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SOME TOPOLOGICAL PROPERTIES OF TOTALLY QUASI-UMBILICAL SUBMANIFOLDS

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We study the topology of a totally quasi-umbilical submanifold isometrically immersed in the Euclidean space. Using Morse theory, we give some restrictions on the Betti numbers of the submanifold . Then, we introduce two canonical normal vector fields, and we give some relations between these two fields and the topology of the submanifold.

1. INTRODUCTION

In " A Survey on Quasi-umbilical Submanifolds " [2] B.Y. Chen and L. Verstraelen summarize the main geometric results on quasi-umbilical submanifolds. On an other hand, J.D. Moore and the author proved recently the following theorem : " Every conformally flat submanifold M^n, $n \geq 7$ of codimension 4 in the Euclidean space is totally quasi-umbilical " [6]. In [7], L. Verstraelen has studied totally quasi-umbilical submanifolds from the geometric point of view. In this paper, using Morse theory [4] and the theory of Bochner-Lichnerowicz [3], we study the totally quasi-umbilical submanifolds from the topological point of view.

2. TOTALLY QUASI-UMBILICAL SUBMANIFOLDS

Let $i : M^n \mapsto E^{n+p}$ be an isometric immersion of a n-dimensional manifold M^n into the Euclidean space E^{n+p}. Let $< \, , >$ be the metric on M^n , $T M^n$ be the tangent bundle of M^n, ∇ the Levi-Civita connection on M^n. We denote by $T^{\perp} M^n$ the normal bundle, and by $\nabla^{\perp}$ the normal connection on $T^{\perp} M^n$.
$\sigma : T M^n \times T M^n \mapsto T^{\perp} M^n$ denotes the second fondamental form, and A_ξ is defined by :

3. BETTI-NUMBERS OF A TOTALLY QUASI-UMBILICAL SUBMANIFOLD

We shall prove the following:

THEOREM 1

Let $i : M^n \to \mathbb{E}^{n+P}$ be an isometric immersion of a n dimensional compact Riemannian manifold M^n in the Euclidean space $\mathbb{E}^{n+p}$.
Assume that :

a) i is totally quasi-umbilical ;

b) the first principal normal space E_1 is of dimension $\leq r$ at every point $m \in M^n$.

Then, 1) M^n possesses a C W decomposition with no cells of dimension k, where $r < k < n-r$

2) If $n-p \geq 2r+1$, M^n is a boundary of a compact manifold.

PROOF OF THE THEOREM.

1) According to remark ii, the second fondamental form σ has the following expression :

(3)' $\sigma(X, Y)_m = \sum_{i=1}^{r} (\alpha^i < X, Y > + \gamma^i < X, T_i > < Y, T_i >) \xi_i$,

where $\xi_1, \ldots, \xi_p$ are orthonormal vectors, $T_1 \ldots T_r \in T_m M^n$, $\xi_1 \ldots \xi_p \in T_m^{\perp} M^n$. Let $\vec{x}$ be the position vector of M^n. It is well known that, for almost every $a \in \mathbb{E}^{n+p}$, the height function $h_a = < \vec{x}, a >$ is a Morse function. Moreover,

$(d^2 h_a(X, Y))_m = < \sigma(X, Y), a >_m \quad \forall X, Y \in T M^n$, if m is a critical point of h_a .

Let $(e_1, \ldots, e_n)$ be a frame of $T_m M^n$ such that, if $j > r$, $e_j \in \{T_i , 1 \leq i \leq r\}^{\perp}$. Using (3)' it is easy to see that $d^2 {h_a}_m$ has the following expression :

$$d^2 {h_a}_m = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & \begin{matrix} c & & \\ & \ddots & \\ & & c \end{matrix} \end{array} \right) \begin{matrix} \}\, r \\ \}\, n-r \end{matrix}$$

where $c \in \mathbb{R}$.

Since $3r < n$, we have :

$$\forall\, i, j \in \{1, \ldots, r\},\ w_i\, w_j \in H^{k_{ij}}(M^n, \mathbb{Z}/2)$$

where $k_{ij} \in \{r+1, \ldots, n-r-1\}$.

Since $H^k(M^n, \mathbb{Z}/2) = 0$ if $k \in \{r+1, \ldots, n-r-1\}$, $w_i\, w_j = 0$, $\forall\, i, j \in \{1, \ldots, r\}$.

Then, it is easy to conclude that $N_{k_1 \ldots k_n} = 0$.

Thus, in every case, $N_{k_1 \ldots k_n} = 0$. All the Stiefel-Whitney numbers are null. Applying a theorem of Thom ([5]) we conclude that M^n is a boundary of a compact manifold. 2) is proved.

CANONICAL NORMAL VECTOR FIELDS ASSOCIATED TO A TOTALLY QUASI-UMBILICAL SUBMANIFOLD

In this paragraph, we shall define two global normal vector fields on a totally quasi-umbilical submanifold, which will give topological properties of the submanifold. We need the following

LEMMA 1

Let $i : M^n \to \tilde{M}^{n+p}$ an isometric immersion of M^n into a Riemannian manifold $\tilde{M}^{n+p}$, $p \leqslant n-2$. We assume that i is totally quasi-umbilical, and that, at $m \in M^n$, the second fundamental form σ has the expressions :

$$(5)\quad \sigma_m(X, Y) = \sum_{i=1}^{r} (\alpha^i < X, Y > + \gamma^i < X, T_i > < Y, T_i >)\, \xi_i$$

$$(6)\quad \sigma_m(X, Y) = \sum_{i=1}^{r} (\alpha'^i < X, Y > + \gamma'^i < X, T'_i > < Y, T'_i >)\, \xi'_i$$

where $\{\xi_1, \ldots, \xi_r\}$ are orthonormal, $\{\alpha^i, \gamma^i\} \in \mathbb{R}$, $\{T_1, \ldots, T_r\}$ are unit tangent vector at m, $\{\xi'_1, \ldots, \xi'_r\}$ are orthonormal, $\{\alpha'^i, j'^i\} \in \mathbb{R}$, $\{T'_1 \ldots T'_r\}$ are unit tangent vectors at m. Then :

a) $\sum_{i=1}^{r} \alpha^i\, \xi_i = \sum_{i=1}^{r} \alpha'^i\, \xi'_i$,

b) If $\{T_1 \ldots T_r\}$ are orthonormal, and $\{T'_1 \ldots T'_r\}$ are orthonormal, $\sum_{i=1}^{r} \gamma^i\, \xi_i = \sum_{i=1}^{r} \gamma'^i\, \xi'_i$.

Let $\{i_1 \dots i_k\}$ the indices such that

$\gamma'^{i_1} \neq 0 \dots \gamma'^{i_k} \neq 0$ $\{T'^{i_1} \dots T'^{i_k}\} \subset [T_1 \dots T_r]$.

Then, $\sum_{j=1}^{r} < T'_{i_\ell} , T_j >^2 = 1$ if $i_\ell \in \{i_1 \dots i_k\}$, since $\{T_1 \dots T_r\}$ are orthonormal. The other indices ℓ satisfy $\gamma'^{i_\ell} = 0$. Then, it follows that

$$\sum_{i=1}^{r} \gamma^j \xi_j = \sum_{i=1}^{r} \gamma'^i \xi'_i .$$

b) is proved.

Consequently, on every totally quasi-umbilical submanifold we can consider the global normal vector field $A = \sum_{i=1}^{r} \alpha^i \xi_i$.

If the normal connection is flat, we can also consider the global normal vector field $B = \sum_{i=1}^{r} \gamma^i \xi_i$ (with the notations of Lemma 1).

We can give the

DEFINITION 2

Let $i : M^n \mapsto E^{n+p}$ a totally quasi-umbilical isometric immersion $p \leq n-2$, such that the second fundamental form has the following expression:

$$\sigma(X, Y) = \sum_{i=1}^{r} (\alpha^i < X, Y > + \gamma^i < X, T_i > < Y, T_i >) \xi_i ,$$

where $\xi_1 \dots \xi_r$ are orthonormal.

$A = \sum_{i=1}^{r} \alpha^i \xi_i$ is called " the first canonical normal field associated to i ". If the normal connection is flat, $B = \sum_{j=1}^{r} \gamma^i \xi_i$ is called " the second canonical normal field associated to i ".

$$K(X, Y) = \sum_{i=1}^{r} [\alpha^i < X, X > + \gamma^i < X, T_i >^2]$$

$$[\alpha^i < Y, Y > + \gamma^i < Y, T_i >^2]$$

$$- \sum_{i=1}^{r} [\alpha^i < X, Y > + \gamma^i < X, T_i > < Y, T_i >]$$

$$[\alpha^i < X, Y > + \gamma^i < X, T_i > < Y, T_i >]$$

$$= \sum_{i=1}^{r} [\alpha^{i^2} + \alpha^i \gamma^i(< X, T_i >^2 + < Y, T_i >^2)]$$

$$= \|A\|^2 + \sum_{i=1}^{r} \alpha^i \gamma^i(< X, T_i >^2 + < Y, T_i >^2) .$$

b) Let $(e_1 \ldots e_n)$ be a local frame on $T M^n$, such that $e_1 = T_1 \ldots e_r = T_r$.

From (1)' we deduce $< R(e_j , X) X, e_j >$

$$= \sum_{i=1}^{r} (\alpha^{i^2} < X, X > < e_j , e_j > + \alpha^i \gamma^i < X, X > < e_j , T_i >^2$$

$$+ \alpha^i \gamma^i < e_j , e_j > < X, T_i >^2)$$

$$- \sum_{i=1}^{r} (\alpha^{i^2} < X, e_j >^2 + 2 \alpha^i \gamma^i < X, T_i > < e_j , T_i > < X, e_j >).$$

Then $\mathrm{Ricc}(X, X) = \sum_{i=1}^{r} (n \alpha_i^2 + \alpha^i \gamma^i + n \alpha^i \gamma^i < X, T_i >^2)$

$$- \sum_{i=1}^{r} (\alpha_i^2 + 2 \alpha^i \gamma^i < X, T_i >^2)$$

$$= (n -1) \sum_{i=1}^{r} \alpha_i^2 + (n -2) \sum_{i=1}^{r} \alpha^i \gamma^i < X, T_i >^2$$

$$+ \sum_{i=1}^{r} \alpha^i \gamma^i .$$

Consequently,

Then, M^n cannot be isometrically immersed in $\mathbb{E}^{n+p}$, $p \leq n-2$, as a totally quasi-umbilical submanifold, with flat normal connection.

COROLLARY 3

Let M^n a conformally flat manifold of dimension $n \geq 7$ which satisfies

$$(n-1)\,\mathrm{Ricc}(X, X) - R > 0$$

for every unit vector X of $T\,M^n$

Then, M^n cannot be isometrically immersed as a submanifold of $\mathbb{E}^{n+4}$, with flat normal connection.

COROLLARY 4

Let $i : M^n \mapsto \mathbb{E}^{n+p}$, $p \leq n-2$ be a totally quasi-umbilical isometric immersion with flat normal connection. If the two canonical normal fields associated to i are orthogonal then the scalar curvature of M^n is non negative.

PROOF OF THE COROLLARIES

1) If $p \leq n-2$, it is always possible to choose two orthonormal vectors X, Y in $\{T_1 \dots T_r\}^{\perp}$. Applying a) we find $K(X, Y) = \|A\|^2 \geq 0$. Then the sectionnal curvature of M^n cannot be negative.

2) If there exists an isometric immersion of M^n into $\mathbb{E}^{n+p}$, which is totally quasi-umbilical, and has a flat normal connection, then the equations (10) and (11) are satisfied. We deduce immediately that $(n-1)\,\mathrm{Ricc}(X, X) - R = (1-n)\,\|A\|^2 \leq 0$ if $X \in \{T_1 \dots T_r\}^{\perp}$. Since $p < n$, $\{T_1 \dots T_r\}^{\perp} \neq \{0\}$. Corollary 2 is proved.

3) Corollary 3 is a direct consequence of Corollary 2 and the result of [6].

4) Corollary 4 is a direct consequence of c.

REMARK

Let $i : M^n \mapsto \mathbb{E}^{n+p}$, $p \leq n-2$, be an isometric immersion of a compact orientable manifold. We assume that i is totally

$$(13)\quad \frac{n-2k}{n-2}\,\mathrm{Ricc}(X,Y) + k\,!\,\frac{k-1}{(n-1)(n-2)}\,R<X,Y>$$

is positive definite, then $\beta_k = 0$.

From proposition 1, we deduce that (13) is equivalent to

$$(14)\quad \begin{cases} \frac{n-2k}{n-2}\,[(n-1)\,\|A\|^2 + <A,B> + (n-2)\,\alpha^j\,\gamma^j] \\ \qquad\qquad + k\,!\,\frac{(k-1)}{(n-2)}\,[n\,\|A\|^2 + <A,B>] > 0, \\ \frac{n-2k}{n-2}\,[(n-1)\,\|A\|^2 + <A,B>] + k\,!\,\frac{(k-1)}{(n-2)} \\ \qquad\qquad [n\,\|A\|^2 + <A,B>] > 0 \end{cases}$$

$\forall\, j \in \{1, \ldots, r\}$ with the evident notations.

We can remark that (14) is satisfied if we have :

$$(15)\quad (n-2k)\,[(n-1)\,\|A\|^2 + <A,B> + (n-2)\,\|A\|\,\|B\|\,] + k\,!\,(k-1)[n\,\|A\|^2 + <A,B>] > 0$$

as soon as $n-2k > 0$.

Suppose that $R > 0$. Then (15) $\Longleftrightarrow$

$$(n-2k)\,[1 + \frac{-\|A\|^2 + (n-2)\,\|A\|\,\|B\|}{R/n-1}] + k\,!\,(k-1) > 0$$

$$\Longleftrightarrow (16)\quad [\frac{\|A\|^2 - (n-2)\,\|A\|\,\|B\|}{R/n-1}] > \frac{k\,!\,(k-1)}{n-2k} + 1 \;.$$

It is clear that if (16) is satisfied with $k = n-r$, then (16) is satisfied with every $k > n-r$.

Then, (16) $\Longleftrightarrow$

$$(17)\quad [\frac{\|A\|^2 - (n-2)\,\|A\|\,\|B\|}{R/n-1}] > \frac{q\,!\,(q-1)}{n-2q} + 1,\; q = n-r \;.$$

Clearly (17) $\Longleftrightarrow$

$$(18)\quad (1+ns)\,\|A\|^2 - (n-2)\,\|A\|\,\|B\| + (n-1)\,s<A,B> > 0$$

7. L. VERSTRAELEN, Foliations of quasi-umbilical submanifolds, Simon Stevin, 51 (II), 1977, p.65-69.

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