$$

3.2.4. Composition of maps

Let (M, g), (M', g') and (M'', g'') be Riemannian manifolds.

Let

$$f : (M,g) \rightarrow (M',g'),$$
$$f' : (M',g') \rightarrow (M'',g'')$$

be two maps of constant rank, of class $\mathfrak{C}^k$ $(k \geq 2)$ and $\mathfrak{C}^{k'}$ $(k' \geq 2)$ respectively.

Let h , h' et h'' be the secund fundamental forms of f, f' et $f'$of respectively.

Then,

$$h''(x,y) = f'_*[h(x,y)] + h'(f_*(x),f_*(y)).$$

The proof can be found in [Ee-Le].

The local expression of h" can be easily given. By using the following notations:
$$f(x^i) = y^a \; ; \; f(y^a) = z^\alpha \, ,$$

we get:

$$h''(x,y) = x^i y^j [z_{ab}{}^\gamma \, y_b{}^i \, y_a{}^j + y_{ij}{}^a \, z_a{}^\gamma + z_a{}^\alpha \, y_i{}^a \, z_b{}^\beta \, y_j{}^b \Gamma''{}^\gamma_{\alpha\beta} - z_a{}^\gamma \, y_k{}^a \Gamma^k_{ij}] \partial'_\gamma \, .$$

3.3. Canonical decomposition of an immersion

(A detailled exposition of this paragraph can be found in [No]).

Let

$$f : (M, g) \to (M', g')$$

be an immersion between two Riemannian manifolds.

The submanifold f(M) in M' can be endowed with the metric g', restriction of the metric of M' on M. We get obviously the following diagram, where f_1 is an isometry :

$$
\begin{array}{ccc}
& f & \\
(M,g) & \to & (M',g') \\
\text{Id} \searrow & & \nearrow f_1 \\
& (M', f^*(g')) &
\end{array}
$$

Applying 3.2.4.1, we obtain, with obvious notations :

$$h(x,y) = f_{1*}[h_{Id}(x,y)] + h_1(x,y) \, ,$$

where h_{Id} is the second fundamental form of Id,
where h_1 is the second fundamental form of f_1, and where we identify x and $Id_*(x)$.

We put :

$$h_0 = f_{1*\circ} h_{Id}.$$

3.4. Geometrical interpretation

The decomposition:

$$h = h_0 + h_1,$$

is interesting for it separates directly the geometrical and analytical properties of f. Roughly speaking, we can say that:
- h_1 measures the shape of f(M) in M'
- h_0 measures how M is sent onto f(M).

Using this point of view, it is clear that h_1 does not depend on the parametrisation, and h_0 does not depend on the shape of $f(M)$. Consequently to control the shape of a submanifold in M', we need to control h_1. If we consider a parametrisation

$$\phi : \mathbb{E}^2 \rightarrow \mathbb{E}^3$$

of a surface into $\mathbb{E}^3$, we can control the shape of $\phi(\mathbb{E}^2)$ with the following technique :

1- Compute $D^2\phi = h$.
2- Decompose $D^2\phi$ into $h_0 + h_1$.
3- Control h_1.

However, we must be careful on the fact that we never consider the particular problems which appear on the boundary. They need a particular treatment. On the other hand, we must also be careful on the fact that if h_1 depends only on the shape of $\phi(U)$, its norm $\| h_1 \|^2$ depends also on the metric g, since

$$\|h_1\|^2 = \Sigma_{i,j} <h(e_i,e_j), \varepsilon_\alpha>^2$$

where (e_i) is an g-orthonormal frame on M, and (ε_α) is a g'-orthonormal frame on M'.

4. The fundamental tensor inequality

In this paragraph, we compare the norm of the second fundamental form of an immersion, and the norm of the second fundamental form of its range. Let f be an immersion from (M, g) to (M', g'). We put, (with the notations of § 3):

$$h_f = h_{f_0} + h_{f(M)},$$

where h_{f_0} is the tangent part of h_f, and $h_{f(M)} = h_{f_1}$ is the second fundamental form of the submanifold $f(M)$ in (M', g'). We put:

$$\left(\|h_{f_0}\|_{(g,g')}\right)^2 = \Sigma_{i,j,\alpha} <h_{f_0}(e_i,e_j), e_\alpha>_{g'}^2),$$

$$\left(\|h_f\|_{(g,g')}\right)^2 = \Sigma_{i,j,\alpha} <h_f(e_i,e_j), e_\alpha>_{g'}^2),$$

$$\left(\|h_{f(M)}\|_{g'}\right)^2 = \Sigma_{i,j,\alpha} <h_{f(M)}(e_i,e_j), e_\alpha>_{g'}^2),$$

where $\{e_i\}$ is a g-orthonormal frame over M, $\{e_\alpha\}$ is a g'-orthonormal frame over M', $\{e_i\}$ is a g-orthonormal frame over of $Tf(M)$, $\{e_\alpha\}$ is a g'-orthonormal frame over of $T^\perp f(M)$.

We shall prove the following

4.1. Theorem

Let (M,g), (M',g') be Riemannian manifolds. Let a, A be two positive constants. Let

$$f : (M, g) \rightarrow (M', g')$$

be an immersion such that, at each point,

$$a \leq \|f_*(x)\|_g^2 \leq A, \forall x \in TM, \|x\|_g^2 = 1.$$

Then, there exist two positive constants c, C (depending only on (a, A)), such that, at each point,

$$\|h_{f_0}\|_{(g,g')}^2 + c\|h_{f(M)}\|_g^2 \leq \|h_f\|_{(g,g')}^2 \leq \|h_{f_0}\|_{(g,g')}^2 + C\|h_{f(M)}\|_g^2.$$

4.2. Proof of the theorem

Consider the decomposition of ϕ given in § 3:

$$\begin{array}{ccc} & \phi & \\ (M, g) & \rightarrow & (M', g') \\ \text{Id} \searrow & & \downarrow \phi \\ & (M, \phi^*(g')) & \end{array}$$

It induces the decomposition

$$h_\phi = h_{\phi_0} + h_{\phi(M)}.$$

Obviously,

$$\|h_\phi\|_{(g,g')}^2 = \|(h_\phi)_0\|_{(g,g')}^2 + \|h_{\phi(M)}\|_{(g,g')}^2$$

Let us look at each term $(h_{\phi(M)})_{ij}^\alpha$

We have:

$$(h_{\phi(M)})_{ij}^\alpha = \langle h_\phi(e_i,e_j), \varepsilon_\alpha \rangle_{g'} = \langle h_{\phi(M)}(e_i,e_j), \varepsilon_\alpha \rangle_{g'}.$$

If we assume that there exist two constants a and A such that

$$a \leq \|\phi_*(e_i)\|_{g'} \leq A, \forall i,$$

we obtain the result.

5. The global problem

5.1. The global frame-work

Let (M,g) be a n-dimensional Riemannian manifold. We denote by E the trivial vector bundle over M, whose fibers are the Euclidean spaces $\mathbb{E}^{n+p}$, and by L(TM, E) the space of fiber-homomorphisms beetween TM and E. L(TM, E) has a canonical Riemannian vector bundle structure over M given by

$$\langle A,B \rangle = \text{Trace} A^t B$$

It is endowed with a canonical connection given by:

$$(\nabla_X A)(y) = \nabla'_X(A(y)) - A(\nabla_X y))$$

where ∇, (resp. ∇') is the Levi-Civita connexion on M, (resp. $\mathbb{E}^{n+p}$).

If ϕ is an immersion (or any map),

$$\phi : M \to \mathbb{E}^{n+p},$$

we can identify ϕ with a section of E. Let $\mathfrak{C}^\infty(E)$ be the space of $\mathfrak{C}^\infty$ sections of E. We can define $\nabla\xi \in \mathfrak{C}^\infty(L(TM,E))$, and by induction,

$$\nabla^j\xi \in \mathfrak{C}^\infty(L^j(TM, E)),$$

where $L^j(TM,E)$ denotes the space of j-linear vector bundle maps between TM and E. $L^j(TM,E)$ has a standard Riemannian structure, and we can define the norm $\mathfrak{C}^k(E)$ and $L^p(E)$ by:

$$\|\xi\|_{C^0} = \mathrm{Sup}_{x \in M}|\xi_x|$$

$$\|\xi\|_{C^k} = \Sigma_{j \le k}\|\nabla^k\xi\|_{C^0} = \Sigma_{j \le k} \mathrm{Sup}_{x \in M}|\nabla^k\xi|_{C^0}$$

$$\|\xi\|^p_{L^p_k} = \Sigma_{j \le k} \int_M (|\nabla^k\xi|^p_x).$$

$\mathfrak{C}^\infty(E)$ is not complete for these norms. We define $\mathfrak{C}^k(E)$ and $L^p_k(E)$ as the completed spaces of $\mathfrak{C}^\infty(E)$ for the norms $\|.\|_{\mathfrak{C}^k(E)}$ and $\|.\|_{L^p_k(E)}$, respectively. (As usual, we put $H_m = L^2_m$). The Sobolev inequalities give

$$L^p_k(E) \subset \mathfrak{C}^s(E),$$

as soon as

$$k - \frac{n}{n + p} > s.$$

In particular, if n = 2,

$$H_2 \subset \mathfrak{C}^0(E).$$

5.2. The minimisation problem

Let

$$\phi : (M, g) \to \mathbb{E}^{n+p}$$

be an immersion, and h_ϕ be its second fundamental form. In general, ϕ is not isometric. So we define on M the new metric

$$g' = \phi^*(<,>),$$

where $<,>$ is the standard scalar product on $\mathbb{E}^{n+p}$.

$$\phi : (M, g') \rightarrow \mathbb{E}^{n+p}$$

is now an isometric immersion. We denote by $h_{\phi(M)}$ its second fundamental form. Using the metric g', we can introduce

$$\|h_{\phi(M)}\|^2 L_0 = \int_M \|h_{\phi(M)}\|^2_g \cdot dv,$$

(where the norm is computed with respect to g').

If $\mathcal{C}$ is a set of constraint, the minimisation problem can be stated as follows:

5.3. Problem ($\mathcal{P}$)

Does there exist an unique class of $\mathcal{C}^2$-immersions ϕ, in $\mathcal{Im}/\sim$, such that ϕ satisfy the constraint $\mathcal{C}$, and

$$\|h_{\phi(M)}\|L_0^2 = \inf_{\phi \in \mathcal{Im}/\sim} \|h_{\phi(M)}\|.$$

6. A study of the problem ($\mathcal{P}$) for surfaces in $\mathbb{E}^3$.

In this paragraph, we precise the problem $\mathcal{P}$ as follows: Let $\mathcal{Im}/\sim$ be the set of all $\mathcal{C}^2$-immersions of $\mathcal{U}$ into $\mathbb{E}^3$, where $\mathcal{U}$ is a bounded domain of $\mathbb{E}^2$, (with smooth boundary). Let F be the operator defined on $\mathcal{Im}/\sim$ by

$$F(\phi) = \int_{\mathcal{U}} \|h_{\phi(\mathcal{U})}\|^2 \cdot da,$$

where h_ϕ is the second fundamental form of the submanifold $\phi(\mathcal{U})$ in $\mathbb{E}^3$, $\|.\|$ denotes the norm in $\mathbb{E}^3$, and da is the area element of $\phi(\mathcal{U})$. Let $\{m_1,...,m_k\}$, $k \geq 3$ be k points in $\mathbb{E}^3$.

6.1. Problem ($\mathcal{P}$)
Does there exist an unique class of immersions ϕ in $\mathcal{Im}/\sim$, such that

$$m_j \in \phi(\mathcal{U}), \ \forall j \in \{1,...,k\},$$

$$\int_{\mathcal{U}} \|h_{\phi(\mathcal{U})}\|^2 da = \inf_{\psi \in \mathcal{Im}/\sim} \int_{\mathcal{U}} \|h_{\psi(\mathcal{U})}\|^2 \cdot da.$$

(where

$$\|h_{\psi(\mathcal{U})}\|^2 = \Sigma_{i,j} <h_{\psi(\mathcal{U})}(\epsilon_i,\epsilon_j),\epsilon_3>,$$

with the previous notations).

First of all, to approach this problem, we shall compare $h_{\psi(\mathcal{U})}$ and the second derivative of ψ. Then, we shall solve a problem $\widetilde{\mathcal{P}}$ which is closed to $\mathcal{P}$.

6.2. Decomposition of the second derivative of an immersion of a surface in $\mathbb{E}^3$

Let $\mathcal{U}$ be a domain of $\mathbb{E}^2$, and

$$\phi : \mathcal{U} \to \mathbb{E}^3,$$

be an immersion. Let (e_1, e_2) be the standard (orthonormal) frame on $\mathcal{U}$, and $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ be an orthonormal frame over $\phi(\mathcal{U})$, such that $(\varepsilon_1, \varepsilon_2)$ is tangent to $\phi(\mathcal{U})$, ε_3 is normal to $\phi(\mathcal{U})$. We denote by

$$(h_\psi)_{ij}{}^\alpha = <h_\psi(e_i, e_j), \varepsilon_\alpha>.$$

We have:

$$\frac{\partial^2 \psi^\alpha}{\partial x_i \partial y_j} = \begin{pmatrix} (h_\psi)^\alpha{}_{11} & (h_\psi)^\alpha{}_{12} \\ (h_\psi)^\alpha{}_{12} & (h_\psi)^\alpha{}_{22} \end{pmatrix}$$

With the notations of 2.4.2, we can write

$$\phi = f_1 \circ \text{Id},$$

(where f_1 is an isometric immersion), and

$$h_\phi = h_0 + h_1,$$

with

$$\text{pr}_{T\phi(\mathcal{U})} h = f_1 \circ h_{\text{Id}} = f_0,$$

$$\text{pr}_{T^\perp \phi(\mathcal{U})} h = h_1 = h_{\phi(\mathcal{U})}.$$

The "shape of $\phi(\mathcal{U})$" is controlled by $h^3{}_\phi$, that is, by $\dfrac{\partial^2 \phi^3}{\partial x_i \partial y_j}$.

6.3. The frame-work

Let $\mathcal{U}$ be a bounded domain of $\mathbb{E}^2$, with smooth boundary $\partial \mathcal{U}$. Let $i = \{i_1, i_2\}$ be a couple of positive integers.

We put

$$|i| = \Sigma_{k \leq 2} \, i_k \, ,$$
$$D^i = \frac{\partial^{i_1 + i_2}}{\partial x_{i_1} \partial x_{i_2}} \, .$$

We introduce the spaces

$$L^2(\mathcal{U}, \mathbb{E}^3) = \{ f : \mathcal{U} \to \mathbb{E}^3, \, f \text{ mesurable, such that } \int_{\mathcal{U}} |f(u)|^2 du < \infty. \}$$

endowed with the L^2-norm $\|.\|_{\mathcal{U}}$, defined by

$$\|f\|_{\mathcal{U}} = (\int_{\mathcal{U}} |f(u)|^2 du)^{1/2} \, ,$$

and the space

$$H^2(\mathcal{U}, \mathbb{E}^3) = \{ f \in L^2(\mathcal{U}, \mathbb{E}^3) \, , \, \forall \, i, \, |i| \leq 2, \, D^i f \in L^2(\mathcal{U}, \mathbb{E}^3) \}$$

endowed with the H^2-norm $\|.\|_{2, \mathcal{U}}$, defined by

$$\|f\|_{2, \mathcal{U}} = (\Sigma_{|i| \leq 2} \, \|D^i f\|^2.)^{1/2}$$

$(H^2(\mathcal{U}, \mathbb{E}^3), \|.\|_{2, \mathcal{U}})$ is complete, and by Sobolev inequalities,

$$H^2(\overline{\mathcal{U}}, \mathbb{E}^3) \subset \mathfrak{C}^0(\overline{\mathcal{U}}, \mathbb{E}^3).$$

Finally we need to introduce the semi-norm $|.|_{2, \mathcal{U}}$ of length 2 defined on $H^2(\mathcal{U}, \mathbb{E}^3)$ by

$$|.|_{2, \mathcal{U}} : H^2(\mathcal{U}, \mathbb{E}^3) \to \mathbb{R}^+$$
$$f \quad \to \quad |f|_{2, \mathcal{U}}$$

where

$$|f|_{2, \mathcal{U}} = (\Sigma_{|i| \leq 2} \, \|D^i f\|^2)^{1/2}. \, \forall \, f \in H^2(\mathcal{U}, \mathbb{E}^3).$$

we get

$$\|f\|_{2, \mathcal{U}} = (|f|^2_{0, \mathcal{U}} + |f|^2_{1, \mathcal{U}} + |f|^2_{2, \mathcal{U}})^{1/2}.$$

We shall prove the following

6.4. Theorem

Let $\mathcal{U}$ be a bounded domain of $\mathbb{E}^2$. Let $\{a_1,...,a_k\}$, $(k\geq3)$, be k points of $\mathcal{U}$. Assume that the points a_i are not aligned. Then, there exists two constants C_1 and C_2 such that

$$C_1\|\phi\|_{2,\mathcal{U}} \leq [|\phi|^2_{2,\mathcal{U}} + \Sigma_{1\leq k}|f(a_i)|^2]^{1/2} \leq C_2\|\phi\|_{2,\mathcal{U}}, \ \forall \ \phi \in H^2(\mathcal{U},\mathbb{E}^3).$$

We need the following

6.5. Lemma

Let $\{a_i\}$, $(i = 1,...,k)$, be k points of $\mathcal{U}$, such that at least three are not aligned. Let L_i, $1\leq i\leq k$, be the linear forms defined on $H^2(\mathcal{U},\mathbb{R})$ by

$$L_i(f) = f(a_i), \ \forall \ f \in H^2(\mathcal{U},\mathbb{R}).$$

Let f be a polynomial of degree ≤ 1 on $\mathcal{U}$. Then
$$\Sigma_{1\leq k}|L_i(\phi)|^2 = 0 \Leftrightarrow \phi = 0$$

6.6. Proof of the lemma

First of all, L_i is well defined since

$$H^2(\overline{\mathcal{U}},\mathbb{E}^3) \subset \mathcal{C}^0(\overline{\mathcal{U}},\mathbb{E}^3).$$

If $\Sigma_{1\leq k}|L_i(\phi)|^2 = 0$, then $L_i(\phi) = 0$, $\forall \ i, 1\leq i\leq k$.

This implies that $\phi(a_i) = 0$, $\forall \ i, 1\leq i\leq k$. Since ϕ is polynomial of degree ≤ 1, and three of the points $\{a_i\}$ are not aligned, this implies that $\phi = 0$. The converse is trivial.

6.7. Proposition

Let $\mathcal{U}$ be a bounded domain of $\mathbb{E}^2$. Let P_1 be the space of polynomials of degree ≤ 1. Let f_i, $1\leq i \leq l$ be functions defined on $H^2(\mathcal{U},\mathbb{R})$ such that, for every $\omega \in P_1$.

$$\Sigma_{1\leq l}|f_i(\omega)|^2 = 0 \Leftrightarrow \omega = 0.$$

Then, there exists two positive constant C_1 and C_2 such that

$$C_1\|v\|_{2,\mathcal{U}} \leq [|v|^2_{2,\mathcal{U}} + \Sigma_{1\leq k}|f(v)|^2]^{1/2} \leq C_2\|v\|_{2,\mathcal{U}} \ , \ \forall \ v \in H^2(\mathcal{U},\mathbb{R}).$$

We do not give the proof of this proposition, which is an obvious consequence of a well-known theorem of equivalence of norms. (See [Ne] p. 110-111).

7. A problem $\tilde{\mathcal{P}}$, closed to $\mathcal{P}$.

Before working on $\mathcal{P}$, we shall use the previous results to solve a problem $\tilde{\mathcal{P}}$, closed to $\mathcal{P}$. We have the following

7.1. Theorem

Let $\mathcal{U}$ be a domain of $\mathbb{E}^2$, (which smooth boundary). Let $\{a_1,...,a_k\}$ be k points of $\mathcal{U}$, $(k \geq 3)$, such that three at least are not aligned. Let $\{x_1,...,x_k\}$ be k points of $\mathbb{E}^2$. Let

$$\mathcal{C} = \{f \in H^2(\mathcal{U}, \mathbb{E}^3), f(a_i) = x_i\}.$$

Then the problem $\tilde{\mathcal{P}}$: "Find f in $\mathcal{C}$ such that

$$\int_{\mathcal{U}} |\phi_{2,\mathcal{U}}|^2 du = \inf_{\mathcal{C}} \int_{\mathcal{U}} |\phi_{2,\mathcal{U}}|^2 du"$$

has a unique solution.

7.2. Proof of the theorem

Let

$$F(\phi) = \Sigma_{|i|=2} \int_{\mathcal{U}} |\phi_{2,\mathcal{U}}|^2 du.$$

$\mathcal{C}$ is an affine space on which

$$F(\phi) + \Sigma_{1 \leq k} |\phi(a_j)|^2 = \|\phi\|^2_{2,\mathcal{U}}.$$

Then, up to a constant, F and $\|.\|^2_{2,\mathcal{U}}$ coincide on $\mathcal{C}$. The solution of the problem of minimizing $\|\phi\|^2_{2,\mathcal{U}}$ in $\mathcal{C}$ exists and is unique. Then $\tilde{\mathcal{P}}$ has an unique solution.

7.3. Remark on the theorem

The solution of $\tilde{\mathcal{P}}$ exists and is unique. However, we don't know if ϕ is a $\mathbb{C}^1$ or $\mathbb{C}^2$-immersion. We only know that $\phi \in H^2(\mathcal{U}, \mathbb{E}^3)$.

7.4. Relations between $\mathcal{P}$ and $\tilde{\mathcal{P}}$

Let ψ be an immersion

$$\psi : \mathcal{U} \to \mathbb{E}^3$$

We know that

$$D^2\psi = h_\psi = h_{\psi_0} + h_{\psi}(\mathcal{U}),$$

Then,

$$\|D^2\psi\|^2 = \|h_{\psi_0}\|^2 + \|h_{\psi(\mathcal{U})}\|^2,$$

and

$$|\psi|^2_{2,\mathcal{U}}.du = \int_{\mathcal{U}} \|h_{\psi_0}\|^2(g,g').du + \int_{\mathcal{U}} \|h_{\psi(\mathcal{U})}\|^2(g,g').du.$$

On the other hand, we have seen that, under the condition:

$$a \leq \|\psi_*(x)\| \leq A , \ \forall \ x , \ \|x\| \leq 1,$$

there exists two positive constants c and C such that

$$\int_{\mathcal{U}} \|h_{\psi_0}\|^2_{(g,g')}du + c\int_{\mathcal{U}} \|h_{\psi(M)}\|^2_g.du \leq \int_{\mathcal{U}} \|h_{\psi_0}\|^2_{(g,g')}du + C\int_{\mathcal{U}} \|h_{\psi(\mathcal{U})}\|^2_g.du.$$

Consequently, using the previous paragraph, we deduce that, if α, β, γ are three positive constants, there exists two positive constants, b and B, depending on $(a,A,\alpha,\beta,\gamma)$ such that :

$$b\|\psi\|^2_{2,\mathcal{U}} \leq \alpha.\Sigma_{i\leq k} |y(x_i)|^2 + \beta\int_{\mathcal{U}} \|h_{\psi_0}\|^2_{(g,g')}du + \gamma\int_{\mathcal{U}} \|h_{\psi(\mathcal{U})}\|^2_g.du$$

$$\leq B\|\psi\|^2_{2,\mathcal{U}}$$

So we can state the

7.5. Theorem

Let a, A, α, β, γ be positive constants. Let ψ be an immersion belonging to $\mathcal{C}$, such that

$$a \leq \|\psi_*(x)\| \leq A , \ \forall \ x , \ \|x\| \leq 1.$$

Then, there exists two positive constants b and B, (depending only on (a, A, α, β, γ), such that

$$b\|\psi\|^2_{2,\mathcal{U}} \leq \alpha\Sigma_{i\leq k} |\psi(x_i)|^2 + \beta\int_{\mathcal{U}} \|h_{\psi_0}\|^2_{(g,g')}du + \gamma\int_{\mathcal{U}} \|h_{\psi(\mathcal{U})}\|^2_g.du \leq B\|\psi\|^2_{2,\mathcal{U}}$$

Now we remark that we can find a minimizing sequence ψ_n for $\tilde{\mathcal{P}}$ in $\mathcal{C}$, but we don't know if this sequence minimizes $\mathcal{P}$. More generally, we don't know if there exist minimizing sequences for $\mathcal{P}$. However remark that

$$\int_{\mathcal{U}} \|h_{\psi(\mathcal{U})}\|^2_g.du$$

is defined on $\mathfrak{Im}/_\sim$, and that

$$\int_{\mathcal{U}} \|h_{\psi_0}\|^2_{(g,g')} du$$

is only defined on $\mathfrak{Im}$. Then, we can apply our method of finding critical points of

$$\int_{\mathcal{U}} \|h_{\psi(\mathcal{U})}\|^2_{g} du$$

to the function f defined by

$$f(\psi) = \int_{\mathcal{U}} \|h_{\psi_0}\|^2_{(g,g')} du.$$

Finally, remark that the term

$$\int_{\mathcal{U}} \|h_{\psi}\|^2_{g} du$$

is defined on $\mathfrak{Im}/\sim$, (and can be interpreted as an F, in the terminology of §.2). On the other hand , the tangent part h_{ψ_0} can be interpreted as ω, (with the same notations) since any "normal perturbation" does not affect it. So we can apply our method of finding critical points to

$$F = \int_{\mathcal{U}} \|h_{\psi(\mathcal{U})}\|^2_{g'} du ,$$

by finding the zeroes of $\omega + dF$.

8. Remarks on the singularities of the limit of a sequence of immersions

We have already seen that the limit of a sequence ψ_n of immersions may have singularities. To find them, we can add assumption of on the curvature of each ψ_n. (The reader can consult [La] for a study of this point of view). In fact, the type of singularity depends on the behaviour on the induced metric on the sequence of surfaces The following theorem, essentially due to S. Gmira [Gm], gives an exemple of this phenomena:

8.1. Theorem

Let
$$f_n : D \to \mathbb{E}^3,$$
be a sequence of conformal immersions, defined on the unit disc $D \subset \mathbb{E}^3$, such that

i) (f_n) converges uniformly $\mathbb{C}^\infty$ over any compactto f, where f is a non constant map.

ii) the sequence of Gauss map (G_{f_n}) associated to (f_n) converges uniformly over any compact.

Then f does not admit any isolated singularity.

8.2. Sketch of proof

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Let

$$f_n : D \to \mathbb{E}^3,$$

be the converging sequence. We put

$$g_n = f_{n*}(g_n) = \lambda_n g_0,$$

where g_0 is the standard metric on D. We need the following

8.3. Lemma

Let k_n be the curvature of the metric g_n. Then, there exists a positive constant C such that, on a neighborhood of 0, we have

$$|k_n| \le \frac{C}{\lambda_n} , \forall n \in \mathbb{N}.$$

8.4. Proof of the lemma

Consider the sequence

$$G_{f_n} : D \to \wedge^2(\mathbb{E}^3)$$
$$m \to (f_{n*}(e_1) \wedge f_{n*}(e_2))_m$$

(where $\{e_1, e_2\}$ is an orthonormal frame of D , and $\wedge^2(\mathbb{E}^3)$ denotes the space of two-vectors in $\mathbb{E}^3$). The second fundamental form of f_n is the normal part of (G_{f_n}). From the assumption we deduce that there exist a constant C_1 such that

$$\|\sigma_n\|^2 < \frac{C_1}{\lambda_n}$$

on a suitable neighborhood U of 0. Using Gauss equation, we deduce that, on this neighborhood, there exists a constant C such that

$$|k_n| \le \frac{C}{\lambda_n} .$$

So the lemma is proved. To end the proof of the theorem, we remark that

$$k_n = - \frac{\Delta \log(\lambda_n)}{\lambda_n}.$$

Suppose that 0 is an isolated singularity, and apply Green formula in a pointed disc $D \backslash \{0\}$, included in U. We get a contradiction.

8.5. Remark on the theorem

A more general theorem of this type can be found in [Gm]. For the case where f_n is pseudo-holomorphic, see [La], from which the technic is transfered.

n	1	2	3	4	5	6	7	8	9
x	0.1	0.1	0.1	2.5	2.5	2.5	4.9	4.9	4.9
y	0.1	2.0	3.9	0.1	2.0	3.9	0.1	2.0	3.9
z	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0
u	u_1	u_1	u_1	u_2	u_2	u_2	u_3	u_3	u_3
v	v_1	v_2	v_3	v_1	v_2	v_3	v_1	v_2	v_3

	u_1	u_2	u_3	v_1	v_2	v_3
a, b	0.10	2.50	4.90	0.10	2.00	3.90
c, d	0.05	1.25	2.45	0.05	1.00	1.95
e, f	0.05	1.25	2.45	0.20	4.00	7.80

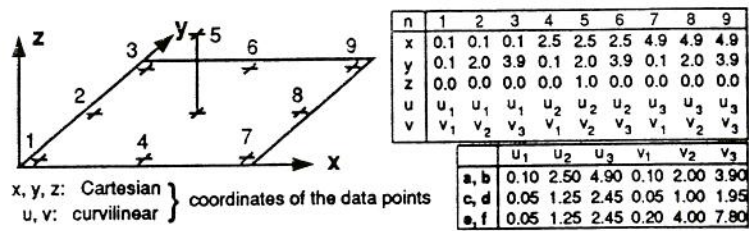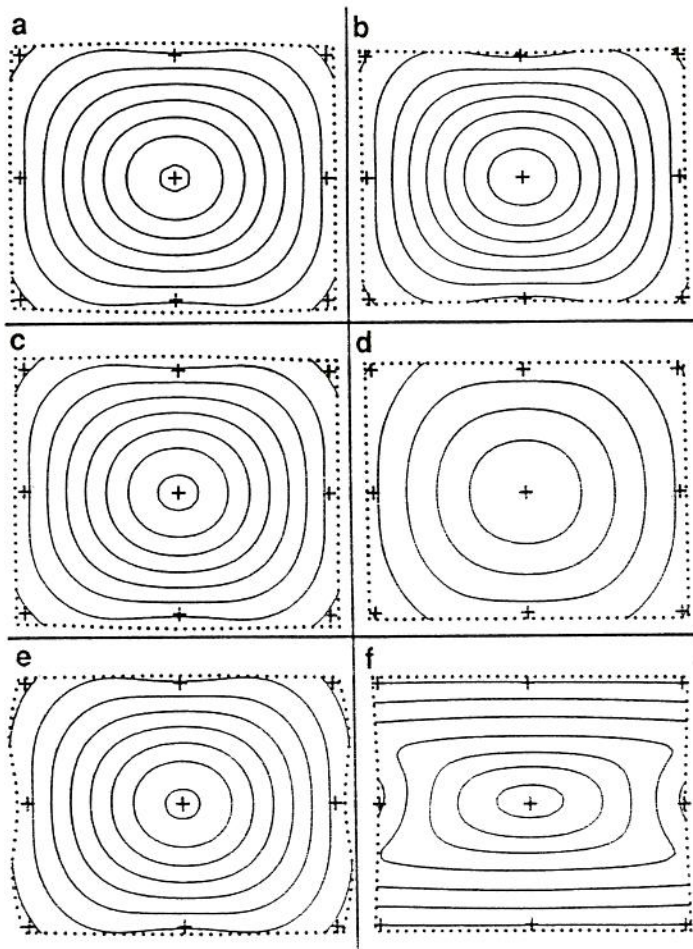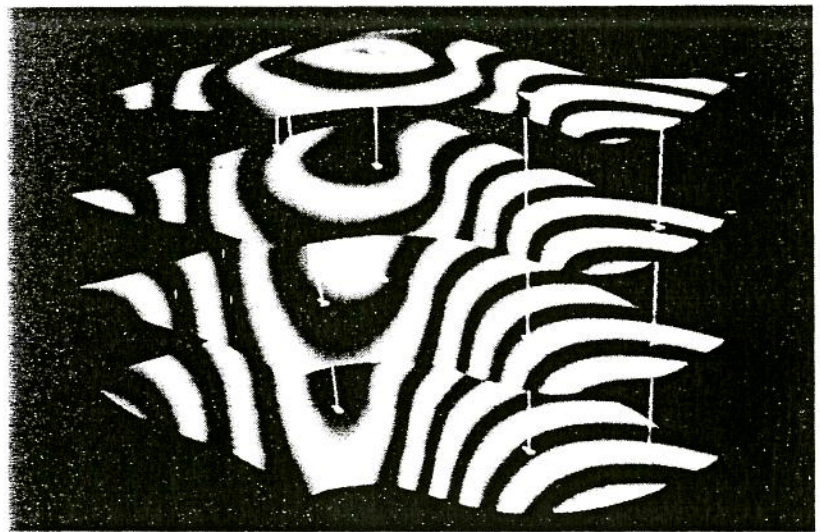x, y, z: Cartesian $\Big\}$ coordinates of the data points
u, v: curvilinear

FIG. 1. Coordinates of the data points used in Figures 2a to 2f.

FIG. 2. Comparison between the minimization of curvature (a, c, e) and second derivatives (b, d, f).
Surface boundaries in dotted lines. Contour interval in z: 0.05. First contour at z=0.

Here is an example of application of the technics that we have described in this article. Different curvatures are minimized to get a foliation of surfaces from which four leaves are visualised.

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