Smooth Surface And Triangular Mesh : Comparison Of The Area, The Normals And The Unfolding

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ABSTRACT

Replacing a smooth surface with a triangular mesh (i.e., a polyedron) "close to it" leads to some errors. The geometric properties of the triangular mesh can be very different from the geometric properties of the smooth surface, even if both surfaces are very close from one another. In this paper, we give examples of "developable" triangular meshes (the discrete Gaussian curvature is equal to 0 at each interior vertex) inscribed in a sphere (whose Gaussian curvature is equal to 1 at every point).

However, if we make assumptions on the geometry of the triangular mesh, on the curvature of the smooth surface and on the Hausdorff distance between both surfaces, we get an estimate of several properties of the smooth surface in terms of the properties of the triangular mesh. In particular, we give explicit approximations of the normals and of the area of the smooth surface. Furthermore, if we suppose that the smooth surface is developable (i.e., "isometric" to a surface of the plane), we give an explicit approximation of the "unfolding" of this surface. Just notice that in some of our approximations, we do not suppose that the vertices of the triangular mesh belong to the smooth surface.

Oddly, the upper bounds on the errors are better when triangles are right-angled (even if there are small angles): we do not need every angle of the triangular mesh to be quite large. We just need each triangle of the triangular mesh to contain at least one angle whose sine is large enough. Besides, approximations are better if the triangles of the triangular mesh are quite small where the smooth surface has a large curvature. Some proofs will be omitted.

Categories and Subject Descriptors

G.1.2 [**Numerical Analysis**]: Approximation—*Approximation of surfaces and contours*

General Terms

THEORY

Keywords

Local feature size, Medial axis, Unfolding, Triangular mesh, Computational geometry, Robustness of geometric computations

1. INTRODUCTION

We are interested in the relationship between a smooth surface and another surface "close to it". We wonder whether we can approximate several properties of a smooth surface (which is not supposed to be known everywhere) with properties of a known-surface which approximates it. In particular, we want to approximate the normals, the area and the curvatures of a smooth surface. Furthermore, if we suppose that the smooth surface can be unfolded (without changing its metric, that is to say by preserving lengths), we want to approximate the unfolding of the smooth surface. This problem appears in many applications, such as in geophysics, where people want to unfold strata.

First note that in general we have no results of convergence. The example of the *"lantern" of Schwarz* ([3]) convinces us easily: we can build a sequence of *"lanterns" of Schwarz* which converges to a finite cylinder (in the Hausdorff sense), but whose area tends to infinity.

Another example concerns the Gaussian curvature and "developable" surfaces: we can build triangular meshes whose discrete Gaussian curvature is equal to 0 at every vertex, and whose vertices belong to a sphere (the Gaussian curvature of the unit sphere is constant and equal to 1).

That is why, without other assumptions, we cannot expect the area (or the total Gauss curvature) of a sequence of triangular meshes to converge to the area (or the total Gauss curvature) of a smooth surface. However, under additional assumptions, J. Fu ([10]) proved results of convergence of the area, of the normals and of the curvatures of a sequence of triangular meshes converging to a surface. In [1], N. Amenta and M. Bern construct an explicit triangular mesh and get a result of convergence of the normals. Our point of view is different: we do not consider the problem of reconstructing a surface. We suppose that a triangular mesh is inscribed in a smooth surface (we do not care about the construction of the triangular mesh) and we get explicit approximations of the smooth surface in terms of geometric data.

Our point of view is different from the one of B. Hamman [12] too: he defines the Gaussian curvature of a triangular mesh thanks to quadratic polynomials. Our definitions (of the curvatures) are linked to the notions of the *normal cycle* ([10]) and of the *reach of a smooth surface*. The *reach* was first introduced by H. Federer [9]. It is interesting to notice that it is in fact linked to the (more recent) notions of *medial axis* and *local feature size*, which are used in some problems of reconstructing a surface from scattered sample points (see [1], [2] or [4]). In [21], F.E. Wolter gives interesting results related to the relationship between *medial axis*, *cut locus* and the *reach*.

Roughly speaking, we evaluate the approximations (of the normals, the area and the unfolding) in terms of the geometry of the triangular mesh, the local curvature of the smooth surface and its reach. We can be more precise : surprisingly, the approximations of the smooth surface do not depend on the *fatness* of the triangular mesh (the results of J. Fu depend on it) but on its *straightness* (see the definition below).

This paper is organized as follows. Section 2 gives classical and usual definitions. Section 3 states a result of convergence of the normals. Section 4 states results of convergence of the area. Section 5 gives some examples which concern the Gaussian curvature. Section 6 deals with "developable" surfaces and gives an approximation of the unfolding of a surface. Section 7 gives the proofs of section 3 and 4. The proofs of the results of sections 5 and 6 can be found in [16] and [20] (or in preprints [18] and [19]).

2. DEFINITIONS

We recall here some classical definitions which concern smooth surfaces, triangular meshes and the relative position of two surfaces. For more details on smooth surfaces, one may refer to [3], [8] or [17]. For more details on triangular meshes, one may refer to [9],[10] or [14].

2.1 Smooth surfaces

- In the following, a smooth surface means a $\mathcal{C}^2$ surface which is regular, oriented, compact with or without boundary. Let S be a smooth surface of the (oriented) Euclidean space $\mathbb{R}^3$. Let ∂S denote the boundary of S. S is endowed with the Riemannian structure induced by the standard scalar product of $\mathbb{R}^3$. We denote by da the area form on S and by ds the canonical orientation of ∂S. Let ν be the unit normal vector field (compatible with the orientation of S) and h be the second fundamental form of S associated with ν. Its determinant at a point p of S is *the Gaussian curvature* G_p, its trace is *the mean curvature* H_p. *The maximal curvature of* S *at* p is $\rho_p = max(|\lambda_p^1|, |\lambda_p^2|)$, where λ_p^1 and λ_p^2 are the eigenvalues of the second fundamental form at p. *The maximal curvature of* S is
$$\rho_S = \sup_{p \in S} \rho_p.$$
We denote by k_p *the geodesic curvature* of ∂S at p.

- We need the following

PROPOSITION 1. *Let S be a smooth compact surface of $\mathbb{R}^3$. Then there exists an open set U_S of $\mathbb{R}^3$ containing S and a continuous map ξ from U_S onto S satisfying the following: if p belongs to U_S, then there exists a unique point $\xi(p)$ of S which is the nearest point of S to p (ξ is nothing but the orthogonal projection onto S).*

A proof of this proposition can be found in [9] (sections 4-8 and 4-12).
The open set U_S depends locally and globally on the smooth surface S. Locally, the normals of S do not intersect in U_S. Globally, U_S depends on points which can be far from one another on the surface, but close in $\mathbb{R}^3$.
We shall also need the notion of the *reach of a surface*, introduced by H. Federer in [9].

DEFINITION 1. *The reach of a surface S is the largest $r > 0$ for which ξ is defined on the open tubular neighborhood of radius r of S.*

Note that the reach r_S of S is smaller than the minimal radius of curvature of S (which is the inverse of the maximal absolute value of the principal curvatures of S). One may refer to [13] or [21]. Thus, we have:
$$\rho_S \; r_S \leq 1,$$
where ρ_S is the maximal curvature of S.

2.2 Triangular meshes

A triangular mesh T is a (finite and connected) union of triangles of $\mathbb{R}^3$, such that the intersection of two triangles is either empty, or equal to a vertex, or equal to an edge.
We denote by $\mathcal{T}_T$ the set of triangles of T and by Δ a generic triangle of T.

- η_Δ denotes the length of the longest edge of Δ, and $\mathcal{A}(\Delta)$ the area of Δ.
- The *fatness* of Δ is the real number
$$\theta(\Delta) = \frac{\mathcal{A}(\Delta)}{{\eta_\Delta}^2}.$$
- The *straightness* of a triangle Δ is the real number
$$\mathrm{str}(\Delta) = \sup_{p \text{ vertex of } \Delta} |\sin(\theta_p)|,$$
where θ_p is the angle at p of Δ.

REMARK 1. In particular, if β is any of the three angles of the triangle Δ, we have:
$$2\theta(\Delta) \leq |\sin\beta| \leq \mathrm{str}(\Delta).$$

We can now define:

- The *area* $\mathcal{A}(T)$ is the sum of the areas of all the triangles of T.
- The *fatness* of T is:
$$\theta(T) = \min_{\Delta \in \mathcal{T}_T} \theta(\Delta).$$
- The *straightness* of T is:
$$\mathrm{str}(T) = \min_{\Delta \in \mathcal{T}_T} \mathrm{str}(\Delta).$$

2.3 Surface close to a smooth surface

Let M denote a surface which is differentiable almost everywhere.

- M is *closely near* a smooth surface S if:
  1. M lies in the tubular neighborhood of radius r of S, where r is the reach of S,
  2. the restriction of ξ to M is one-to-one.
- We say that a triangular mesh of $\mathbb{R}^3$ is *inscribed* in a smooth surface S if all its vertices belong to S.
- A triangular mesh T is *closely inscribed* in a smooth surface S if:
  1. T is closely near S,
  2. all the vertices of T belong to S.
- One can define at almost every point m of M the angle $\alpha_m \in \left[0, \frac{\pi}{2}\right]$ between the normals ν_m^M and $\nu_{\xi(m)}^S$. We put

  $$\alpha_{\max} = \sup_{m \in M} \alpha_m \quad \text{and} \quad \alpha_{\min} = \inf_{m \in M} \alpha_m.$$

  $\alpha_{\max}$ is called *the maximal angle between the normals of S and M.*
- Let M be closely near a smooth surface S. *The relative curvature of S to M* is the real defined by:

  $$\omega_S(M) = \sup_{m \in M \setminus \partial M} \|\xi(m) - m\| \ \rho_{\xi(m)}.$$
- Let T be a triangular mesh closely inscribed in a smooth surface S. *The relative height of T to S* is the real defined by:

  $$\pi_S(T) = \sup_{\Delta \in \mathcal{T}_T} \sup_{m \in \Delta} \eta_\Delta \ \rho_{\xi(m)}.$$
- The *Hausdorff distance* between M and S is given by:

  $$\delta_{\text{Hauss}}(M, S) = max(\sup_{x \in M} d(x, S), \sup_{y \in S} d(M, y)).$$

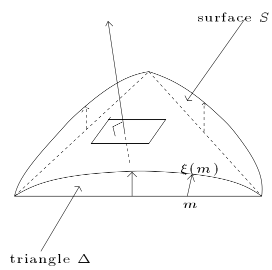


Figure 1: A triangle Δ closely near S : there is a one-to-one map which is linked to the normals of S

REMARK 2.
If M is compact and closely near a smooth surface S, we get:

$$\omega_S(M) < 1.$$

A triangular mesh T closely inscribed in a smooth surface S satisfies:

$$\omega_S(T) \leq \pi_S(T).$$

3. APPROXIMATION OF THE NORMALS

Theorem 1 gives an approximation of the normals of a smooth surface with the normals of a triangular mesh close enough to it. The upper bound depends locally on the relative curvature $\omega_S(T)$, on the relative height $\pi_S(T)$ and on the straightness $\text{str}(T)$ of T.

THEOREM 1. *Let S be a smooth surface and T a triangular mesh closely inscribed in S. Then the maximal angle $\alpha_{\max}$ between the normals of S and T satisfies*

$$\sin \alpha_{\max} \leq \left(\frac{\sqrt{10}}{2 \ str(T)} + 1 \right) \ \frac{\pi_S(T)}{1 - \omega_S(T)}.$$

COROLLARY 1. *Let S be a smooth surface and T a triangular mesh closely inscribed in S.*

$$\textit{If} \quad \pi_S(T) \leq \frac{1}{2},$$

then the maximal angle $\alpha_{\max}$ between the normals of S and T satisfies

$$\sin \alpha_{\max} \leq \left(\frac{4}{str(T)} + 2 \right) \ \pi_S(T).$$

Oddly, the upper bound on the error on the normals is better when the straightness is large. It means that we do not need each angle of the triangular mesh to be large. We just need one angle in each triangle of the triangular mesh to be large. For us, a right-angled triangle is a "good triangle", even if it is very thin (the sine of one angle is small).

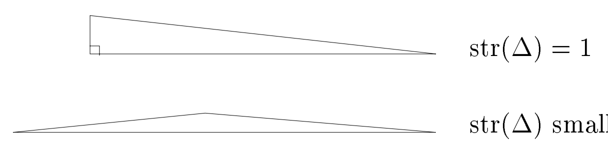


Figure 2: The straightness of a triangle Δ

Furthermore, the upper bound is better if the relative height of T to S is small, that is to say if the triangles of T are small where the curvature of S is large. We directly get

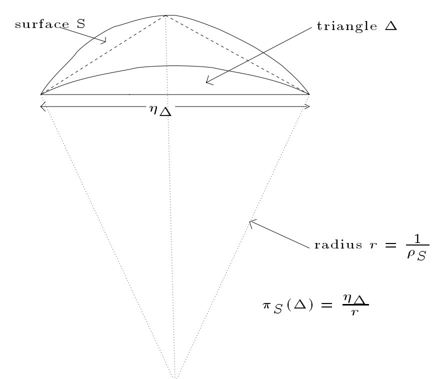


Figure 3: The relative height of a triangle Δ to S in the particular case in which S is a sphere of radius r. $\pi_S(\Delta)$ is small if the diameter η_Δ is small compared to r.

a result of convergence for a sequence of triangular meshes.

COROLLARY 2. *Let S be a smooth surface and $(T_n)_{n\geq 0}$ be a sequence of triangular meshes closely inscribed in S such that*

1. *the straightness $str(T_n)$ is uniformly bounded from below by a strictly positive constant,*
2. *the lengths of the edges of T_n tend to zero when n tends to infinity,*

then the normals of $(T_n)_{n\geq 0}$ tend to the normals of S.

The approximation of the normals of a smooth surface S with the normals of a triangular mesh T close to it is "good" if :
• the vertices of T belong to S,
• there is an homeomorphism between T and S (thanks to ξ),
• the straightness str(T) is large,
• the relative height $\pi_S(T)$ is small.

4. APPROXIMATION OF THE AREA

In this section, we are interested in the approximation of the area of a smooth surface by the area of a surface differentiable almost everywhere and close enough. We first consider the example of the half "lantern" of Schwarz. Then we give results of approximation.

4.1 Half "lantern" of Schwarz

The well known *"lantern" of Schwarz phenomenon* ([3]) shows that we can build a sequence of triangular meshes inscribed in a fixed smooth surface, which converges in the Hausdorff sense to the smooth surface and such that the area tends to infinity.

Let us describe succinctly a *half "lantern" of Schwarz*. Let C be a half cylinder of finite height H and of radius R in $\mathbb{R}^3$. For every positive integers n and N, let $P(n, N)$ denote the triangular mesh whose vertices $v_{i,j}$ (with $i \in \{0, ..n-1\}$ and $j \in \{0, ..N\}$) belong to C and are defined as follows:

$$v_{i,j} = (R\ \cos(i\alpha), R\ \sin(i\alpha), jh) \text{ if } j \text{ is even,}$$
$$v_{i,j} = (R\ \cos(i\alpha + \tfrac{\alpha}{2}), R\ \sin(i\alpha + \tfrac{\alpha}{2}), jh) \text{ if } j \text{ is odd,}$$

and whose faces are:

$$v_{i,j}\ v_{i+1,j}\ v_{i,j+1},$$
$$v_{i,j}\ v_{i-1,j+1}\ v_{i,j+1},$$

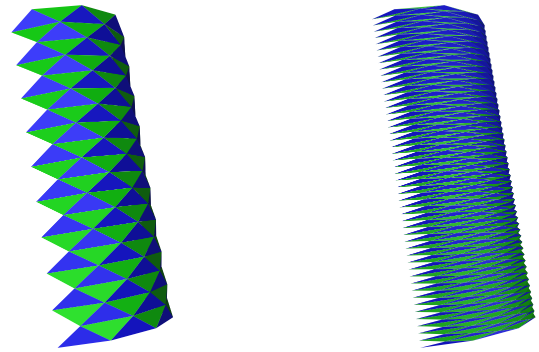
where $\alpha = \frac{\pi}{n}$ and $h = \frac{H}{N}$.

Then, when n tends to infinity, the area $\mathcal{A}(P(n, n^3))$ of $P(n, n^3)$ tends to infinity (although $P(n, n^3)$ tends to the half-cylinder in the Hausdorff sense).

That is why, without other assumptions, we cannot expect results of convergence of the area of a sequence of triangular meshes closely inscribed in a smooth surface.

4.2 Result of approximation

The following result shows that if a surface M is closely near S and has enough regularity to have a tangent plane almost everywhere, then the area of S is bounded from above and from below by quantities depending on **the area of M, the Hausdorff distance between M and S, the curvature of S, and the angle between the normals**.



(a) P(5-19) (b) P(5-99)

Figure 4: Examples of *Half "lantern" of Schwarz*

THEOREM 2. *Let S be a (compact orientable) C^2 surface in $\mathbb{R}^3$. Let M be a surface which is differentiable almost everywhere, and closely near S. Then the area $\mathcal{A}(S)$ satisfies:*

$$\mathcal{A}(S) = \int_M \frac{cos\ \alpha_m}{1 + \delta_m \epsilon_m H_{\xi(m)} + \delta_m^2 G_{\xi(m)}} da_M(m),$$

where $\epsilon_m =< \nu_{\xi(m)}, \frac{\xi(m)-m}{||\xi(m)-m||} >\in \{-1, +1\}$, α_m is the angle between the normals at the points m and $\xi(m)$, $\delta_m = \|\xi(m) - m\|$, da_M is the area form of M, $H_{\xi(m)}$ and $G_{\xi(m)}$ are the mean and the Gaussian curvature of S at the point $\xi(m)$.

In fact, Theorem 2 is a consequence of the following important lemma :

LEMMA 1. *Let S be a smooth surface of $\mathbb{R}^3$ without boundary, U_S an open subset of $\mathbb{R}^3$ where the map $\xi : U_S \to S$ is well defined. Let $M \subset U_S$ be a smooth surface. Then the Jacobian of the differential of $\xi_{|M}$ is given by:*

$$|D\xi_{|M}(m))| = \frac{cos\alpha_m}{(1 + \delta_m\epsilon_m\lambda^1_{\xi(m)})(1 + \delta_m\epsilon_m\lambda^2_{\xi(m)})}.$$

As an obvious consequence of the theorem, one has the following

COROLLARY 3. *Let M be a surface closely near S and differentiable almost everywhere. Then,*

$$\frac{\cos\alpha_{\max}}{(1 + \omega_S(M))^2}\mathcal{A}(M) \leq \mathcal{A}(S) \leq \frac{\cos\alpha_{\min}}{(1 - \omega_S(M))^2}\mathcal{A}(M),$$

where $\omega_S(M)$ is the relative curvature of S to M.

It is worth noticing that in Theorem 2 and Corollary 3, we do not make the assumption that the vertices of the triangular mesh belong to the smooth surface S**.** We just need the vertices to be not to far from the smooth surface S.

However, if we use corollary 1, we get an other approximation for which we suppose that the vertices belong to the smooth surface S.

COROLLARY 4. *Let S be a (compact orientable) C^2 surface in $\mathbb{R}^3$ and T a triangular mesh closely inscribed in S.*

If

$$\left(\frac{4}{str(T)}+2\right)\ \pi_S(T)\leq 1,$$

then the area of S satisfies:

$$\frac{\sqrt{1-\left(\frac{4}{str(T)}+2\right)^2\ \pi_S(T)^2}}{(1+\pi_S(T))^2}\mathcal{A}(T)\leq\mathcal{A}(S)\leq\frac{1}{(1-\pi_S(T))^2}\mathcal{A}(T).$$

As an obvious consequence, we get the following convergence result:

COROLLARY 5. *Let S be a (compact orientable) C^2 surface in $\mathbb{R}^3$. Let $(T_n)_{n\geq 0}$ be a sequence of triangular meshes closely inscribed in S. If*
- *the lengths of the edges of T_n tend to zero when n goes to infinity,*
- *the straightness of T_n is uniformly bounded from below by a strictly positive constant, then*

$$\lim_{n\to\infty}\mathcal{A}(T_n)=\mathcal{A}(S).$$

Note that, since S is compact, the first condition may be weakened by asking that $\pi_S(T_n)$ tends to zero when n goes to infinity *(in some sense, the lengths of the edges may be "large" when the curvature is "small"...).*

The approximation of the area of a smooth surface S with the area of a triangular mesh T close to it is "good" if :
- **the vertices of T belong to S,**
- **there is an homeomorphism between T and S (thanks to ξ),**
- **the straightness str(T) is large,**
- **the relative height $\pi_S(T)$ is small.**

5. GAUSSIAN CURVATURE AND DEVELOPABLE SURFACES

In this section, thanks to several examples, we compare the Gaussian curvature of a smooth surface with the Gaussian curvature of a triangular mesh closely inscribed in it (i.e., close to it and whose vertices belong to the smooth surface). In particular, we give examples of *developable triangular meshes* (which have the property of having their discrete Gaussian curvature identically equal to 0) closely inscribed in smooth surfaces with strictly constant positive Gaussian curvature. We first need to give some definitions.

5.1 Definitions

The global definitions about discrete curvature are the consequence of a global theory which uses an important notion: *the normal cycle*. For more details on it, we can refer to [10] or [5]. Let T denote a triangular mesh, p a vertex of T, $\mathcal{T}_T(p)$ the set of triangles of T which contain p as a vertex. Let S^o_T denote the set of interior vertices of T and $S_{\partial T}$ the set of vertices of the boundary ∂T of T.

- We call the *angle to the vertex* p the real:

$$\alpha_T(p)=\sum_{\sigma\in\mathcal{T}_T(p)}\alpha_\sigma(p),$$

  where $\alpha_\sigma(p)$ is the angle at p to the triangle σ.

- The *discrete Gaussian curvature at a vertex* $p\in S^o_T$ is:

$$G_T(p)=\frac{3\ (2\pi-\alpha_T(p))}{A(T,p)},$$

  where $A(T,p)$ is the sum of the areas of the triangles of $\mathcal{T}_T(p)$.

- The *discrete geodesic curvature at a vertex* $p\in S_{\partial T}$ is:

$$k(p)=\frac{2\ (\pi-\alpha_T(p))}{l(\partial T,p)},$$

  where $l(\partial T,p)$ is the sum of the lengths of the two edges of ∂T which contain p as a vertex.

- The *total interior Gaussian curvature* of T is:

$$G_{int}(T)=\sum_{p\in S^o_T T}G_T(p)\ \frac{A(T,p)}{3}=\sum_{p\in S^o_T T}(2\pi-\alpha_T(p)).$$

- The *total geodesic curvature* of ∂T is:

$$\mathcal{K}(\partial T)=\sum_{p\in S_{\partial T}}k(p)\ \frac{l(\partial T,p)}{2}=\sum_{p\in S_{\partial T}}(\pi-\alpha_T(p)).$$

- A smooth surface (or a triangular mesh) M is *developable* if there exists an homeomorphism (of class $\mathcal{C}^2$ if M is smooth)

$$f:M\to\mathcal{U}(M)\subset\mathbb{R}^2$$

  which preserves distances. The surface $\mathcal{U}(M)$ is called *an unfolding* of S.

REMARK 3. We could have taken other pointwise definitions. For instance, M. Desbrun et al. [7] propose an other definition for the discrete Gauss curvature. They divide $2\pi-\alpha_T(p)$ by an area which is different from $\frac{A(T,p)}{3}$. In fact these two definitions are coherent with the continuous case (they are the inverse of an area). We could also consider the pointwise discrete curvature as a measure linked to vertices. In any case, it is important to define the total curvatures ($G_{int}(T)$ and $\mathcal{K}(\partial T)$) as numbers without dimension. Thus it will make sense to compare them with the Euler characteristic (see remark 5). It is worth noticing that whatever the definition we take, the sign is always the same (positive, negative or equal to 0).

REMARK 4. Note that *theorema egregium* of Gauss implies that a developable smooth surface S satisfies:

$$\forall p\in S\setminus\partial S\quad G_p=0,$$

Similarly, a developable triangular mesh T satisfies:

$$\forall p\in S^o_T\quad G_T(p)=0.$$

Just note that for us a developable surface can be isometrically embedded in the plane (and not only immersed).

REMARK 5. The main result on the total Gaussian curvature is the Gauss-Bonnet theorem (see [8]). It states that the Euler characteristic $\chi(S)$ of a smooth compact surface S (whose boundary ∂S is composed by $C_1,...,C_n$ positively oriented closed curves of class $\mathcal{C}^2$) satisfies:

$$2\pi\chi(S)=\int_S G_p\ da(p)\ +\ \sum_{i=1}^n\int_{C_i}k_p\ ds(p)\ +\ \sum_{i=1}^p\theta_i,$$

where $\{\theta_1, .., \theta_p\}$ is the set of all external angles of the curves $C_1, ..., C_n$.
The discrete analogous result for the Euler characteristic $\chi(T)$ of a triangular mesh T is the following:

$$2\pi\chi(T) = G_{int}(T) + \mathcal{K}(\partial T).$$

Since the Euler characteristic of a smooth surface S and of a triangular mesh T closely inscribed in S is the same, the sum of the total Gaussian curvature and of the total curvature of the boundary of both surfaces is the same.

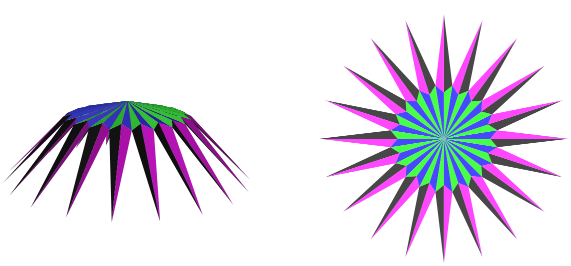
5.2 Examples

Theorem 3 gives examples of developable triangular meshes (the Gaussian curvature is thus 0 at each interior vertex) closely inscribed in smooth surfaces with strictly positive Gaussian curvature.

THEOREM 3. *For every integer $n \geq 3$, there exists $\alpha_0 \in]0,1]$ such that and for every $\alpha \in]0, \alpha_0]$, there exists a triangular mesh T^n_α such that :*

1. *T^n_α is closely inscribed in S^n_α (an open disk of sphere $\mathbb{S}^2$);*
2. *T^n_α contains $(3n + 1)$ vertices ($(n + 1)$ of them are interior) and $4n$ faces;*
3. *T^n_α is developable.*

The proof of theorem 3 can be found in [18].



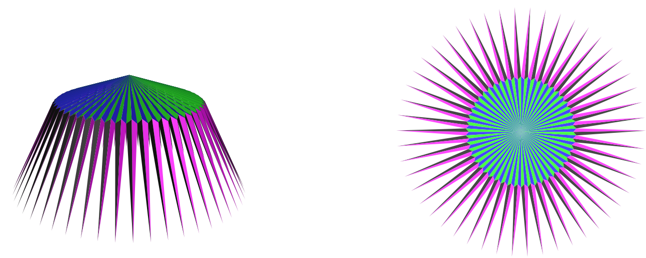
(a) Triangular mesh (b) Unfolded triangular mesh

Figure 5: case "$n = 20$ $\alpha = 0.4$"

REMARK 6. We give here a sketch of the construction of T^n_α :
We first consider the points $z, z_0, ..., z_{n-1}$:

$$z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad z_0 = \begin{pmatrix} \alpha \cos \frac{\pi}{n} \\ \alpha \sin \frac{\pi}{n} \\ \sqrt{1-\alpha^2} \end{pmatrix} \quad \textit{and} \quad z_i = r^i(z_0),$$

where r^i is the rotation of angle $\frac{2\pi i}{n}$ of axis (O, z).
Then, we build a point $u_0 \in \mathbb{S}^2$ which is at equal distance from z_0 and z_{n-1} such that the angle $\langle u_0, z, z_1 \rangle = \frac{\pi}{n}$. In fact, there exists two points satisfying these properties, and



(a) Triangular mesh (b) Unfolded triangular mesh

Figure 6: case "$n = 50$ $\alpha = 0.6$"

we take the one which is the farthest from z. We put for every $i \in \{1, ..., n-1\}$ $u_i = r^i(u_0)$.
Then, we build a point $v_0 \in \mathbb{S}^2$ which is at equal distance from u_0 and u_2 such that the angle $\langle u_0, z_0, v_0 \rangle$ is equal to the angle $\langle u_0, z_0, z \rangle$. In fact, there exists two points satisfying these properties, and we take the one which is the farthest from z. We put for every $i \in \{1, ..., n-1\}$ $v_i = r^i(v_0)$.
The vertices of the triangular mesh T^n_α are $z, z_0, ..., z_{n-1}$, $u_0, ..., u_{n-1}$ and $v_0, ..., v_{n-1}$. The triangles are $r^i(zz_0u_0)$, $r^i(zz_0u_2)$, $r^i(v_0z_0u_0)$ and $r^i(v_0z_0u_2)$ where $i \in \{0, ..., n-1\}$. S^n_α is nothing but $\xi(T^n_\alpha)$ where ξ is the map defined in proposition 1. In that particular case, if $m \in T^n_\alpha$, $\xi(m) = \frac{m}{\|m\|}$.
The integer n is linked to the number of vertices. The real α is small if the vertices are close to the vertex z. It measures the distance to the point z.

Since S^n_α is a smooth surface whose Gaussian curvature is 1 at every interior point, S^n_α is not developable. However theorem 3 tells us that the triangular mesh T^n_α, which is closely inscribed in S^n_α, is developable.

This implies that without other assumptions, **the knowledge of the Gaussian curvature of a triangular mesh closely inscribed in a smooth surface does not give information on the Gaussian curvature of the smooth surface**.

In particular, the knowledge of a developable triangular mesh closely inscribed in a smooth surface does not allow us to conclude whether the smooth surface is developable. It implies that the fact of building a developable triangular mesh inscribed in a smooth surface does not allow us to check an assumption of unfoldness made a priori on the smooth surface.

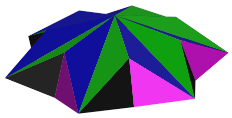
REMARK 7. Thanks to Gauss-Bonnet theorem, we know that the total geodesic curvature of ∂T^n_α is equal to 2π and is larger than the total geodesic curvature of ∂S^n_α

$$\sum_{i=1}^{n} \int_{C_i} k_p \; ds(p) \; + \; \sum_{i=1}^{p} \theta_i,$$

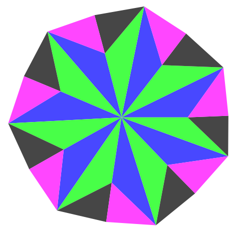
which is 2π minus the area of S^n_α.

We present here some of those triangular meshes T^n_α which are inscribed in sphere $\mathbb{S}^2$ and we unfold them. We use

Geomview [11] to visualize examples. The triangular mesh of



(a) Developable triangular mesh

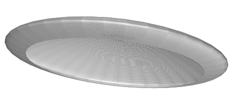


(b) Unfolded triangular mesh

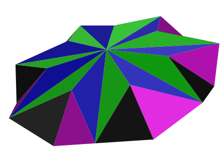
Figure 7: Developable triangular mesh inscribed in a smooth surface with strictly positive Gaussian curvature

figure 7 is developable and its boundary *"is quite regular"*, in the sense that the discrete geodesic curvature at each vertex of the boundary is not too large. This triangular mesh is not inscribed in a sphere, but in a smooth surface of revolution, whose Gaussian curvature is strictly positive at each interior point.

The triangular mesh of figure 8 is not developable. More precisely, the discrete Gaussian curvature at each interior vertex is strictly negative (in fact $(2\pi - \alpha_T(p)) \leq -0.02$ at each interior vertex p). However, this triangular mesh is closely inscribed in a smooth surface of revolution, whose Gaussian curvature is strictly positive at each interior point. Thus we have a triangular mesh with strictly negative Gaussian curvature inscribed in a smooth surface with strictly positive Gaussian curvature.



(a) Smooth surface with strictly positive Gaussian curvature



(b) Triangular mesh

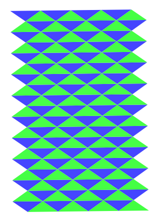
Figure 8: Triangular mesh of strictly negative Gaussian curvature inscribed in a smooth surface with strictly positive Gaussian curvature

6. UNFOLDING OF A SURFACE

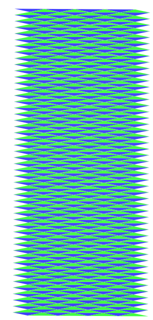
In this section, we consider the problem of the approximation of the unfolding of a smooth developable surface. We suppose that we have a smooth developable surface S and that the triangular mesh T closely near the smooth surface S is developable too. We want to answer to the question: **does the unfolding of T give a "good approximation" of the unfolding of S?**

We first give a "counter-example" : it is still the *half "lantern" of Schwarz*. Then we give results of approximation.

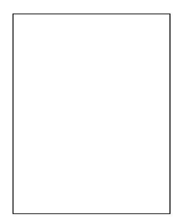
6.1 Half "lantern" of Schwarz



(a) Unfolded P(5-19)



(b) Unfolded P(5-99)



(c) Unfolded half cylinder

Figure 9: Unfolding of C and of two *half "lanterns" of Schwarz* closely inscribed in C (the scale is the same)

The *half "lantern" of Schwarz* $P(n, N)$ has the property of *being developable*. Thanks to part 4, we know that we can construct a *half "lantern" of Schwarz* whose area is as large as we wish. Therefore, if we unfold such a "lantern", its unfolding is "very different" from the unfolding of the half cylinder.

In figure 9, we unfold the two *half "lanterns" of Schwarz* of figure 4. Note that the boundaries of the two unfolded *half "lanterns" of Schwarz* are very different from one another and are very different from the unfolding of the half cylinder C. The unfolding of C is a rectangle of height H. The height of the two unfolded *half "lanterns" of Schwarz* are:

$$h(P(5-19)) \approx 1.026H,$$
$$h(P(5-99)) \approx 1.57H.$$

The height of a *half "lantern" of Schwarz* is getting larger when it is unfolded. For example, the height of the unfolded $P(5-99)$ is more than one and an half the height of the cylinder C.

Furthermore, if we consider the problem of the convergence of a sequence of triangular meshes, one may notice that the height of the unfolding of the *half "lantern" of Schwarz* $P(n, n^3)$ tends to infinity when n tends to infinity.

That is why, without other assumptions, we cannot expect the unfolding of a sequence of triangular meshes to give us a good approximation of the unfolding of the smooth surface.

6.2 Approximation of the unfolding of a smooth surface

The following result gives an explicit approximation of the unfolding of a smooth surface S in terms of the unfolding of a triangular mesh T. We notice that if the normals of S are close enough to the normals of T, then the two unfoldings are quite similar.

THEOREM 4. *Let S be a smooth compact connected and developable surface of $\mathbb{R}^3$ and T be a developable triangular mesh closely near S. Then the Hausdorff distance between two unfoldings $\mathcal{U}(S)$ and $\mathcal{U}(T)$ of S and T satisfies (up to a motion of $\mathbb{R}^2$):*

$$\delta_{\text{Hauss}}(\mathcal{U}(T), \mathcal{U}(S)) \leq$$

$$\epsilon\sqrt{(2R(\mathcal{D}+ka))^2 + \left(\frac{8R(\mathcal{D}+ka)}{\sqrt{3}}(1+2R) + \frac{4(\mathcal{D}+ka)^2}{a\sqrt{3}}\epsilon\right)^2},$$

$$\textit{with } \epsilon = sup\left(1 - \frac{\cos(\alpha_{\max})}{1+\omega_S(T)}, \frac{1}{1-\omega_S(T)} - 1\right),$$

where $\alpha_{\max}$ is the maximal angle between the normals of S and T, $\omega_S(T)$ is the relative curvature of S to T,
a is the length of a side of an equilateral triangle included in $\mathcal{U}(T)$, $\mathcal{D}$ is the length of a longest geodesic of $\mathcal{U}(T)$, $R = \frac{Diam}{a}$, where $Diam$ is the diameter of $\mathcal{U}(T)$ in $\mathbb{R}^2$ and k is the real number given by:

$$k = k(\epsilon) =$$

$$\sqrt{\left(3\left(1+\frac{\epsilon}{2}\right)\right)^2 + \left(\frac{10}{\sqrt{3}} + \frac{19\epsilon}{\sqrt{3}} + 6\sqrt{3}\epsilon^2 + \frac{3\sqrt{3}\epsilon^3}{2}\right)^2}.$$

For the proof of theorem 4, we can refer to [16] or [19].

REMARK 8. In the first part of the proof of Theorem 4, we compare the geodesics of T (linking two points p and q) with the geodesics of S (linking $\xi(p)$ and $\xi(q)$). We obtain :

$$(1-\epsilon)l_T(p,q) \leq l_S(\xi(p), \xi(q)) \leq (1+\epsilon)l_T(p,q),$$

where $l_T(p,q)$ is the distance between p and q on T and $l_S(\xi(p), \xi(q))$ is the distance between $\xi(p)$ and $\xi(q)$ on S.
In a second step, we consider an equilateral triangle (the vertices are A, B and C) included in $\mathcal{U}(T)$. Then (up to a motion of the plane of $\mathcal{U}(T)$), we can have :

$$\tilde{A} = A \quad \|\tilde{B} - B\| \leq \epsilon a \quad \|\tilde{C} - C\| \leq \epsilon k(\epsilon)a,$$

where $a = AB$, $f_S : S \to \mathcal{U}(S)$ and $f_T : S \to \mathcal{U}(T)$ are two unfoldings and for every $M \in \mathcal{U}(T)$, $\tilde{M} = (f_S \circ \xi \circ f_T^{-1})(M)$.
In a last step, we give an upper bound of $\|M - \tilde{M}\|$ for every $M \in \mathcal{U}(T)$.

The following corollary then follows directly:

COROLLARY 6. *Let S be a smooth compact connected developable surface of $\mathbb{R}^3$ and $(T_n)_{n\geq 0}$ a sequence of developable triangular meshes closely near S such that:*

- *the angle between the normals of S and T_n tends to 0 when n tends to infinity;*
- *T_n tends to S in the Hausdorff sense when n tends to infinity;*

then a sequence $(\mathcal{U}(T_n))_{n\geq 0}$ of unfoldings of $(T_n)_{n\geq 0}$ tends in the Hausdorff sense to an unfolding $\mathcal{U}(S)$ of S.

It is worth noticing that in theorem 4 and corollary 6, we do not suppose that the vertices of the triangular mesh belong to the smooth surface S. We just need the vertices to be not to far from the smooth surface S.

In the case in which the triangular mesh is closely inscribed in the smooth surface (vertices of T belong to S), thanks to part 3, we get the following result:

COROLLARY 7. *Let S be a smooth compact connected developable surface of $\mathbb{R}^3$ and $(T_n)_{n\geq 0}$ a sequence of developable triangular meshes closely inscribed in S such that:*
- *the straightness of the sequence $(T_n)_{n\geq 0}$ is uniformly bounded from below by a strictly positive constant;*
- *the height of T_n tends to 0 when n tends to infinity;*

then a sequence $(\mathcal{U}(T_n))_{n\geq 0}$ of unfoldings of $(T_n)_{n\geq 0}$ tends in the Hausdorff sense to an unfolding $\mathcal{U}(S)$ of S.

The approximation of the unfolding of a smooth surface S with the unfolding of a triangular mesh T close to it is "good" if :
- **the vertices of T belong to S,**
- **there is an homeomorphism between T and S (thanks to ξ),**
- **the straightness str(T) is large,**
- **the relative height $\pi_S(T)$ is small.**

7. SOME PROOFS

We first need a result on the differential of ξ (defined in proposition 1).

LEMMA 2. *Let S be a smooth surface of $\mathbb{R}^3$ without boundary, U_S an open subset of $\mathbb{R}^3$ where the map $\xi : U_S \to S$ is well defined. Then,*

1. *the map ξ is C^1 in U_S and satisfies for every $m \in U_S$:*

$$D\xi(m)(Z_m) = 0, \forall Z_m \textit{ orthogonal to } T_{\xi(m)}S,$$

$$D\xi(m)(X_m) = (Id + \delta_m\epsilon_m A_{\xi(m)})^{-1}(X_m),$$
$$\forall X_m \textit{ parallel to } T_{\xi(m)}S,$$

where $A_{\xi(m)} = -D\nu(\xi(m))$ is the Weingarten endomorphism of S at the point $\xi(m)$.

2. *In particular, the matrix of $D\xi(m) : \mathbb{R}^3 \to T_{\xi(m)}S$ (in local orthonormal frames $(e^1_{\xi(m)}, e^2_{\xi(m)}, \nu^S_{\xi(m)})$ and $(e^1_{\xi(m)}, e^2_{\xi(m)})$) is given by:*

$$\begin{pmatrix} \frac{1}{1+\delta_m\epsilon_m\lambda^1_{\xi(m)}} & 0 & 0 \\ 0 & \frac{1}{1+\delta_m\epsilon_m\lambda^2_{\xi(m)}} & 0 \end{pmatrix}.$$

where $e^1_{\xi(m)}$ and $e^2_{\xi(m)}$ are unit principal vectors and $\nu^S_{\xi(m)}$ is the oriented normal of S at $\xi(m)$.

Sketch of proof of Lemma 2

- For every $m \in U_S$, the point $\xi(m)$ satisfies the following relation:

  $$\forall m \in U_S, \forall X \in T_{\xi(m)}S, < \xi(m) - m, X_{\xi(m)} >= 0.$$

  Consequently, for every $m \in T_mS$, the function ξ is constant on the orthogonal of T_mS.

- Consider now a vector $X_m \in T_mU_S$ which is parallel to $T_{\xi(m)}S$. We have:

  $$D\xi(m)(X) = X + \delta_m\epsilon_m D\nu(\xi(m)) \circ D\xi(m)(X).$$

  The endomorphism $(I+\delta_m\epsilon_m A_{\xi(m)})$ is clearly inversible, and consequently,

  $$D\xi(m)(X) = (I + \delta_m\epsilon_m A_{\xi(m)})^{-1}(X).$$

- The rest of the proof is obvious.

7.1 Proof of theorem 1

The proof is organized in two steps:

• first of all we compare the normals of the surface and the triangle at a vertex,

• then, we compare the normal of the surface at a vertex and at a point on the surface "close to this vertex".

We need before technical lemmas. The following lemma is a purely geometric result in $\mathbb{R}^3$:

LEMMA 3. *Let Δ be a triangle whose vertices are p, p_1 and p_2. If $\alpha_p \in \left[0, \frac{\pi}{2}\right]$ denotes the angle between a normal to the triangle and the axis (O, z), then*

1.

$$cos^2(\alpha_p) = \frac{\cos^2\theta_1\cos^2\theta_2 - \sin^2\theta_1\sin^2\theta_2 - \cos^2\gamma + 2\cos\gamma\sin\theta_1\sin\theta_2}{\sin^2(\theta_2-\theta_1) + 2\sin\theta_1\sin\theta_2\cos(\theta_2-\theta_1) + \cos^2\theta_1\cos^2\theta_2 - \sin^2\theta_1\sin^2\theta_2 - \cos^2\gamma},$$

where $\theta_i \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the angle between $p\vec{p}_i$ and the orthogonal projection of $p\vec{p}_i$ onto the plane orthogonal to (O, z) which contains p ($\theta_i \geq 0$ iff the third composante of $p\vec{p}_i$ is positive) and $\gamma \in]0, \pi[$ is the angle of Δ at p.

2. *In particular, if $|\sin\theta_1| \leq \epsilon$ and $|\sin\theta_2| \leq \epsilon$, then*

$$\sin(\alpha_p) \leq \frac{\sqrt{10}\epsilon}{\sin\gamma},$$

where $\gamma \in]0, \pi[$ is the angle of Δ at p.

Proof of Lemma 3

1. We notice that the angle α_p does not depend on the lengths pp_1 and pp_2. Therefore, one may suppose that $p = 0$ and that p_1 and p_2 belong to sphere $\mathbb{S}^2$.

We put $N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

In spherical coordinates, we get:

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad p_i = \begin{pmatrix} \cos\phi_i\cos\theta_i \\ \sin\phi_i\cos\theta_i \\ \sin\theta_i \end{pmatrix},$$

$$\text{with} \begin{cases} \theta_i \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \\ \phi_i \in [0, 2\pi], \\ i \in \{1, 2\}. \end{cases}$$

We can suppose that we have $\phi_1 = 0$ (even if it means composing by a rotation). We put

$$p\vec{p}_1 \wedge p\vec{p}_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Thus

$$\cos(\alpha_p) = \left| < N, \frac{p\vec{p}_1 \wedge p\vec{p}_2}{||p\vec{p}_1 \wedge p\vec{p}_2||} > \right| = \sqrt{\frac{c^2}{a^2+b^2+c^2}}.$$

We have:

$$\begin{aligned} a &= -\cos\theta_2\sin\phi_2\sin\theta_1, \\ b &= \cos\theta_2\cos\phi_2\sin\theta_1 - \cos\theta_1\sin\theta_2, \\ c &= \cos\theta_1\cos\theta_2\ \sin\phi_2. \end{aligned}$$

This implies that

$$\begin{aligned} & a^2 + b^2 \\ =\ & \cos^2\theta_1\sin^2\theta_2 + \cos^2\theta_2\sin^2\theta_1 \\ & -2\cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2\cos\phi_2 \\ =\ & \sin^2(\theta_2-\theta_1) \\ & +2\cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2(1-\cos\phi_2). \end{aligned}$$

On the other hand, we get:

$$\begin{aligned} & \cos\gamma \\ =\ & < p\vec{p}_1, p\vec{p}_2 > \\ =\ & \cos\theta_1\cos\theta_2\cos\phi_2 + \sin\theta_1\sin\theta_2. \end{aligned}$$

Suppose that $\cos\theta_1 \neq 0$ and $\cos\theta_2 \neq 0$. Then

$$\cos\phi_2 = \frac{\cos\gamma - \sin\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2}.$$

Therefore

$$\begin{aligned} & a^2 + b^2 \\ =\ & \sin^2(\theta_2-\theta_1) \\ & +2\cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2\left(1 - \frac{\cos\gamma-\sin\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2}\right) \\ =\ & \sin^2(\theta_2-\theta_1) \\ & +2\sin\theta_1\sin\theta_2\cos(\theta_2-\theta_1) - 2\cos\gamma\sin\theta_1\sin\theta_2, \\ & c^2 \\ =\ & \cos^2\theta_1\cos^2\theta_2\left(1 - \left(\frac{\cos\gamma-\sin\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2}\right)^2\right) \\ =\ & \cos^2\theta_1\cos^2\theta_2 - \sin^2\theta_1\sin^2\theta_2 - \cos^2\gamma \\ & +2\cos\gamma\sin\theta_1\sin\theta_2, \\ & a^2+b^2+c^2 \\ =\ & \sin^2(\theta_2-\theta_1) + 2\sin\theta_1\sin\theta_2\cos(\theta_2-\theta_1) \\ & +\cos^2\theta_1\cos^2\theta_2 - \sin^2\theta_1\sin^2\theta_2 - cos^2\gamma. \end{aligned}$$

Suppose now that $\cos\theta_1 = 0$. Thus the angle α_p is equal to $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ and $\cos^2\alpha_p = 0$. Since $\cos\gamma = \sin\theta_2$, the result is still true.

2. We have
$$|\sin\theta_i| \leq \epsilon \quad \text{and} \quad |\cos\theta_i| \geq \sqrt{1-\epsilon^2}.$$
Thus
$$\begin{array}{rl} & \cos^2\theta_1\cos^2\theta_2 - \sin^2\theta_1\sin^2\theta_2 - \cos^2\gamma \\ & +2\cos\gamma\sin\theta_1\sin\theta_2 \\ \geq & (1-\epsilon^2) - \epsilon^4 - \cos^2\gamma - 2\epsilon^2 \\ = & \sin^2\gamma - 4\epsilon^2, \end{array}$$
and
$$\begin{array}{rl} & \sin^2(\theta_2-\theta_1) + 2\sin\theta_1\sin\theta_2\cos(\theta_2-\theta_1) \\ & +\cos^2\theta_1\cos^2\theta_2 - \sin^2\theta_1\sin^2\theta_2 - cos^2\gamma \\ \leq & 4\epsilon^2 + 2\epsilon^2 + 1 - cos^2\gamma \\ = & \sin^2\gamma + 6\epsilon^2. \end{array}$$
Then
$$\cos^2(\alpha_p) \geq \frac{\sin^2\gamma - 4\epsilon^2}{\sin^2\gamma + 6\epsilon^2}.$$
Suppose now that $\frac{6\epsilon^2}{\sin^2\gamma} < 1$. Then, we get
$$\cos^2(\alpha_p) \geq \left(1 - \frac{4\epsilon^2}{\sin^2\gamma}\right)\left(1 - \frac{6\epsilon^2}{\sin^2\gamma}\right) \geq 1 - \frac{10\epsilon^2}{\sin^2\gamma},$$
and
$$sin(\alpha_p) \leq \frac{\sqrt{10}\epsilon}{\sin\gamma}.$$
If $\frac{6\epsilon^2}{\sin^2\gamma} \geq 1$, we get :
$$\frac{\sqrt{10}\epsilon}{\sin\gamma} \geq \frac{\sqrt{6}\epsilon}{\sin\gamma} \geq sin(\alpha_p).$$

7.1.1 Comparing the length of a geodesic and its chord

LEMMA 4. *Let S be a smooth compact surface of $\mathbb{R}^3$, U_S a neighborhood of S where the map $\xi : U_S \to S$ is well defined, p and q two points on S such that the line-segment of $\mathbb{R}^3$ $[p,q] \subset U_S$ and $\xi(]p,q[) \subset S \setminus \partial S$. Then the distance l_{pq} between p and q on S satisfies:*
$$l_{pq} \leq \frac{1}{1-\omega}\ pq,$$
where $\omega = \sup_{m\in]p,q[} \|\xi(m) - m\|\ \rho_{\xi(m)}$ and pq is the the length of $[p,q]$.

Proof of Lemma 4 Since $\xi([p,q])$ is a curve on S, its length is larger than the length l_{pq} of the geodesic whose ends are p and q. Therefore:
$$l_{pq} \leq l(\xi([p,q])) \leq \sup_{m\in]p,q[} |D\xi(m)|\ pq.$$
On the other hand, for every $m \in]p,q[$, we have $\|\xi(m) - m\|\ \rho_{\xi(m)} < 1$. Since $\xi(p) = p$ and $\xi(q) = q$, $\omega < 1$. Therefore Lemma 2 implies that:
$$|D\xi(m)| \leq \frac{1}{1 - \|\xi(m) - m\|\ \rho_{\xi(m)}} \leq \frac{1}{1-\omega},$$
and lemma 4 is proved.

7.1.2 Comparing the normals at a vertex

We shall prove the following

PROPOSITION 2. *Let S be a smooth surface, Δ a triangle closely inscribed in S and p a vertex of Δ. Then the angle $\alpha_p \in \left[0, \frac{\pi}{2}\right]$ between the normals of S and Δ at p satisfies:*
$$\sin(\alpha_p) \leq \frac{\sqrt{10}\pi_S(\Delta)}{2\ \sin\gamma_p\ (1-\omega_S(\Delta))},$$
where γ_p is the angle of Δ at p.

This proposition is a consequence of the following

LEMMA 5. *Let S be smooth surface, Δ a triangle closely inscribed in S, p and q two vertices of Δ. Then the angle $\theta \in \left[0, \frac{\pi}{2}\right]$ between $\vec{pq}$ and T_pS satisfies:*
$$\sin\theta \leq \frac{\rho_S\ l}{2},$$
where l is the distance on S between p and q.

Proof of Lemma 5 Let $\mathcal{C}$ denote a geodesic of S linking p and q. $\mathcal{C}$ is parameterized by arc length by:
$$\gamma : [0,l] \to S,$$
with $\gamma(0) = p$ and $\gamma(l) = q$. A simple calculation gives:
$$\begin{array}{rl} & \gamma(l) - \gamma(0) = l\gamma'(0) + \int_0^l (l-t)\gamma''(t)dt, \\ \text{thus} & \gamma'(0) \ = \frac{\vec{pq}}{l} - \frac{1}{l}\int_0^l (l-t)\gamma''(t)dt. \end{array}$$
Let $\vec{v} = \frac{1}{l}\int_0^l (l-t)\gamma''(t)dt$ and $\vec{e} = \frac{\vec{pq}}{pq}$. $|\gamma''(t)|$ is the curvature of the curve $\mathcal{C}$ at the point $\gamma(t)$. Since $\mathcal{C}$ is a geodesic of S, $|\gamma''(t)|$ is the absolute value of the curvature of S at the point $\gamma(t)$ in the direction $\gamma'(t)$ and is smaller than ρ_S. Therefore we have:
$$\gamma'(0) = \frac{pq}{l}\vec{e} - \vec{v} \text{ with } \left\{ \begin{array}{l} \|\vec{e}\| = 1, \\ \|\vec{v}\| \leq \frac{l\rho_S}{2}, \end{array} \right.$$
and
$$\sin\theta = \inf_{\substack{\vec{u}\in T_pS \\ \|\vec{u}\|=1}} \|\vec{u} \wedge \vec{e} \leq \|\gamma'(0) \wedge \vec{e}\| \leq \frac{l\rho_S}{2}.$$

Proof of Proposition 2 Let p_1 and p_2 be the two vertices of Δ different from p and denote by l_i the distance on S between p and p_i. Since T is closely inscribed in S, thanks to lemma 4 we get:
$$l_i \leq \frac{pp_i}{1-\omega_S(\Delta)} \leq \frac{\eta_\Delta}{1-\omega_\Delta}.$$
Therefore, lemma 5 implies that
$$\sin\theta_i \leq \frac{\rho_{\xi(\Delta)} l_i}{2} \leq \frac{\pi_S(\Delta)}{2(1-\omega_S(\Delta)}$$
Then lemma 3 implies
$$\sin(\alpha_p) \leq \frac{\sqrt{10}}{\sin\gamma_p}\ \frac{\pi_S(\Delta)}{2(1-\omega_S(\Delta))} = \frac{\sqrt{10}\ \pi_S(\Delta)}{2\ \sin\gamma_p\ (1-\omega_S(\Delta))}.$$

7.1.3 Comparing the normals of a smooth surface

PROPOSITION 3. *Let S be a smooth compact oriented surface of $\mathbb{R}^3$, Δ a triangle closely inscribed in S, p and s two* *points on Δ. Then the angle $\alpha_{sp} \in \left[0, \frac{\pi}{2}\right]$ between two normals $\nu^S_{\xi(p)}$ and $\nu^S_{\xi(s)}$ at $\xi(p)$ and $\xi(s)$ satisfies:*

$$\sin(\alpha_{sp}) \leq \frac{\pi_S(\Delta)}{1-\omega_\Delta},$$

where η_Δ is the height of Δ and ω_Δ is the relative curvature of Δ with respect to $\xi(\Delta)$.

This proposition is the consequence of the following

LEMMA 6. *Let S be a smooth compact oriented surface of $\mathbb{R}^3$, a and b two points of S. The angle $\alpha_{ab} \in \left[0, \frac{\pi}{2}\right]$ between two normals ν^S_a and ν^S_b at a and b satisfies:*

$$\sin(\alpha_{ab}) \ \leq \rho_S \ L_S(a,b),$$

where ρ_S the maximal curvature of S and $L_S(a,b)$ the distance on S between a and b.

Proof of Lemma 6 Using the mean-value theorem we have:

$$\|\nu^S_a - \nu^S_b\| \leq |DN|_\infty \ L_S(a,b) = \rho_S \ L_S(a,b).$$

Thus,

$$\begin{aligned}\sin(\alpha_{ab}) \ &\leq 2 \ \sin(\tfrac{\alpha_{ab}}{2})\\ &= \|\nu^S_a - \nu^S_b\|\\ &\leq \rho_S \ L_S(a,b).\end{aligned}$$

Proof of Proposition 3 Lemma 6 implies:

$$\sin(\alpha_{sp}) \ \leq \rho_S \ L_S(\xi(p),\xi(s)).$$

$L_S(\xi(p),\xi(s))$ is smaller than the length $L(\xi([p,s]))$ of the curve $\xi([p,s])$ which joins $\xi(p)$ and $\xi(s)$ on S. Thus:

$$L_S(\xi(p),\xi(s)) \leq L(\xi([p,s])) \leq \sup_{m\in\Delta} |D\xi(m)| \ ps.$$

Lemma 2 implies:

$$\sin(\alpha_{sp}) \ \leq \rho_S \ \sup_{m\in\Delta} |D\xi(m)| \ ps \leq \frac{\pi_S(\Delta)}{1-\omega_\Delta}.$$

7.1.4 End of proof of theorem 1

The proof of this theorem uses propositions 2 and 3.
Let $s \in \Delta$ and p be a vertex of Δ. The angle α_s is at most $\alpha_p + \alpha_{sp}$. Furthermore α_s, α_p and α_{sp} belong to $\left[0, \frac{\pi}{2}\right]$. Thus we get:

$$\begin{aligned}\sin(\alpha_s) \ &\leq \sin(\alpha_p) + \sin(\alpha_{sp})\\ &\leq \left(\frac{\sqrt{10}}{2 \ str(\Delta) \ (1-\omega_\Delta)} + \frac{1}{1-\omega_\Delta}\right) \ \pi_S(\Delta),\end{aligned}$$

and then,

$$\sin(\alpha_s) \leq \left(\frac{\sqrt{10}}{2 \ str(T) \ (1-\omega_S(T))} + \frac{1}{1-\omega_S(T)}\right) \ \pi_S(T).$$

7.1.5 Proof of Corollary 1

Thanks to Remark 2, we get $\omega_S(T) \leq \pi_S(T) \leq \frac{1}{2}$. We conclude by using Theorem 1.

7.2 Proof of theorem 2

Proof of Lemma 1:
It is obvious, by considering the restriction $\xi_{|M}$ of ξ to M and by using Lemma 2.
Proof of Theorem 2:
Consider the surface M lying in U_S. By assumption, the restriction $\xi_{|M}$ of ξ to M is one to one, and then is a diffeomorphism. We shall apply the classical area formula:

$$\mathcal{A}(S) = \int_S da_S = \int_M \xi^*_{|M} da_S = \int_M |D\xi_{|M}(m))| \ da_S(\xi(m)),$$

which is pertinent since we are in the case where ξ is a diffeomorphism. The following lemma evaluate the Jacobian of $\xi_{|M}$. Therefore:

$$\begin{aligned}\mathcal{A}(S) \ &= \int_M |D\xi_{|M}(m))| \ da_M(m)\\ &= \int_M \frac{cos \ \alpha_m}{(1+\delta_m\epsilon_m\lambda^1_{\xi(m)})(1+\delta_m\epsilon_m\lambda^2_{\xi(m)})} \ da_M(m).\end{aligned}$$

8. CONCLUSION

The properties of a smooth surface and of a triangular mesh "close to it" can be very different. In particular, the fact of having a developable triangular mesh closely inscribed in a smooth surface does not allow us to conclude on the "unfoldness" of the smooth surface.

However, we get explicit approximations of the area and of the unfolding (when it is developable) of a smooth surface in terms of a triangular mesh which is close enough, even if the vertices of the triangular mesh do not belong to the smooth surface. It is interesting to notice that those approximations depend on the approximation of the normals.

Therefore, if the vertices of the triangular mesh belong to the smooth surface, we can have all the approximations in terms of the straightness and of the relative height. The condition on the straightness implies that the upper bounds on the approximations can be "good" even if there are small angles: we just need each triangle of the triangular mesh to contain at least one angle whose sine is large enough. The condition on the relative height implies that the triangles of the triangular mesh have to be small where the curvature of the smooth surface is large.

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