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ON ISOMETRIC LAGRANGIAN IMMERSIONS

JOHN DOUGLAS MOORE AND JEAN-MARIE MORVAN

Abstract. This article uses Cartan-Kähler theory to show that a small neighborhood of a point in any surface with a Riemannian metric possesses an isometric Lagrangian immersion into the complex plane (or by the same argument, into any Kähler surface). In fact, such immersions depend on two functions of a single variable. On the other hand, explicit examples are given of Riemannian three-manifolds which admit no local isometric Lagrangian immersions into complex three-space. It is expected that isometric Lagrangian immersions of higher-dimensional Riemannian manifolds will almost always be unique. However, there is a plentiful supply of flat Lagrangian submanifolds of any complex n-space; we show that locally these depend on $\frac{1}{2}n(n+1)$ functions of a single variable.

1. Introduction

This note is concerned with the question of which n-dimensional Riemannian manifolds can be immersed isometrically as Lagrangian submanifolds of $\mathbb{C}^n$. Recall that an immersed submanifold $M^n \subset \mathbb{C}^n$ is *Lagrangian* if the complex structure J maps the tangent space T_pM at an arbitrary point $p \in M$ isometrically onto the corresponding normal space N_pM.

In the special case $n = 2$, we use Cartan-Kähler theory to prove:

Theorem 1. *Let M^2 be a real-analytic Riemannian manifold of dimension two. If $p \in M^2$, then there is an open neighborhood U of p which possesses an isometric Lagrangian immersion into $\mathbb{C}^2$. Indeed, the local isometric Lagrangian immersions depend upon three functions of a single variable.*

This local theorem should be contrasted with the fact that there are obstructions to the existence of global isometric Lagrangian immersions of Riemannian surfaces into $\mathbb{C}^2$. For example, although the two-sphere S^2 possesses a Lagrangian immersion into $\mathbb{C}^2$ as the Whitney sphere (see [6] for instance), there is no isometric Lagrangian immersion of S^2 with any metric of strictly positive curvature into $\mathbb{C}^2$, by an argument that we recall in Section 2.

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It is to be expected that (even locally) most Riemannian manifolds of dimension $n \geq 3$ possess no isometric Lagrangian immersions into $\mathbb{C}^n$, because the system of partial differential equations that would need to be solved is overdetermined. Indeed, when $n \geq 4$, the curvature tensor at any given point must satisfy some quite explicit conditions: the Pontrjagin forms of a Lagrangian submanifold M^n of $\mathbb{C}^n$ must vanish, because the Lagrangian immersion provides a geometric trivialization of the complexification of the tangent bundle TM; the Chern forms of the complexification must vanish identically and these are just the Pontrjagin forms of TM. In the case $n = 3$, explicit counterexamples to the existence of isometric Lagrangian immersions are provided by the "Berger spheres" defined at the beginning of Section 5:

Theorem 2. *The Berger spheres are three-dimensional Riemannian manifolds which admit no local isometric Lagrangian immersions in* $\mathbb{C}^3$.

In addition, when $n \geq 3$ one expects that for most Riemannian metrics the isometric Lagrangian immersion, when it exists, will be unique up to rigid motion. However, manifolds with special curvature properties can exhibit more flexibility. In the special case of flat Riemannian manifolds, we will apply Cartan-Kähler theory to show that there does exist a plentiful supply of local isometric Lagrangian immersions:

Theorem 3. *Let p be a point in $\mathbb{E}^n$. The isometric Lagrangian immersions from an open neighborhood U of p into $\mathbb{C}^n$ depend upon $\frac{1}{2}n(n+1)$ functions of a single variable.*

The simplest explicit example of a flat Lagrangian submanifold in $\mathbb{C}^n$ is the Clifford torus

$$S^1 \times S^1 \times \cdots \times S^1 \subset \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C} = \mathbb{C}^n.$$

More general flat Lagrangian submanifolds have been studied by several authors. Indeed, Pinkall [18] has shown that every conformal structure on the two-torus can be realized by a flat Lagrangian immersion into $S^3 \subset \mathbb{C}^2$ (see also [19]). More recently, Chen, Dillen, Verstraelen and Vranken [10] have given explicit constructions of flat Lagrangian immersions in higher dimensions in terms of twistor forms.

It follows from the proof that a flat Lagrangian submanifold of $\mathbb{C}^n$ is determined by Cauchy data on a curve by a succession of applications of the Cauchy-Kowalewski theorem. This is quite similar to what happens in the theory of flat n-manifolds in the constant curvature $(2n-1)$-sphere S^{2n-1}, or n-manifolds of constant curvature -1 in $\mathbb{E}^{2n-1}$ (see [15]). In [3] Cartan proved that such submanifolds depend upon $n(n-1)$ functions of a single variable, and in [1] Berger, Bryant and Griffiths extended this result to certain "quasi-hyperbolic" submanifolds of $\mathbb{E}^{2n-1}$.

Theorems 1 and 3 extend an earlier theorem due to Chen and Houh [11]. We will prove these theorems in Sections 4 and 6 after a brief exposition of Cartan-Kähler theory in Section 3. These theorems are complemented by nonexistence theorems. We will describe some global obstructions to the existence of isometric Lagrangian immersions in Section 2 and prove Theorem 2 in Section 5.

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2. Preliminaries and global restrictions

We consider the space $F_{\mathbb{C}}(\mathbb{C}^n)$ of unitary frames on $\mathbb{C}^n$. An element of $F_{\mathbb{C}}(\mathbb{C}^n)$ is a pair $(p, (e_1, \ldots, e_{2n}))$, where $p \in \mathbb{C}^n$ and $(e_1, \ldots, e_{2n})$ is a real orthonormal frame such that $Je_i = e_{n+i}$ for $1 \le i \le n$, J being the complex structure on $\mathbb{C}^n$. Note that after choice of a base frame, $F_{\mathbb{C}}(\mathbb{C}^n)$ can be identified with the trivial bundle $\mathbb{C}^n \times U(n) \to \mathbb{C}^n$, where $U(n)$ is the unitary group. On $F_{\mathbb{C}}(\mathbb{C}^n)$, we define differential forms $\widetilde{\omega}_{IJ}$ and $\widetilde{\theta}_I$, for $1 \le I, J \le 2n$, so that

$$de_J = \sum_{I=1}^{2n} e_I \widetilde{\omega}_{IJ}, \qquad dp = \sum_{I=1}^{2n} e_I \widetilde{\theta}_I; \tag{1}$$

these forms must satisfy the Cartan structure equations

$$d\widetilde{\theta}_I = -\sum \widetilde{\omega}_{IJ} \wedge \widetilde{\theta}_J, \qquad d\widetilde{\omega}_{IJ} = -\sum \widetilde{\omega}_{IK} \wedge \widetilde{\omega}_{KJ}.$$

We write $i^* = n + i$ for $1 \le i \le n$, so that $e_{i*} = Je_i = e_{n+i}$. Since the matrix-valued one-form $\omega = (\widetilde{\omega}_{IJ})$ takes its values in the Lie algebra of the unitary group, we must have

$$\widetilde{\omega}_{ij} = \widetilde{\omega}_{i*j*} = -\widetilde{\omega}_{ji}, \qquad \widetilde{\omega}_{i*j} = -\widetilde{\omega}_{ji*} = \widetilde{\omega}_{j*i}. \tag{2}$$

Suppose that $f\colon M^n \to \mathbb{C}^n$ is an isometric Lagrangian immersion. An *adapted moving frame* over an open subset U of M^n is a lifting $\tilde{f}\colon U \to F_C(C^n)$ of $f|U$ such that $(e_1 \circ \tilde{f}, \ldots, e_n \circ \tilde{f})$ are tangent to $f(M^n)$. Let

$$\omega_{IJ} = \tilde{f}^* \widetilde{\omega}_{IJ}, \qquad \theta_I = \tilde{f}^* \widetilde{\theta}_I.$$

Then $(\theta_1, \dots, \theta_n)$ is an orthonormal coframe on U, $\theta_{i*} = 0$ and the ω_{IJ} determine the Levi-Civita connection

$$\nabla_X e_j = \sum_{i=1}^n e_i \omega_{ij}(X)$$

and the second fundamental form

$$\alpha(\cdot, \cdot) = \sum_{i,j=1}^n e_{j*} \omega_{j*i} \otimes \theta_i = \sum_{i,j,k=1}^n e_{j*} h_{jik} \theta_i \otimes \theta_k.$$

It follows from (2) that $h_{jik} = h_{ijk}$ and from (1) that

$$\sum_{j=1}^n \omega_{i*j} \wedge \theta_j = 0, \quad \text{and hence} \quad h_{ijk} = h_{ikj},$$

so h_{ijk} is symmetric in all three indices. The structure equations (1) imply that the curvature

$$\Omega_{ij} = \sum_{k,l} \frac{1}{2} R_{ijkl} \theta_k \wedge \theta_l = dw_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj}$$

must satisfy the Gauss equation

$$\Omega_{ij} = \sum_k \omega_{k*i} \wedge \omega_{k*j} = \frac{1}{2} \sum_{k,r,s} (h_{kir} h_{kjs} - h_{kis} h_{kjr}) \theta_r \wedge \theta_s. \tag{3}$$

The structure equations also imply that

$$d\left[\sum_{i=1}^n \omega_{i*i}\right] = -\sum_{i,j=1}^n \omega_{i*j} \wedge \omega_{ji} - \sum_{i,j=1}^n \omega_{i*j*} \wedge \omega_{j*i} = -2 \sum_{i,j=1}^n \omega_{i*j} \wedge \omega_{ji} = 0,$$

since ω_{ji} is skew-symmetric in i and j while ω_{i*j} is symmetric in these indices. The closed form

$$\mu = \sum_{i=1}^n \omega_{i*i}$$

is called the *Maslov form*, and as described in [17], it is related to the mean curvature

$$H = \sum_{i,j=1}^n e_{j*} h_{iij} \quad \text{by the formula} \quad \mu = \langle JH, \cdot \rangle.$$

These facts can be used to prove that if M^n is an n-dimensional compact Riemannian manifold with finite fundamental group or nonzero Euler characteristic which possesses a Lagrangian isometric immersion into $\mathbb{C}^n$, then M^n must have a point of nonpositive scalar curvature. We recall the argument for this fact (which was presented in the proof of Theorem 4.6 of [8]). If M^n is a compact Lagrangian submanifold of $\mathbb{C}^n$ with finite fundamental group, then $H^1(M^n; \mathbb{R}) = 0$. Hence the Maslov form μ is exact, $\mu = df$ for some smooth function $f \colon M^n \to R$. If p is a maximum for f, df and hence H must vanish at p. On the other hand, if the Euler characteristic of M^n is nonzero, then one can reason directly that the vector field JH has a zero, and hence again there is a point $p \in M^n$ at which H vanishes. In either case, it follows from the Gauss equations (3) that

$$\begin{aligned}\sum_{i,j} R_{ijji}(p) &= \sum_{i,j,k} \left[h_{iik}(p)h_{jjk}(p) - h_{ijk}(p)^2\right] \\ &= H(p)^2 - \sum_{i,j,k} h_{ijk}(p)^2 = -\sum_{i,j,k} h_{ijk}(p)^2 \leq 0,\end{aligned}$$

and hence the scalar curvature at p is nonpositive.

The Chern-Simons invariant provides another obstruction to the existence of isometric or conformal Lagrangian immersions. This obstruction is a consequence of the Corollary on page 139 of [16], but we give a simple direct argument. The key fact is that the closed differential form

$$\Phi = \frac{-1}{48\pi^2} \sum_{I,J,K=1}^{2n} \widetilde{\omega}_{IJ} \wedge \widetilde{\omega}_{JK} \wedge \widetilde{\omega}_{KI}$$

on the unitary group $U(n)$ lies in the image of the coefficient homomorphism $H^3(U(n);\mathbb{Z}) \to H^3(U(n);\mathbb{R})$, and hence its integral over any cycle is an integer. To check that the coefficient is correct, one can restrict to $SU(2) = S^3$, and check that the restricted form integrates to ± 1.

It is well-known that a compact oriented three-manifold possesses a trivial tangent bundle. Hence if M^3 is a compact oriented three-manifold which is immersed as a Lagrangian submanifold of $\mathbb{C}^3$, we can construct a global moving frame $\tilde{f}\colon M^3 \to U(3)$. But then

$$\begin{aligned}\tilde{f}^*(\Phi) = \frac{-1}{48\pi^2} \Bigg[&\sum_{i,j,k=1}^{n} \omega_{ij} \wedge \omega_{jk} \wedge \omega_{ki} + 3 \sum_{i,j,k=1}^{n} \omega_{ij*} \wedge \omega_{j*k} \wedge \omega_{ki} \\ &+ 3 \sum_{i,j,k=1}^{n} \omega_{ij*} \wedge \omega_{j*k*} \wedge \omega_{k*i} + \sum_{i,j,k=1}^{n} \omega_{i*j*} \wedge \omega_{j*k*} \wedge \omega_{k*i*} \Bigg] \\ = \frac{-1}{24\pi^2} &\left[\sum_{i,j,k=1}^{n} \omega_{ij} \wedge \omega_{jk} \wedge \omega_{ki} - 3 \sum_{i,j=1}^{n} \Omega_{ij} \wedge \omega_{ij} \right] = TP_1(\omega),\end{aligned}$$

$TP_1(\omega)$ being the transgressed Pontrjagin form considered by Chern and Simons [12], pulled back via the trivialization of TM. We conclude that the reduction mod $\mathbb{Z}$ of $TP_1(\omega)$ is zero (but not the somewhat finer mod $\mathbb{Z}$ reduction of $(1/2)TP_1(\omega)$ also considered by Chern and Simons).

The Chern-Simons invariant depends only on the conformal structure. Since lens spaces with constant positive curvature have a nontrivial Chern-Simons invariant, they do not have conformal Lagrangian immersions into $\mathbb{C}^3$, even though a small neighborhood of any point is conformally equivalent to an open subset of Euclidean space and hence possesses a conformal Lagrangian immersion as part of a flat Lagrangian torus. Thus none of the Riemannian metrics within the conformal equivalence class of the constant curvature lens space admits an isometric Lagrangian immersion into $\mathbb{C}^3$.

3. Cartan-Kähler theory

Let W be an N-dimensional smooth manifold, $\Omega^k(W)$ the vector space of smooth differential k-forms on W,

$$\Omega^*(W) = \sum_{k=0}^{N} \Omega^k(W)$$

the algebra of differential forms on W. We say that an ideal $\mathcal{A} \subset \Omega^*(W)$ is *homogeneous* if each homogeneous component of an element in $\mathcal{A}$ lies within $\mathcal{A}$, or equivalently,

$$\mathcal{A} = \sum_{k=0}^{N} \mathcal{A}^k, \quad \text{where} \quad \mathcal{A}^k = \mathcal{A} \cap \Omega^k(W).$$

A *differential ideal* is a homogeneous ideal $\mathcal{A} \subset \Omega^*(W)$ which satisfies the condition $d\mathcal{A} \subset \mathcal{A}$.

An n-dimensional submanifold $M \subset W$ is called an *integral submanifold* for the differential ideal $\mathcal{A}$ if $i^*\mathcal{A} = 0$, where $i\colon M \to W$ is the inclusion.

If $p \in W$, let G_p^n denote the set of n-dimensional linear subspaces of T_pW. An element $E_p^n \in G_p^n$ is an *integral element* for $\mathcal{A}$ if

$$\omega \in \mathcal{A}^n \quad \Rightarrow \quad \omega(v_1, \dots, v_n) = 0,$$

when $(v_1, \dots, v_n)$ is a basis for E_p^n. Thus an integral submanifold is just a submanifold whose tangent spaces are integral elements.

Suppose now that E_p^n is an integral element for $\mathcal{A}$ and that $(v_1, \dots v_n)$ is a basis for E_p^n. The *polar space* for E_p^n is the linear space

$$H(E_p^n) = \{v \in T_pW : \omega(v, v_1, \dots, v_n) = 0 \text{ for all } \omega \in \mathcal{A}^{n+1}\},$$

or equivalently,

$$H(E_p^n) = \{v \in T_pW : \text{the restriction of } \iota_v\omega \text{ to } E_p^n \text{ is zero, for all } \omega \in \mathcal{A}\},$$

where ι_v denotes the interior product. Note that $E_p^{n+1} \in G_p^{n+1}$ is an integral element containing E_p^n if and only if $E_p^{n+1} \subset H(E_p^n)$. Following the notation of Kähler [14], we let

$$r_{n+1}(E_p^n) = \dim H(E_p^n) - (n+1).$$

A zero-dimensional integral element E_p^0 is said to be *regular* if r_1 assumes its minimum value at E_p^0 and $r_1(E_p^0) \geq 0$. Inductively, we say that an n-dimensional integral element E_p^n is *ordinary* if it contains a regular integral element of dimension $n-1$, and *regular* if in addition, r_{n+1} assumes its minimum value at E_p^n and $r_{n+1}(E_p^n) \geq 0$. Finally, an integral submanifold is *regular* if all of its tangent spaces are regular integral elements.

Cartan-Kähler Theorem. *Let M be a connected regular integral submanifold for the differential ideal $\mathcal{A}$ on W, F an $(n - r_{n+1})$-dimensional submanifold of W containing M such that*

$$\dim(T_pF \cap H(T_pM)) = n+1, \quad \text{for } p \in N.$$

Then there is an $(n+1)$-dimensional integral submanifold $\widetilde{M}$ for $\mathcal{A}$, unique up to extension, such that $M \subset \widetilde{M} \subset F$.

The proof of this theorem can be found in [2], Chapter III, Section 2 or in the classical references [5] and [14].

By a *system of n independent variables*, we simply mean a decomposable n-form $\Theta = \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_n$. If $\mathcal{A}$ is a differential ideal and Θ is a system of n independent variables, we say that the pair $(\mathcal{A}, \Theta)$ is *in involution* if there is an n-dimensional ordinary integral element E_p^n with basis $(v_1, \ldots, v_n)$ such that $\Theta(v_1, \ldots, v_n) \neq 0$. The Cartan-Kähler theorem implies that if $(\mathcal{A}, \Theta)$ is in involution, there exist integral submanifolds of $\mathcal{A}$ on which the restriction of Θ is nonzero.

4. Lagrangian surfaces

To study Lagrangian surfaces in $\mathbb{C}^2$, we use a differential ideal quite similar to that used in Cartan's proof of the Janet-Cartan Theorem (see [4]). We let $W = M^2 \times F_{\mathbb{C}}(\mathbb{C}^2)$, where M^2 is a given two-dimensional Riemannian manifold and $F_{\mathbb{C}}(\mathbb{C}^2)$ is the bundle of complex unitary frames over $\mathbb{C}^2$. Our strategy is to construct a submanifold N of W which can serve as the graph of a mapping $\tilde{f}\colon M^2 \to F_{\mathbb{C}}(\mathbb{C}^2)$, which defines not only an isometric Lagrangian immersion into $\mathbb{C}^2$, but also a corresponding adapted moving frame along the immersion.

Since the problem is local, we can assume that M^2 is parallelizable and choose a fixed moving frame (e_1, e_2) for TM with corresponding coframe (θ_1, θ_2). There is a unique connection one-form $\omega_{12} = -\omega_{21}$ on M^2 which satisfies the equations

$$d\theta_1 = -\omega_{12} \wedge \theta_2, \qquad d\theta_2 = -\omega_{21} \wedge \theta_1.$$

Let $\Omega_{12} = d\omega_{12}$ denote the curvature two-form on M^2.

We pull the differential forms $\widetilde{\theta}_I$, $\widetilde{\omega}_{IJ}$, θ_i, ω_{12} back to the product manifold W. On W, we will take the ideal $\mathcal{A}$ which is generated by the differential one-forms

$$\widetilde{\theta}_i - \theta_i, \qquad \widetilde{\theta}_{i*}, \qquad \widetilde{\omega}_{12} - \omega_{12},$$

and the differential two-forms

$$d\theta_{i*}, \qquad \widetilde{d\omega}_{12} - d\omega_{12},$$

or equivalently, the two-forms

$$\sum_j \widetilde{\omega}_{i*j} \wedge \widetilde{\theta}_j, \qquad \sum_{i*} \widetilde{\omega}_{i*1} \wedge \widetilde{\omega}_{i*2} - \Omega_{12}.$$

It is quickly checked that $\mathcal{A}$ is closed under the exterior derivative, and is therefore a differential ideal.

We claim that $(\mathcal{A}, \theta_1 \wedge \theta_2)$ is in involution. Since there are no zero-forms in $\mathcal{A}$, an arbitrary point $q = E^0_q$ of W can be taken as a zero-dimensional integral element. Its polar space $H(E^0_q)$ is the collection of vectors $v_1 \in T_qW$ which are annihilated by the one-form generators in $\mathcal{A}$. Once $\theta_i(v_1)$ are chosen, $\omega_{12}(v_1)$ is determined, and the one-form generators for $\mathcal{A}$ determine

$$\tilde{\theta}_I(v_1), \qquad \widetilde{\omega}_{12}(v_1),$$

while $\widetilde{\omega}_{i*j}(v_1)$ can be chosen at will, subject to the requirement that $\widetilde{\omega}_{41}(v_1) = \widetilde{\omega}_{32}(v_1)$. We have two degrees of freedom in choosing $\theta_i(v_1)$ and an additional three in choosing $\widetilde{\omega}_{31}(v_1)$, $\widetilde{\omega}_{41}(v_1)$ and $\widetilde{\omega}_{42}(v_1)$, so the polar space $H(E^0_p)$ has constant dimension 5 and all zero-dimensional integral elements are regular.

Suppose now that we choose a one-dimensional integral element E^1_q containing E^0_q and generated by a nonzero vector $v_1 \in T_qW$. The polar space $H(E^1_q)$ is the set of vectors $v_2 \in T_pW$ which satisfy the linear equations

$$L_1: \quad \tilde{\theta}_i(v_2) = \theta_i(v_2), \qquad \tilde{\theta}_{i*}(v_2) = 0, \qquad \widetilde{\omega}_{12}(v_2) = \omega_{12}(v_2),$$

together with the three equations

$$L_2: \quad \sum_j (\widetilde{\omega}_{i*j} \wedge \widetilde{\theta}_j)(v_1, v_2) = 0, \qquad \sum_{i*} (\widetilde{\omega}_{i*1} \wedge \widetilde{\omega}_{i*2})(v_1, v_2) = \Omega_{12}(v_1, v_2).$$

Once $\theta_i(v_2)$ are chosen, the linear system L_1 determines

$$\tilde{\theta}_I(v_2), \qquad \omega_{12}(v_2),$$

and all that remain to be determined are the three elements $\widetilde{\omega}_{31}(v_2)$, $\widetilde{\omega}_{41}(v_2)$ and $\widetilde{\omega}_{42}(v_2)$. We can regard L_2 as a linear system in these unknowns, and since L_2 contains three equations, its maximal possible rank is three. If we can show that the three equations are independent, then E^1_q will be a regular integral element with $\dim H(E^1_q) = 2$, $r_2(E^1_q) = 0$, and E^1_q will lie in a unique two-dimensional ordinary integral element E^2_q.

But we can choose v_1 so that $\theta_1(v_1) = 1$ and $\theta_2(v_1) = 0$, and set $h_{i*j1} = \widetilde{\omega}_{i*j}(v_1)$. If $h_{i*j2} = \widetilde{\omega}_{i*j}(v_2)$, the system L_2 can be rewritten as

$$h_{312} = h_{321}, \qquad h_{412} = h_{421}, \qquad \begin{vmatrix} h_{311} & h_{312} \\ h_{321} & h_{322} \end{vmatrix} + \begin{vmatrix} h_{321} & h_{322} \\ h_{421} & h_{422} \end{vmatrix} = K,$$

where K is the Gaussian curvature of M^2. If h_{311} is nonzero, this linear system for the unknowns h_{312}, $h_{412} = h_{322}$ and h_{422} does indeed have rank three, so the conclusions at the end of the preceding paragraph do indeed hold. If we choose v_2 so that $\theta_1(v_2) = 0$ and $\theta_2(v_2) = 1$, then $\theta_1 \wedge \theta_2$ will be nonzero on the ordinary integral element E^2_q, and we conclude that our differential system is indeed in involution.

Proof of Theorem 1. Suppose that C is a real analytic curve in M^2 which contains an arbitrary point $p \in M^2$ and suppose that (e_1, e_2) is chosen so that along C, e_1 is tangent to C. Given arbitrary functions h_{311}, $h_{321} = h_{411}$ and h_{421} along C, we can construct a real analytic moving frame

$$f\colon C \to F_{\mathbb{C}}(\mathbb{C}^2) \quad \text{such that} \quad f^*(\widetilde{\omega}_{12}) = \omega_{12}, \quad f^*(\widetilde{\omega}_{i*j}) = h_{i*j1}\theta_1.$$

We can lift this to a real analytic map

$$\tilde{f}\colon C \to \mathbb{C}^2 \times F_{\mathbb{C}}(\mathbb{C}^2) \quad \text{such that} \quad \tilde{f}^*\tilde{\theta}_1 = \theta_1, \quad \tilde{f}^*\tilde{\theta}_2 = 0.$$

The graph of $\tilde{f}$ is an integral submanifold $\widehat{C} \subset W$ for $\mathcal{A}$ which projects to C via the obvious projection $\pi\colon W \to M$ and satisfies the condition

$$\widetilde{\omega}_{31}(\hat{e}_1) = h_{311}, \qquad \widetilde{\omega}_{32}(\hat{e}_1) = h_{321} \qquad \widetilde{\omega}_{42}(\hat{e}_1) = h_{421},$$

where $\hat{e}_1$ projects to e_1. If h_{321} is never zero, the tangent spaces to $\widehat{C}$ will be regular integral elements, and since $r_2 = 0$ the Cartan-Kähler theorem will guarantee the existence of a two-dimensional integral submanifold N of W, unique up to extension, which contains $\widehat{C}$. Since $\theta_1 \wedge \theta_2$ is nonzero along N, the inverse function theorem implies that the projection $\pi|N\colon N \to M$ possesses a local inverse map $\sigma\colon U \to N$ from an open neighborhood U of p in M^2. The composition of σ with the obvious projection $W \to \mathbb{C}^2$ yields the desired isometric Lagrangian immersion from U into $\mathbb{C}^2$. The immersions so constructed depend upon the functions h_{311}, $h_{321} = h_{411}$ and h_{421} which can be completely arbitrary, except for the requirement that the one-dimensional integral elements they determine be regular. Thus we see that the isometric Lagrangian immersions do indeed depend upon three functions of a single variable, as claimed. $\square$

Remark. From the proof, it is clear that we could replace the ambient space $\mathbb{C}^2$ by any Kähler manifold of complex dimension two, or more generally any Riemannian four-manifold with compatible almost complex structure J satisfying $\nabla J = 0$ (in which case a Lagrangian submanifold would be regarded as a submanifold for which J interchanges tangent and normal spaces).

5. Berger spheres

In higher dimensions, the system of equations for isometric Lagrangian immersions is overdetermined, so we expect that most Riemannian manifolds would not admit such immersions. When $n \geq 3$, the analog of the differential system with independent variables considered in the previous section is no longer in involution. For any given Riemannian metric, we might try to solve the local problem by means of Cartan's method of prolongation (see Chapter

VI in [5]), and this would typically lead to algebraic problems that might be difficult to solve. An interesting example is provided by left-invariant metrics on the Lie group S^3 of unit quaternions. In this section, we will show that many of these metrics admit no isometric Lagrangian immersion into $\mathbb{C}^3$.

We regard S^3 as the double cover of $SO(3)$, which possesses the standard left-invariant one-forms ϕ_{ij}, for $1 \leq i,j \leq 3$, which satisfy the structure equations

$$d\phi_{ij} = -\phi_{ik} \wedge \phi_{kj},$$

where (i,j,k) is a permutation of $(1,2,3)$. if we set

$$\alpha_i = \phi_{jk}, \quad \text{for } (i,j,k) \text{ a positive permutation of } (1,2,3),$$

we can rewrite the structure equations as

$$d\alpha_i = \alpha_j \wedge \alpha_k, \quad \text{for } (i,j,k) \text{ a positive permutation of } (1,2,3).$$

If we set

$$\theta_1 = \epsilon\alpha_1, \qquad \theta_2 = \alpha_2, \qquad \theta_3 = \alpha_3,$$

where $\epsilon > 0$, then

$$ds^2 = \theta_1^2 + \theta_2^2 + \theta_3^2$$

is a left-invariant metric on $SO(3)$ which lifts to a left-invariant metric on S^3. We call the resulting manifold a *Berger sphere* in the case where $0 < \epsilon < 1$ (see [7], Example 3.35).

According to the fundamental theorem of Riemannian geometry, the connection forms ω_{ij} on $SO(3)$ are determined by the structure equations

$$d\theta_i = -\sum_j \omega_{ij} \wedge \theta_j, \qquad \omega_{ij} + \omega_{ji}.$$

A straightforward calculation shows that

$$\omega_{23} = [1 - (\epsilon^2/2)]\alpha_1, \qquad \omega_{31} = (\epsilon/2)\theta_2, \qquad \omega_{12} = (\epsilon/2)\theta_3,$$

and from this we can determine the curvature:

$$R_{1212} = R_{1313} = (1/2)\epsilon^2, \qquad R_{2323} = 1 - (3/4)\epsilon^2,$$
$$R_{ijik} = 0 \quad \text{when } i,j,k \text{ are distinct.}$$

Proof of Theorem 2. We first seek the solutions h_{ijk} to the Gauss equations. Using the symmetry in three variables, we can write these as

$$h_{1jk} = \begin{pmatrix} x & a & f \\ a & d & u \\ f & u & c \end{pmatrix}, \qquad h_{2jk} = \begin{pmatrix} a & d & u \\ d & y & b \\ u & b & e \end{pmatrix}, \qquad h_{3jk} = \begin{pmatrix} f & u & c \\ u & b & e \\ c & e & z \end{pmatrix}.$$

We have the freedom of rotating the part of the coframe (θ_2, θ_3), and using this freedom, we can arrange that $u = 0$ at a given point.

The Gauss equations divide into two groups, those of the form $R_{ijij} =$ something, and those of the form $R_{ijik} =$ something, where i, j, k are distinct. The first group of these is

$$dx + ay = (1/4)\epsilon^2 + a^2 + d^2 - bf, \qquad cx + fz = (1/4)\epsilon^2 + c^2 + f^2 - ae,$$
$$ey + bz = 1 - (3/4)\epsilon^2 + b^2 + e^2 - cd,$$

while the second group is

$$-af + ab + fe = 0, \quad -df + db - bc = 0, \qquad ac + de - ce = 0.$$

We can analyze this system of six quadratic equations by means of Maple or Mathematica; in particular, the calculations can be guided by finding a Groebner basis as described in [13]. This procedure leads to the conclusion that if $\epsilon \neq 1$, then automatically $a = f = 0$.

In more detail, an explicit calculation could proceed as follows. We can look at the first group of Gauss equations as a linear system for the unknowns (y, z, ϵ^2). This system will possess a unique solution in terms of the other variables so long as the determinant of the coefficient matrix is nonzero. Using a computer to do the calculations (if necessary), we can check that the determinant of coefficients is af, and that if this determinant is nonzero, one piece of the unique solution is $\epsilon^2 = 1$. Therefore if $\epsilon^2 \neq 1$, we see that $af = 0$. Since a and f are interchanged under interchange of θ_2 and θ_3, we can assume without loss of generality that $a = 0$. It then follows from the second group of Gauss equations that $ef = 0$. If $f \neq 0$, then $e = 0$ and we can analyze the first group of Gauss equations as a linear system for the unknowns (x, z, ϵ^2). This time we find that either $\epsilon^2 = 1$ or the determinant of coefficients df vanishes. Thus if $f \neq 0$, we must have $d = e = 0$, which implies that $b\,c = 0$. We leave it to the reader to check that either case, $b = 0$ or $c = 0$, leads to a contradiction with the Gauss equations. By this procedure, we are finally forced to conclude that $a = f = 0$.

The upshot is that the second fundamental form simplifies to

$$h_{1jk} = \begin{pmatrix} x & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & c \end{pmatrix}, \qquad h_{2jk} = \begin{pmatrix} 0 & d & 0 \\ d & y & b \\ 0 & b & e \end{pmatrix}, \qquad h_{3jk} = \begin{pmatrix} 0 & 0 & c \\ 0 & b & e \\ c & e & z \end{pmatrix}.$$

The Codazzi equations are equivalent to the assertion that the functions h_{ijkl} are symmetric in all indices, where

$$\sum_l h_{ijkl}\theta_l = dh_{ijk} - h_{mjk}\omega_{mi} - h_{imk}\omega_{mj} - h_{ijm}\omega_{mk}.$$

Since $h_{11i} = 0$ unless $i = 1$, the Codazzi equations in the case where $(i, j, k) = (1, 1, 1)$ imply that

$$h_{111i} = e_i(h_{111}) = e_1(x).$$

In the case where $(i,j,k) = (1,1,2)$, the Codazzi equations yield

$$h_{1121}\theta_1 + h_{1122}\theta_2 + h_{1123}\theta_3 = (2h_{122} - h_{111})\omega_{12}. \tag{4}$$

Since ω_{12} is a multiple of θ_3, we conclude that $h_{1121} = h_{1122} = 0$. Similarly, we can show that $h_{1131} = h_{1133} = 0$, and by polarization we have $h_{1123} = 0$ as well. Applying (4) once again now yields

$$2h_{122} - h_{111} = 0, \quad \text{or} \quad 2d = x.$$

It then follows from the first of the Gauss equations that $d^2 = \epsilon^2/4$. In particular, h_{111} and h_{122} are constant and hence $h_{1111} = 0$. We now know that $h_{11ij} = 0$, for all i and j. Similarly, we show that $2c = x$ and hence

$$h_{1jk} = \pm \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & (1/2)\epsilon & 0 \\ 0 & 0 & (1/2)\epsilon \end{pmatrix}.$$

In the cases where $(i,j,k) = (1,2,2)$, $(1,3,3)$ or $1,2,3)$ the Codazzi equations become

$$\begin{aligned} h_{1222}\theta_2 + h_{1223}\theta_3 &= (\epsilon/2)(h_{222}\theta_3 - h_{322}\theta_2), \\ h_{1332}\theta_2 + h_{1333}\theta_3 &= (\epsilon/2)(h_{233}\theta_3 - h_{333}\theta_2), \\ h_{1232}\theta_2 + h_{1233}\theta_3 &= (\epsilon/2)(h_{223}\theta_3 - h_{323}\theta_2) + (\dots)\theta_1, \end{aligned}$$

where we have used the facts that $\omega_{12} = (\epsilon/2)\theta_3$, $\omega_{31} = (\epsilon/2)\theta_2$ and $\omega_{23} = (\dots)\theta_1$. These equations imply that

$$h_{222} = -h_{323}, \qquad h_{333} = -h_{223},$$

or equivalently $y = -e$ and $b = -z$. It therefore follows from the third of the Gauss equations that

$$-y^2 - z^2 = 1 - (3/4)\epsilon^2 + y^2 + z^2 - cd \quad \Rightarrow \quad 1 - \epsilon^2 \leq 0.$$

In particular, we see there are no local solutions for $0 < \epsilon < 1$, or in other words, the Berger spheres do not possess isometric Lagrangian immersion in $\mathbb{C}^3$, even locally. On the other hand, in the constant curvature case ($\epsilon = 1$), it is known that there do exist local isometric Lagrangian immersions (see [9]). □

6. Flat Lagrangian submanifolds

Since h_{ijk} is symmetric in its three indices, the trilinear form

$$(x,y,z) \mapsto \langle \alpha(x,y), Jz \rangle$$

is symmetric in its three arguments. For $x \in T_pM$, define a linear transformation

$$A(x) \colon T_pM \to T_pM \quad \text{by} \quad A(x)(y) = J\alpha(x,y).$$

Then

$$\begin{aligned}\langle A(x)y, z\rangle &= \langle J\alpha(x,y)z\rangle = -\langle \alpha(x,y), Jz\rangle \\ &= -\langle \alpha(x,z), Jy\rangle = \langle y, J\alpha(x)z\rangle = \langle y, A(x)z\rangle,\end{aligned}$$

so $A(x)$ is symmetric with respect to the inner product $\langle\cdot,\cdot\rangle$. Moreover, if R denotes the Riemann-Christoffel curvature tensor of M^n, it follows from the Gauss equation

$$\langle R(x,y)z, w\rangle = \langle \alpha(y,z), \alpha(x,w)\rangle - \langle \alpha(x,z), \alpha(y,w)\rangle$$

that

$$\begin{aligned}\langle A(x)A(y)z, w\rangle &= \langle A(y)z, A(x)w\rangle = \langle J\alpha(y,z), J\alpha(x,w)\rangle \\ &= \langle \alpha(y,z), \alpha(x,w)\rangle = \langle \alpha(x,z), \alpha(y,w)\rangle + \langle R(x,y)z, w\rangle \\ &= \cdots = \langle A(y)A(x)z, w\rangle + \langle R(x,y)z, w\rangle,\end{aligned}$$

and hence we can reformulate the Gauss equations as

$$[A(x), A(y)] = R(x,y), \quad \text{for } x, y \in T_pM.$$

Thus in the special case in which M^n is flat, the linear transformations $A(x)$ will all commute with each other and we can prove:

Lemma. *If M^n is a flat Lagrangian submanifold of $\mathbb{C}^n$ and $p \in M^n$, there is an orthonormal basis $(e_1, \ldots, e_n)$ for T_pM such that if $e_{i*} = Je_i$, for $1 \leq i \leq n$, then*

$$\langle \alpha(e_i, e_j), e_{k*}\rangle = 0, \quad \textit{unless } i = j = k.$$

Proof. Since the linear transformations $A(x)$ commute, we can find an orthonormal basis $(e_1, \ldots, e_n)$ for T_pM which simultaneously diagonalizes them, so that

$$A(x)e_i = \lambda_i(x)e_i,$$

where each $\lambda_i \colon T_pM \to R$ is a linear functional. In particular, we find that

$$\langle J\alpha(x, e_i), e_j\rangle = \langle A(x)e_i, e_j\rangle = 0, \quad \text{unless } i = j.$$

It follows by trisymmetry that

$$\langle \alpha(e_i, e_j), e_{k*}\rangle = -\langle J\alpha(e_i, e_j), e_k\rangle = 0, \quad \text{unless } i = j = k,$$

which is exactly what we needed to prove. □

In terms of the differential forms described in Section 2, the lemma implies that

$$\omega_{i*j} = \begin{cases} u_i\theta_i, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

To prove the existence of flat Lagrangian submanifolds of $\mathbb{C}^n$, we will need to use a differential ideal which takes into account this special nature of the second fundamental form. We will choose an appropriate differential ideal on the manifold

$$W = F(M^n) \times F_{\mathbb{C}}(\mathbb{C}^n) \times \mathbb{R}^n,$$

where $F(M^n)$ is the bundle of orthonormal frames on M^n and $F_{\mathbb{C}}(\mathbb{C}^n)$ is the space of unitary frames on $\mathbb{C}^n$, whose integral submanifolds can represent the graphs of maps which represent isometric Lagrangian immersions together with adapted moving frames.

On the frame bundle $F(M^n)$, we have the familiar tautological one-forms $(\theta_1, \ldots, \theta_n)$ and connection one-forms $\omega_{ij} = -\omega_{ji}$ which satisfy the equations

$$d\theta_i = -\omega_{ij} \wedge \theta_j, \qquad d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj}, \tag{5}$$

the last of these holding because M^n is assumed to have curvature zero.

We pull these differential forms and the differential forms defined in Section 2 back to the product manifold W. Our ideal $\mathcal{A}$ on W will be generated by the differential one-forms

$$\widetilde{\theta}_i - \theta_i, \qquad \widetilde{\theta}_{i*}, \qquad \widetilde{\omega}_{ij} - \omega_{ij}, \qquad \widetilde{\omega}_{i*i} - u_i\theta_i, \qquad \widetilde{\omega}_{i*j} \quad \text{if } i \neq j,$$

where $(u_1, \ldots u_n)$ are the coordinates on the $\mathbb{R}^n$ factor in W, together with certain differential two-forms which will make $\mathcal{A}$ closed under d. It follows from the structure equations (1) and (5) that we can take the two-form generators to be

$$d(u_i\theta_i), \qquad \widetilde{\omega}_{i*i} \wedge \widetilde{\omega}_{ij} + \widetilde{\omega}_{ij} \wedge \widetilde{\omega}_{j*j}.$$

Modulo the one-form generators, we can replace these by

$$\omega_{ij} \wedge (u_i\theta_i - u_j\theta_j), \qquad du_i \wedge \theta_i - u_i \sum_j \omega_{ij} \wedge \theta_j. \tag{6}$$

As a final simplification, we substitute the first of these into the second, so that the two-form generators become

$$\omega_{ij} \wedge (u_i\theta_i - u_j\theta_j), \qquad \left(du_i - \sum_j \frac{u_i^2}{u_j}\omega_{ij} \right) \wedge \theta_i. \tag{7}$$

Once again, we claim that $(\mathcal{A}, \theta_1 \wedge \cdots \wedge \theta_n)$ is in involution. As before, an arbitrary point $q = E_q^0$ of W can be taken as a zero-dimensional integral element for $\mathcal{A}$. The polar space $H(E_q^0)$ of this integral element is the collection of vectors $v_1 \in T_qW$ which are annihilated by the one-form generators in $\mathcal{A}$. Once $\theta_i(v_1)$, $\omega_{ij}(v_1)$ and $du_i(v_1)$ are chosen, the one-form generators for $\mathcal{A}$ determine

$$\tilde{\theta}_I(v_1), \qquad \widetilde{\omega}_{ij}(v_1), \quad \text{and} \quad \widetilde{\omega}_{i*j}(v_1).$$

We have n degrees of freedom in choosing $\theta_i(v_1)$, an additional $(1/2)n(n-1)$ in choosing $\widetilde{\omega}_{ij}(v_1)$, and yet another n degrees of freedom in choosing $du_i(v_1)$, so the polar space $H(E_q^0)$ has constant dimension $2n + (1/2)n(n-1)$ and, as before, all zero-dimensional integral elements are regular.

Suppose now that E^1_q is a one-dimensional integral element containing E^0_q and generated by a nonzero vector $v_1 \in T_qW$. The polar space $H(E^1_q)$ is the set of vectors $v_2 \in T_qW$ which satisfy two sets of linear equations which come from the one-form generators of $\mathcal{A}$,

$$L_1\colon \quad \widetilde{\theta}_i(v_2) = \theta_i(v_2), \quad \widetilde{\theta}_{i*}(v_2) = 0, \quad \widetilde{\omega}_{i*j}(v_2) = \begin{cases} u_i\theta_i(v_2), & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

$$L_2\colon \quad \widetilde{\omega}_{ij}(v_2) = \omega_{ij}(v_2),$$

together with two sets of equations which come from the two-form generators (6),

$$L_3\colon \quad (u_i\theta_i(v_1) - u_j\theta_j(v_1))\omega_{ij}(v_2) = (u_i\theta_i(v_2) - u_j\theta_j(v_2))\omega_{ij}(v_1),$$

$$L_4\colon \quad \theta_i(v_1)du_i(v_2) = \theta_i(v_2)du_i(v_1) - u_i\sum_j \omega_{ij}\wedge\theta_j(v_1, v_2).$$

Once $\theta_i(v_2)$ are chosen, the linear system L_1 determines $\tilde{\theta}_I(v_2)$ and $\omega_{i*j}(v_2)$. If

$$u_i\theta_i(v_1) - u_j\theta_j(v_1) \neq 0,$$

then L_3 determines $\omega_{ij}(v_2)$, and L_2 determines $\widetilde{\omega}_{ij}(v_2)$. Finally, if $\theta_i(v_1) \neq 0$, the equations L_4 determine $du_i(v_2)$. In other words, once $\theta_i(v_2)$ is chosen, the entire basis

$$\theta_i(v_2), \quad \omega_{ij}(v_2), \quad \widetilde{\theta}_I(v_2), \quad \widetilde{\omega}_{IJ}(v_2), \quad du_i(v_2)$$

is completely determined, so long as the condition

$$u_i\theta_i(v_1) - u_j\theta_j(v_1) \neq 0, \qquad \theta_i(v_1) \neq 0 \tag{8}$$

is satisfied. Thus we see that E^1_q is a regular element exactly when (8) holds, and in this case $\dim H(E^1_q) = n$.

We claim that the polar space $H(E^1_q)$ is itself an integral element of dimension n on which, moreover, $\theta_1\wedge\cdots\wedge\theta_n$ is nonzero. Since the one-form generators of $\mathcal{A}$ automatically vanish on $H(E^1_q)$, we need only check that the two-form generators given by (7) vanish when evaluated on arbitrary elements $v_2, v_3 \in H(E^1_q)$. Alternatively, it suffices to show that there exist linear functionals, which we denote by $\omega_{ij}, du_i\colon H(E^1_q) \to \mathbb{R}$, which satisfy the equations

$$\omega_{ij}\wedge(u_i\theta_i - u_j\theta_j) = 0, \qquad \left(du_i - \sum_j \frac{u_i^2}{u_j}\omega_{ij}\right)\wedge\theta_i = 0,$$

and take the appropriate values on v_1. But such solutions do exist and are given by the explicit formulae

$$\omega_{ij} = a_{ij}(u_i\theta_i - u_j\theta_j), \qquad du_i = \sum_j \frac{u_i^2}{u_j}\omega_{ij} + b_i\theta_i,$$

where the a_{ij}'s and b_i's are determined so as to yield the correct values for $\omega_{ij}(v_1)$ and $du_i(v_1)$.

It follows immediately that a regular integral element E^1_q lies in a flag of integral elements

$$E^1_q \subset E^2_q \subset \cdots \subset E^{n-1}_q \subset E^n_q = H(E^1_q),$$

all of which have the same polar space $H(E^1_q)$. Moreover,

$$r_{k+1}(E^k_q) = \dim H(E^k_q) - (k+1) = \dim H(E^1_q) - (k+1) = n - k - 1,$$

and each of the integral elements is regular except for the last one which is ordinary. Since $\theta_1 \wedge \cdots \wedge \theta_n$ is nonzero on $H(E^1_q)$, we see that the differential system $\mathcal{A}$ is indeed in involution.

Proof of Theorem 3. Suppose that C is a real analytic curve in M^n which contains an arbitrary point $p \in M^n$ and choose submanifolds

$$C = F^1 \subset F^2 \subset \cdots \subset F^{n-1} \subset M^n$$

so that F^i has dimension i. Let $\widetilde{F}^i = \pi^{-1}(F^i)$, where $\pi\colon W \to M^n$ is the obvious projection. Given arbitrary functions $f_{ij} = -f_{ji}$ and g_i along C, we can construct an integral submanifold $\widehat{C} \subset W$ for $\mathcal{A}$ which projects to C via the projection $\pi\colon W \to M$ and satisfies the condition

$$\omega_{ij}(\hat{e}_1) = f_{ij}, \qquad du_i(\hat{e}_1) = g_i,$$

where $\hat{e}_1$ projects to e_1, just as we did in the case of Lagrangian surfaces. Moreover, we can choose $\widehat{C}$ so that

$$u_i\theta_i(\hat{e}_1) - u_j\theta_j(\hat{e}_1) \neq 0, \qquad \theta_i(\hat{e}_1) \neq 0$$

along $\widehat{C}$, where $\hat{e}_1$ projects to the unit speed tangent e_1 to C, which will ensure that $\widehat{C}$ is a regular integral submanifold. Then the Cartan-Kähler theorem guarantees the existence of a two-dimensional integral submanifold $N^2 \cap \widetilde{F}^2$ of W, unique up to extension, which contains $\widehat{C}$. If $2 < n$, this integral submanifold will be regular, so a second application of the Cartan-Kähler theorem yields a three-dimensional integral submanifold $N^3 \cap \widetilde{F}^3$ of W, unique up to extension, which contains $N^2 \cap \widetilde{F}^2$. Continuing in this manner, we finally obtain an n-dimensional integral submanifold N^n of W. Since $\theta_1 \wedge \cdots \wedge \theta_n$ is nonzero along N^n, the inverse function theorem implies that the projection $\pi|N\colon N \to M$ possesses an inverse map $\sigma\colon U \to N$ from an open neighborhood U of p in M^n. The composition of σ with the obvious projection $W \to \mathbb{C}^n$ yields the desired isometric Lagrangian immersion from U into $\mathbb{C}^n$. The immersions so constructed depend upon the functions $f_{ij} = -f_{ji}$ and g_i which can be completely arbitrary, except for the requirement that the one-dimensional integral elements they determine be regular. Thus the local isometric Lagrangian immersions depend upon $(1/2)n(n+1)$ functions of a single variable, as claimed.

References

[1] E. Berger, R. Bryant, and P. Griffiths, *The Gauss equations and rigidity of isometric embeddings*, Duke Math. J. **50** (1983), 803–892.

[2] R. Bryant, S. S. Chern, R. Gardner, H. Goldschmidt, and P. Griffiths, *Exterior differential systems*, Springer-Verlag, New York, 1991.

[3] É. Cartan, *Sur les variétés de courbure constante d'un espace euclidien ou non-euclidien*, Bull. Soc. Math. France **47** (1919), 125–160; ibid. **48** (1920), 132–208.

[4] ———, *Sur la possibilité de plonger un espace riemannien donné dans un espace euclidien*, Ann. Soc. Polon. Math. **6** (1927), 1–7.

[5] ———, *Les systemes différentiels extérieurs et leurs applications géométriques*, Hermann, Paris, 1945.

[6] I. Castro and C. Urbano, *Lagrangian surfaces in the complex Euclidean plane with conformal Maslov form*, Tohoku Math. J. **45** (1993), 565–582.

[7] J. Cheeger and D. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland, Amsterdam, 1975.

[8] B. Y. Chen, *A Riemannian invariant for submanifolds in space forms and its applications*, Geometry and topology of submanifolds, VI (Leuven, 1993), World Scientific Publishing, River Edge, NJ, 1994, pp. 58–81,

[9] ———, *Complex extensors of Lagrangian submanifolds in complex euclidean spaces*, Tohoku Math J. (2) **49** (1997), 277–297.

[10] B. Y. Chen, F. Dillen, L. Verstraelen, and L. Vranken, *Lagrangian isometric immersions of a real-space-form $M^n(c)$ into a complex-space-form $\tilde{M}^n(4c)$*, Proc. Camb. Phil. Soc. **124** (1998), 107–125.

[11] B. Y. Chen and C. S. Houh, *On totally real flat surfaces*, Boll. Un. Mat. Ital. A (5) **15** (1978), 370–378.

[12] S. S. Chern and J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. **99** (1974), 48–69.

[13] D. Cox, J. Little, and D. O'Shea, *Ideals, varieties and algorithms*, Springer-Verlag, New York, 1998.

[14] E. Kähler, *Einführung in die Theorie der Systeme von Differentialgleichungen*, Chelsea, New York, 1949.

[15] J. D. Moore, *Isometric immersions of space forms in space forms*, Pacific J. Math. **40** (1972), 157–166.

[16] J. D. Moore and J. H. White, *Normal characteristic forms: conformal invariance and duality*, Math. Z. **164** (1978), 125–141.

[17] J. M. Morvan, *Classe de Maslov d'une immersion lagrangienne et minimalité*, C. R. Acad. Sci. Paris **292** (1981), 633–636.

[18] U. Pinkall, *Hopf tori in S^3*, Invent. Math. **81** (1985), 379–386.

[19] J. Weiner, *Flat tori in the 3-sphere*, Proc. London Math. Soc. **62** (1991), 54–76.

John Douglas Moore, Department of Mathematics, University of California, Santa Barbara, CA 93106, USA

E-mail address: moore@math.ucsb.edu

Jean-Marie Morvan, Université Claude Bernard Lyon 1, U.F.R. de Mathématiques, U.M.R. 5028, Institut Girard Desargues, 43, Bd. du 11 Novembre 1918, 69622 Villeurbanne CEDEX, France

E-mail address: morvan@desargues.univ-lyon1.fr