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# On the approximation of a smooth surface with a triangulated mesh $\stackrel{\text{\tiny{triangulated}}}{\to}$

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#### Abstract

We approximate the normals and the area of a smooth surface with the normals and the area of a triangulated mesh whose vertices belong to the smooth surface. Both approximations only depend on the triangulated mesh (which is supposed to be known), on an upper bound on the smooth surface's curvature, on an upper bound on its reach (which is linked to the local feature size) and on an upper bound on the Hausdorff distance between both surfaces.

We show in particular that the upper bound on the error of the normals is better when triangles are right-angled (even if there are small angles). We do not need every angle to be quite large. We just need each triangle of the triangulated mesh to contain at least one angle whose sinus is large enough.

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## 1. Introduction

Replacing a smooth surface with a triangulated mesh appears in many applications. In this paper, we are interested in the relationship between a smooth surface and a triangulated mesh inscribed in it (i.e., whose vertices belong to the smooth surface). In particular we wonder whether we can approximate the normals and the area of the smooth surface with the normals and the area of a triangulated mesh. Remark that the normals and the area of the triangulated mesh can be very different from the normals and the area

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Fig. 1. Examples of Lampion de Schwarz. (a) P(19-5); (b) P(99-5).

of the smooth surface, even if the mesh is "very close" to it. The famous *lampion de Schwarz* is a typical example: let *C* be a half cylinder of finite height *H* and of radius *R*. Let P(n, N) denote the triangulated mesh whose vertices  $S_{ij}$  belong to *C* and are defined as follows:

$$\forall i \in \{0, \dots, n-1\} \quad S_{i,j} = \left(R\cos(i\alpha), R\sin(i\alpha), jh\right) \quad \text{if } j \text{ is even,} \\ \forall j \in \{0, \dots, N\} \qquad S_{i,j} = \left(R\cos\left(i\alpha + \frac{\alpha}{3}\right), R\sin\left(i\alpha + \frac{\alpha}{2}\right), jh\right) \quad \text{if } j \text{ is odd,}$$

and whose faces are

$$S_{ij}S_{i+1,j}S_{i,j+1}, \qquad S_{i,j}S_{i-1,j+1}S_{i,j+1}$$

where  $\alpha = \pi/n$  and h = H/N.

Then for example, when *n* tends to infinity, the area  $\mathcal{A}(P(n, n^3))$  of  $P(n, n^3)$  tends to infinity and the normals of  $P(n, n^3)$  tend to be orthogonal to the normals of the surface *C*.

That is why, without other assumptions, we cannot expect the mesh to give us a good approximation of the normals and of the area of the smooth surface.

Under suitable additional assumptions, J. Fu already proved in [8] convergence results of the area and curvatures of a sequence of triangulated meshes converging to a smooth surface. The assumptions are related to the *fatness* of the sequence of triangulated meshes, which must be uniformly bounded from below by a strictly positive constant. In [1], N. Amenta and M. Bern construct an explicit triangulated mesh inscribed in a smooth surface and obtain an approximation of the normals of the smooth surface. In [3], N. Amenta et al. construct an explicit triangulated mesh for which the circumradii of the triangles are small compared to the *local feature size* and they deduce an approximation of the normals.

Our point of view is different: in particular, we do not look for convergence results and we do not consider the problem of reconstructing a surface. We suppose that a triangulated mesh is inscribed in a smooth surface (we do not care about its construction) and we get explicit approximations of the area and of the normals of the smooth surface in terms of geometric data. If we consider the triangulated mesh constructed by the algorithm of [3], our approximation of the normals is similar to the approximation of the normals of [3].

The notion of the *reach of a smooth surface* is one of the main tools of this paper. It was first introduced by H. Federer [7]. It is interesting to notice that the *reach* is in fact linked to the (more recent) notions of *medial axis* and *local feature size*, which are used in some problems of reconstructing a surface from scattered sample points (see [1,2] or [5]). In [13], F.E. Wolter gives many interesting results related to the relationship between *medial axis, cut locus* and the *reach*.

Roughly speaking, we evaluate these approximations in terms of the geometry of the triangulated mesh, the local curvature of the smooth surface, its local reach and the local Hausdorff distance between the two surfaces. We can be more precise: surprisingly, the approximation of the normals of the smooth surface does not depend on the *fatness* of the triangulated mesh but on its *straightness* (see the definition below).

This paper is organized as follows. Section 2 gives classical and usual definitions. Section 3 states our main results. Section 4 sketches the proofs of results.

# 2. Definitions

We recall here some classical definitions which concern smooth surfaces, triangulated meshes and the relative position of two surfaces. For more details on smooth surfaces, one may refer to [4] or [12]. For more details on triangulated meshes, one may refer to [7,8] or [11].

# 2.1. Smooth surfaces

In the following, a smooth surface means a C<sup>2</sup> surface which is regular, oriented, compact with or without boundary. Let S be a smooth surface of the (oriented) euclidean space R<sup>3</sup>. Let ∂S denote the boundary of S. S is endowed with the Riemannian structure induced by the standard scalar product of R<sup>3</sup>. We denote by *da* the area form on S and by *ds* the canonical orientation of ∂S. Let ν be the unitary normal vector field (compatible with the orientation of S) and *h* be the second fundamental form of S associated with ν. Its determinant at a point p of S is *the Gauss curvature* G(p), its trace is *the mean curvature* H(p). We put ρ(p) = max(|λ<sub>1</sub>(p)|, |λ<sub>2</sub>(p)|), where λ<sub>1</sub>(p) and λ<sub>2</sub>(p) are the eigenvalues of the second fundamental form at p, and

$$\rho_S = \sup_{p \in S} \rho(p)$$

• We need the following

**Proposition 1.** Let S be a smooth compact surface of  $\mathbb{R}^3$ . Then there exists an open set  $U_S$  of  $\mathbb{R}^3$  containing S and a continuous map  $\xi$  from  $U_S$  onto S satisfying the following: if p belongs to  $U_S$ , then there exists a unique point  $\xi(p)$  realizing the distance from p to S ( $\xi$  is nothing but the orthogonal projection onto S).

A proof of this proposition can be found in [7]. Remark that  $U_S$  is an open set where the normals of S do not intersect. We can now define the *reach*.

**Definition 1.** The reach of a smooth compact surface S is the largest real r > 0 for which  $\xi$  is defined on the tubular neighborhood  $U_r$  of radius r of S.

Just notice that if p is a point of  $\mathbb{R}^3$  which is at a distance less than r to S (i.e.,  $p \in U_r$ ) and if  $\xi(p)$  is an interior point of S, then  $p - \xi(p)$  is a vector normal to S at the point  $\xi(p)$ .

## Remark 1.

- The open set  $U_S$  depends locally and globally on the smooth surface S. Globally,  $U_S$  depends on points which are far from one another on the surface, but close in  $\mathbb{R}^3$ . Locally, the normals of S do not intersect in  $U_S$ . This implies that the reach  $r_S$  of S is smaller than the minimal radius of curvature of S (see [10] or [13]). Thus, we have

 $\rho_S r_S \leqslant 1$ ,

where  $\rho_S$  is the maximal curvature of *S*.

- The reach is linked to the notion of medial axis and local feature size (see [13]). The reach  $r_S$  is a global notion which measures the distance of *S* to the medial axis of *S* along the normals of *S*. The local feature size at a point *p* of *S* is the distance from *p* to the medial axis.

# 2.2. Triangulated meshes

# 2.2.1. Generalities

A triangulated mesh T is a (finite and connected) union of triangles of  $\mathbb{R}^3$ , such that the intersection of two triangles is either empty, or equal to a vertex, or equal to an edge.

We denote by  $\mathcal{T}_T$  the set of triangles of T and by  $\Delta$  a generic triangle of T.

- $\eta(\Delta)$  denotes the length of the longest edge of  $\Delta$ , and  $\mathcal{A}(\Delta)$  its area.
- The fatness of  $\Delta$  is the real number

$$\theta(\Delta) = \frac{\mathcal{A}(\Delta)}{\eta(\Delta)^2}.$$

• The straightness of a triangle  $\Delta$  is the real number

$$\operatorname{str}(\Delta) = \sup_{p \text{ vertex of } \Delta} \left| \sin(\theta_p) \right|,$$

where  $\theta_p$  is the angle at p of  $\Delta$ .

**Remark 2.** In particular, if  $\beta$  is any of the three angles of the triangle  $\Delta$ , we have

 $2\theta(\Delta) \leq |\sin\beta| \leq \operatorname{str}(\Delta).$ 

We can now define

- The area  $\mathcal{A}(T)$  is the sum of the areas of all the triangles of T.
- The height of T is

$$\eta(T) = \sup_{\Delta \in \mathcal{T}_T} \eta(\Delta).$$

• The fatness of T is

$$\theta(T) = \min_{\Delta \in \mathcal{T}_T} \theta(\Delta).$$

• The straightness of *T* is

$$\operatorname{str}(T) = \min_{\Delta \in \mathcal{T}_T} \operatorname{str}(\Delta).$$

**Remark 3.** If  $R_{\Delta}$  denotes the circumradius of the triangle  $\Delta$ , we get the following:

$$\frac{\eta(\Delta)}{\operatorname{str}(\Delta)} = 2R_{\Delta}$$

2.2.2. Hausdorff distance between two subsets of  $\mathbb{R}^3$ 

The Hausdorff distance  $\Delta_{\text{Hauss}}$  between two subsets A and B of  $\mathbb{R}^3$  is

$$\delta_{\text{Hauss}}(A, B) = \max\left(\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\right).$$

2.2.3. Triangulated mesh closely inscribed in a smooth surface

- We say that a triangulated mesh of  $\mathbb{R}^3$  is inscribed in a smooth surface S if all its vertices belong to S.
- A triangulated mesh T is closely inscribed in a smooth surface S if
  - (1) all the vertices of T belong to S,
  - (2) all the vertices of  $\partial T$  belong to  $\partial S$ ,
  - (3)  $\xi$  is defined on *T*,
  - (4)  $\xi_{|T}$  is bijective.
- Let *T* be an oriented triangulated mesh closely inscribed in a smooth surface *S*. Every  $m \in T$  belongs to *d* triangles  $(d \ge 1)$ . Let  $N_1, \ldots, N_d$  denote the unitary normals to all those triangles. Let

$$\alpha_m = \sup_{1 \leqslant i \leqslant d} \left| \left( N_i, \widehat{N}^S_{\xi(m)} \right) \right|,$$

where  $N_{\xi(m)}^S$  is the normal of *S* at  $\xi(m)$ . So we can define the real number  $\alpha = \sup_{m \in T} \alpha_m$ .  $\alpha$  is called *the maximal angle between the normals of S and T*.

# 3. Results

## 3.1. Approximation of the area of a smooth surface

The following result shows that the knowledge of a triangulated mesh closely inscribed in a smooth surface can give an estimation of its area with respect to the fatness of the triangulated mesh and the maximal angle  $\alpha$  between the normals defined above.

**Theorem 1.** Let S be a smooth compact oriented surface of  $\mathbb{R}^3$  and T a triangulated mesh closely inscribed in S. If the maximal angle a between the normals of S and T is less than  $\pi/6$ , then the area  $\mathcal{A}(S)$  of S satisfies the following inequality:

$$\left(1 - \frac{3\tan^2\alpha}{(1 - \tan^2\alpha)\theta(T)}\right)\mathcal{A}(T) \leqslant \mathcal{A}(S) \leqslant \frac{1}{\cos(\alpha)}\left(1 + \frac{3\tan^2\alpha}{(1 - \tan^2\alpha)\theta(T)}\right)\mathcal{A}(T),$$

where  $\theta(T)$  is the fatness of T and  $\mathcal{A}(T)$  the area of T.

#### 3.2. Approximation of the normals of a smooth surface

Let *S* be a smooth surface and *T* a triangulated mesh closely inscribed in *S*. The map  $\xi$  (defined in Proposition 1) induces a bijection between any triangle  $\Delta$  of *T* and  $\xi(\Delta) \subset S$ .

The following result compares the normals of a triangle  $\Delta$  with the normals of  $\xi(\Delta)$ . The upper bound depends on the triangle  $\Delta$ , on the curvature of  $\xi(\Delta)$  and on the reach of  $\xi(\Delta)$ .

**Theorem 2.** Let *S* be a smooth compact oriented surface of  $\mathbb{R}^3$ ,  $\Delta$  a triangle inscribed in *S*, such that the map  $\xi$  induces a bijection between  $\Delta$  and  $\xi(\Delta) \subset S$ . If

(1)  $\eta_{\Delta} < r_{\xi(\Delta)}$ ,

(2)  $\operatorname{str}(\Delta) \ge 20(1 + \operatorname{str} 2)\rho_{\xi(\Delta)}\eta_{\Delta}$ ,

then for every  $p \in \Delta$ , the angle  $\alpha_p$  between a normal to the triangle  $\Delta$  and the normal to S at the point  $\xi(p)$  satisfies

$$\sin(\alpha_p) \leqslant \eta_{\Delta} \rho_{\xi(\Delta)} \left( 2 + \frac{20}{\operatorname{str}(\Delta) - 20\eta_{\Delta} \rho_{\xi(\Delta)}} \right)$$

where  $\eta_{\Delta}$  is the height of  $\Delta$ ,  $\rho_{\xi(\Delta)}$  is the maximal curvature of  $\xi(\Delta)$ ,  $r_{\xi(\Delta)}$  is the reach of  $\xi(\Delta)$  and str( $\Delta$ ) the straightness of  $\Delta$ .

We notice that the assumptions of Theorem 2 are always satisfied if the height  $\eta_{\Delta}$  of the triangle is small enough and if the straightness str( $\Delta$ ) is large enough. More precisely, the assumptions are satisfied if  $\eta_{\Delta}/\text{str}(\Delta)$  is small compared to  $1/\rho_{\xi(\Delta)}$ . Thanks to Remark 3, we see that this condition means that the circumradius  $R_{\Delta}$  of the triangle  $\Delta$  is small compared to  $1/\rho_{\xi(\Delta)}$ . In the context of surface reconstruction with an  $\varepsilon$ -sample, N. Amenta et al. (see [3]) already have a similar result: they prove that  $R_{\Delta}$  is small compared to the *local feature size* (which is linked to the curvature of *S*) and they deduce that the normals of the triangulated mesh are close to the normals of the smooth surface *S*.

The following corollary is global (it concerns the whole surface *S*).

**Corollary 1.** Let S be a smooth compact oriented surface of  $\mathbb{R}^3$  and T a triangulated mesh closely inscribed in S. If

(1) 
$$\eta_T < r_S$$
,

(2) 
$$str(T) \ge 20(1 + \sqrt{2})\rho_S \eta_T$$
,

then the maximal angle  $\alpha$  between S and T satisfies

$$\sin(\alpha) \leqslant \eta_T \rho_S \bigg( 2 + \frac{20}{\operatorname{str}(T) - 20\eta_T \rho_S} \bigg),$$

where  $\eta_T$  is the height of T,  $\rho_S$  is the maximal curvature of S,  $r_S$  is the reach of S and str(T) the straightness of T.

Remark that in [8], J. Fu assumes that the fatness of a sequence of triangulated meshes which converge to a surface is bounded from below by a strictly positive constant. In Corollary 1, we do not need the fatness to be quite large: we only need the straightness to be large enough, which is a weaker condition (cf. Remark 2) and leads to a much more precise approximation. For instance, the fatness of a right-angled triangle can be very small and its straightness is always 1.

Oddly, the upper bound is better when triangles are right-angled and not equilateral. This is due to the straightness. Remark that a condition on the *fatness* would have implied that the upper bound is better when triangles are equilateral.

Theorem 2 can be refined in a more complicated but more general and precise approximation as follows.

**Theorem 3.** Let S be a smooth compact oriented surface of  $\mathbb{R}^3$  and  $\Delta$  a triangle inscribed in S, such that  $\xi$  induces a bijection between  $\Delta$  and  $\xi(\Delta)$ . In  $\eta_{\Delta}$  and  $\delta_{\Delta}$  are small enough and str( $\Delta$ ) is large enough to satisfy the following conditions:

(1)  $\eta_{\Delta} < r_{\xi(\Delta)}$ ,

(2)  $4 \operatorname{str}(\Delta) (1 - \delta_{\Delta} \rho_{\xi(\Delta)})^4 - \rho_{\xi(\Delta)}^2 \eta_{\Delta}^2 - 4\rho_{\xi(\Delta)} \eta_{\Delta} > 0$ , then  $\delta_{\Delta} \rho_{\xi(\Delta)} < 1$  and  $\alpha$  the maximal angle between  $\xi(\Delta)$  and  $\Delta$  satisfies

$$\sin(\alpha) \leqslant \eta_{\Delta} \rho_{\xi(\Delta)} \left( \frac{1}{1 - \delta_{\Delta} \rho_{\xi(\Delta)}} + \frac{\rho_{\xi(\Delta)} \eta_{\Delta} + 4}{4 \operatorname{str}(\Delta)(1 - \delta_{\Delta} \rho_{\xi(\Delta)})^4 - \rho_{\xi(\Delta)}^2 \eta_{\Delta}^2 - 4\rho_{\xi(\Delta)} \eta_{\Delta}} \right),$$

where  $\eta_{\Delta}$  is the height of  $\Delta$ ,  $\delta_{\Delta}$  the Hausdorff distance between  $\Delta$  and  $\xi(\Delta)$ ,  $\rho_{\xi(\Delta)}$  the maximal curvature of  $\xi(\Delta)$ ,  $r_{\xi(\Delta)}$  the reach of  $\xi(\Delta)$  and str $(\Delta)$  the straightness of  $\Delta$ .

#### 3.3. An example

We wish to approximate the normals and the area of a smooth surface S. We do not need any parameterization of S. In applications, we have no parameterization of the smooth surface. We make the assumption that this surface is smooth, that its curvature is bounded by a known constant  $\rho_s$  and that its reach is bounded by a known constant  $r_s$ .

Furthermore there is a triangulated mesh which is supposed to be closely inscribed in that surface and we make the assumption that the Hausdorff distance between the two surfaces is less than  $\delta$ . Thus we can use our results to approximate the area and the normals of the smooth surface.

In that example, we have an image of a surface. This image has been obtained by slightly modifying a well-known parameterized surface and we therefore do not know any parameterization of the surface. An estimation of an upper bound of  $\delta$ ,  $\rho_s$  and  $r_s$  could be



Fig. 2. Example. (a) Smooth surface. (b) Triangulated mesh.

Furthermore a calculation on the triangulated mesh gives

 $\begin{cases} \eta_T < 0.075, \\ \text{str}(T) > 0.98, \\ \theta(T) > 0.21. \end{cases}$ 

Then Theorem 3 tells us that the normals of the smooth surface are close to the normals of the triangulated mesh and that the maximal angle  $\alpha$  between the two surfaces is less than  $0.013\pi$ .

Furthermore Theorem 1 tells us that the area of the smooth surface is more than 4.68 and less than 4.89.

**Remark 4.** By definition,  $1/\rho_S$  bounds from below all the radii of curvature of the surface. Geometrically, this means that a sphere  $S(m, 1/\rho_S)$  of radius  $1/\rho_S$  tangent to the smooth surface S at a point m does not intersect S in a neighborhood of m (except at the point m itself). Using this, we have a geometrical idea of  $\rho_S$ .

## 4. Proofs

## 4.1. Proof of Theorem 1

We need the following proposition:

**Proposition 1.** Let S be a smooth surface parameterized by the map

 $F: U \subset \mathbb{R}^2 \to \mathbb{R}^3.$ 

For every  $m \in S$ , let  $\alpha_m$  denote the angle between the normal  $N_m^S$  of S at m and the vertical  $(Oz_0)(\alpha_m \in [0, \pi/2])$ . If

$$\sup_{m\in S} |\alpha_m| \leqslant \alpha < \frac{\pi}{2},$$

then

$$\mathcal{A}(P(S)) \leq \mathcal{A}(S) \leq \frac{1}{\cos(\alpha)} \mathcal{A}(P(S)).$$

where P(S) is the orthogonal projection of S onto  $\mathbb{R}^2 \times \{0\}$ ,  $\mathcal{A}(P(S))$  the area of P(S) and  $\mathcal{A}(S)$  the area of S.

**Proof of Proposition 2.** Let  $m = F(x, y) \in S$ . Let  $N_m^S$  be the normal vector defined at m by

$$N_m^S = N(x, y) = \partial_x F(x, y) \wedge \partial_y F(x, y) = \begin{pmatrix} \beta(x, y) \\ \gamma(x, y) \\ \delta(x, y) \end{pmatrix}.$$

The area  $\mathcal{A}(S)$  of S is given by (see [6] for instance)

$$\mathcal{A}(S) = \int_{U} \sqrt{\beta^2 + \gamma^2 + \delta^2} \, \mathrm{d}x \, \mathrm{d}y.$$



Fig. 3.  $\Delta$ ,  $\xi(\Delta)$  and  $P(\xi(\Delta))$ .

On the other hand, let *P* be the orthogonal projection onto  $\mathbb{R}^2 \times \{0\}$ , and define the application  $\widetilde{F} = P \circ F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ . If we define by  $D\widetilde{F}$  the differential of  $\widetilde{F}$ , we get

$$\mathcal{A}(P(S)) = \mathcal{A}(\widetilde{F}(U)) = \int_{U} |\det(D\widetilde{F})| \, \mathrm{d}x \, \mathrm{d}y = \int_{U} |\delta| \, \mathrm{d}x \, \mathrm{d}y.$$

• Clearly,

$$\mathcal{A}(S) \ge \int_{U} |\delta| \, \mathrm{d}x \, \mathrm{d}y = \mathcal{A}(P(S)).$$

• On the other hand, remark that

$$\frac{\sqrt{\beta^2 + \gamma^2}}{|\delta|} = \left| \tan(\alpha_m) \right| \leq \tan(\alpha).$$

Then

$$\mathcal{A}(S) = \int_{U} \sqrt{1 + \left(\frac{\sqrt{\beta^2 + \gamma^2}}{|\delta|}\right)^2} |\delta| \, \mathrm{d}x \, \mathrm{d}y \leqslant \frac{1}{\cos \alpha} \int_{U} |\delta| \, \mathrm{d}x \, \mathrm{d}y.$$

Finally,

$$\mathcal{A}(P(S)) \leq \mathcal{A}(S) \leq \frac{1}{\cos(\alpha)} \mathcal{A}(P(S)).$$

We now give the proof of Theorem 1. Let  $\Delta$  be a triangle of the triangulated mesh T.  $\xi_{|\Delta}$  is a bijection between  $\Delta$  and  $\xi(\Delta)$ . Suppose without restriction that  $\Delta \subset \mathbb{R}^2 \times \{0\}$ , and denote by P the orthogonal projection onto  $\mathbb{R}^2 \times \{0\}$ . Let  $N^{\Delta}$  denote the oriented normal of  $\Delta$ ,  $N^S_{\xi(m)}$ , the normal of S at  $\xi(m)$  and  $\alpha_m$  denote the angle  $(N^{\Delta}, \widehat{N}^S_{\xi(m)})$ . 1. *Comparison between*  $\mathcal{A}(\xi(\Delta))$  and  $\mathcal{A}(P(\xi(\Delta)))$ . Using the last proposition, we have

$$\mathcal{A}(P(\xi(\Delta))) \leq \mathcal{A}(\xi(\Delta)) \leq \frac{1}{\cos(\alpha)} \mathcal{A}(P(\xi(\Delta)))$$



Fig. 4. Comparison of  $\Delta$  with  $P(\xi(\Delta))$ .

2. *Comparison between*  $\mathcal{A}(\Delta)$  *and*  $\mathcal{A}(P(\xi(\Delta)))$ . Let *s* be a vertex of  $\Delta$ . We get  $P(\xi(s)) = s.$ 

We take a point *m* to of  $\Delta$  and we let  $m' = P(\xi(m))$ . The angle  $\xi(m)sm'$  is bounded from above by  $\alpha$ . Therefore,

$$m'\xi(m) = \tan(\xi(m)sm')sm' \leq \tan(\alpha)sm' \leq \tan(\alpha)(sm + mm')$$
  
=  $\tan(\alpha)(sm + \tan(\alpha_m)m'\xi(m)) \leq \tan(\alpha)(sm + \tan(\alpha)m'\xi(m)),$ 

then

$$m'\xi(m) \leqslant \frac{\tan \alpha}{1 - \tan^2 \alpha} sm$$
 and  $mm' = \tan(\alpha_m)m'\xi(m) \leqslant \frac{\tan^2 \alpha}{1 - \tan^2 \alpha} sm \leqslant \frac{\tan^2 \alpha}{1 - \tan^2 \alpha} \eta_{\Delta}$ .

where  $\eta_{\Delta}$  is the length of the longest side of  $\Delta$ . Let  $d_{\Delta} = \frac{\tan^2 \alpha}{1 - \tan^2 \alpha} \eta_{\Delta}$ . Since  $\alpha \leq \pi/6$ , we have  $\frac{\tan^2 \alpha}{1 - \tan^2 \alpha} \leq 1$ . Then we get

 $\begin{cases} mm' \leq d_{\Delta}, \\ mm' \leq sm & \text{for every vertex } s \text{ of } \Delta. \end{cases}$ 

It implies that if m belongs to an edge A of  $\Delta$ , then m' belongs to the rectangle  $D_A$  (Fig. 4). This implies

$$\begin{aligned} \left| \mathcal{A}(\Delta) - \mathcal{A}(P(\xi(\Delta))) \right| &\leq d_{\Delta} \text{per}(\Delta) \leq d_{\Delta} 3\eta_{\Delta} \\ &= \frac{\tan^2 \alpha}{1 - \tan^2 \alpha} 3\eta_{\Delta}^2 = \frac{3 \tan^2 \alpha}{(1 - \tan^2 \alpha)\theta(\Delta)} \mathcal{A}(\Delta), \end{aligned}$$

where  $\theta(\Delta) = \mathcal{A}(\Delta)/\eta_{\Delta}^2$  is the fatness of  $\Delta$  and per( $\Delta$ ) the perimeter of  $\Delta$ .

Since  $\theta(T)$  is the fatness of the triangulated mesh, we have

$$\left|\mathcal{A}(\Delta) - \mathcal{A}(P(\xi(\Delta)))\right| \leq \frac{3\tan^2 \alpha}{(1 - \tan^2 \alpha)\theta(T)}\mathcal{A}(\Delta).$$

Consequently,

$$\left(1 - \frac{3\tan^2\alpha}{(1 - \tan^2\alpha)\theta(T)}\right)\mathcal{A}(\Delta) \leqslant \mathcal{A}(\xi(\Delta)) \leqslant \frac{1}{\cos(\alpha)}\left(1 + \frac{3\tan^2\alpha}{(1 - \tan^2\alpha)\theta(T)}\right)\mathcal{A}(\Delta).$$

Since  $\xi$  induces a bijection between the mesh T and the smooth surface S, we obtain the expected result by adding areas of all triangles  $\Delta$  and of all surfaces  $\xi(\Delta)$ .

## 4.2. Proof of Theorem 3

Since Theorem 3 directly implies Theorem 2 and Corollary 1, we only prove Theorem 3 in that section. The proof follows from Proposition 3 (Section 4.2.2) and from Proposition 4 (Section 4.2.3). Let  $\Delta$  be a triangle of *T*, *p* a vertex of  $\Delta$  and *s* a point in  $\Delta$ . The proof of Theorem 3 needs two steps.

- (1) In a first step, we compare the normal  $N_p^S$  of the smooth surface *S* at *p* (which is a vertex of  $\Delta$ ) with the normal  $\vec{N}_{\Delta}$  of  $\Delta$  (Section 4.2.2).
- (2) Then, we compare the two normals  $N_p^S$  and  $N_{\xi(s)}^S$  of S at the points p and  $\xi(s)$  (Section 4.2.3).

First of all, we need to compare a geodesic on a smooth surface with its chord (Section 4.2.1).

4.2.1. Comparing the lengths of a geodesic and of its chord

In the following, if u is a linear map, we denote by |u| its norm defined by

$$|u| = \sup_{X \neq 0} \frac{||u(X)||}{||X||}.$$

We denote by  $D\xi$  the differential of  $\xi$ .

**Lemma 1.** Let *S* be a smooth oriented surface without boundary and  $U_S$  be a neighborhood where  $\xi$  is well defined. For all  $m \in U_S$ , if  $m \in S \setminus \partial S$  then  $\xi$  is differentiable at *m* and we have

$$\left|D\xi(m)\right| \leq \frac{1}{1 - \|\xi(m) - m\|\rho_{\xi(m)}},$$

where  $\rho_{\xi(m)}$  is the maximal curvature of *S* at  $\xi(m)$ .

The calculation of the differential of  $\xi$  can be found in [9] in the case where *S* is the boundary of a convex set of  $\mathbb{R}^3$ .

**Proof of Lemma 1.** Let  $m \in \Delta$ . Let  $N_{\xi(m)}$  denote the unitary oriented normal to S at  $\xi(m)$ . We have

$$N_{\xi(m)} = \varepsilon \frac{\xi(m) - m}{\|\xi(m) - m\|} \quad \text{with } \varepsilon \in \{-1, +1\},$$

thus

$$\xi(m) - m = \langle \xi(m) - m, N_{\xi(m)} \rangle N_{\xi(m)}.$$

Then we obtain for all  $X \in T_m U_S$ ,

$$D\xi(m)X = X + \langle \xi(m) - m, N_{\xi(m)} \rangle DN_{\xi(m)} (D\xi(m)X) + \langle D\xi(m)X - X, N_{\xi(m)} \rangle N_{\xi(m)} + \langle \xi(m) - m, DN_{\xi(m)} (D\xi(m)X) \rangle N_{\xi(m)} = X + \varepsilon \|\xi(m) - m\| DN_{\xi(m)} (D\xi(m)X) - \langle X, N_{\xi(m)} \rangle N_{\xi(m)} = pr_{|T_{\xi(m)}S}(X) + \varepsilon \|\xi(m) - m\| DN_{\xi(m)} (D\xi(m)X),$$

where  $pr_{|T_{\xi(m)}S}(X)$  the orthogonal projection of X onto  $T_{\xi(m)}S$ . Furthermore,  $m \in U_S$  implies that  $||\xi(m)m||$  is smaller than the radius of curvature. Then  $||\xi(m) - m||\rho_{\xi(m)} < 1$ . Thus

$$\|D\xi(m)X\| \le \|X\| + \|\xi(m) - m\|\rho_{\xi(m)}\|D\xi(m)X\| \quad \text{and} \quad \|D\xi(m)X\| \le \frac{\|X\|}{1 - \|\xi(m) - m\|\rho_{\xi(m)}\|}$$

Lemma 2. Let S be a smooth compact surface, p and q two points on S. If

$$pq \leq r \quad and \quad \xi(]p,q[) \subset S \setminus \partial S, \quad then \ l_{pq} \leq \frac{1}{1 - \partial \rho_S} pq,$$

where  $l_{pq}$  is the length between p and q on S,  $\rho_S$  the maximal curvature of S, r is the reach of S and  $\delta = \sup_{x \in [a,b]} d(x, S)$ .

**Proof of Lemma 2.**  $pq \leq r$  implies that  $[p,q] \subset U_S$ .  $\xi([p,q])$  is a curve on S. Its length is larger than the length  $l_{pq}$  of the geodesic whose ends are p and q. Thus

$$l_{pq} \leq l(\xi([p,q])) \leq \sup_{m \in [p,q]} |D\xi(m)| pq \leq \frac{1}{1 - \partial \rho_S} pq.$$

4.2.2. Comparing the normals at a vertex

**Lemma 3.** Let *S* be a smooth compact oriented surface of  $\mathbb{R}^3$ ,  $\Delta$  a triangle whose vertices p, q, r belong to *S*. Let R > 0 be such that

- (1) the length  $l_1$  on S between p and q satisfies  $l_1 R \leq pq$ ,
- (2) the length  $l_2$  on S between p and r satisfies  $l_2 R \leq pr$ .

If we suppose that

$$4\left|\sin(\theta_p)\right|R^4 - \rho_S^2\eta_\Delta^2 - 4\rho_S\eta_\Delta R > 0,$$

then the angle  $\alpha_p$  between the normals of S and  $\Delta$  at the point p satisfies

$$\sin(\alpha_p) \leqslant \frac{\rho_S^2 \eta_\Delta^2 + 4\rho_S \eta_\Delta R}{4|\sin(\theta_p)|R^4 - \rho_S^2 \eta_\Delta^2 - 4\rho_S \eta_\Delta R}$$

where  $\rho_S$  is the maximal curvature of S,  $\eta_{\Delta}$  the height of the triangle  $\Delta$  and  $\theta_p$  the angle of  $\Delta$  at p.

**Proof of Lemma 3.** Let  $c_1$  denote a geodesic of S linking p and q.  $c_1$  is parameterized by arc length by

$$\gamma_1: [0, l_1] \to S,$$

with  $\gamma_1(0) = p$  and  $\gamma_1(l_1) = q$ . A simple calculation gives

$$\gamma_1(l_1) - \gamma_1(0) = l_1 \gamma_1'(0) + \int_0^{l_1} (l_1 - t) \gamma_1''(t) dt,$$

thus

$$\gamma_1'(0) = \frac{\vec{pq}}{l_1} - \frac{1}{l_1} \int_0^{l_1} (l_1 - t) \gamma_1''(t) \, \mathrm{d}t.$$

Let  $\vec{u}_1 = \frac{\vec{p}q}{l_1}$ ,  $\vec{v}_1 = \frac{1}{l_1} \int_0^{l_1} (l_1 - t) \gamma_1''(t) dt$  and  $\vec{e}_1 = \frac{\vec{p}q}{pq}$ . We have  $\|\vec{u}_1\| \le 1$ .

$$\gamma_1'(0) = \vec{u}_1 - \vec{v}_1 \quad \text{with } \begin{cases} \|\vec{u}_1\| \leqslant 1, \\ \|\vec{v}_1\| \leqslant \frac{l_1 \rho_S}{2}. \end{cases}$$

Similarly, if we denote by  $c_2$  a geodesic of *S* linking *p* and *r* and parameterized by arc length by  $\gamma_2$ , we get

$$\gamma_2'(0) = \vec{u}_2 - \vec{v}_2 \quad \text{with } \begin{cases} \|\vec{u}_2\| \leq 1, \\ \|\vec{v}_2\| \leq \frac{l_2 \rho_S}{2}. \end{cases}$$

The normal  $N_p^S$  to the smooth surface S at the point p is proportional to the vector  $\gamma'_1(0) \wedge \gamma'_2(0)$ .

$$\gamma_1'(0) \wedge \gamma_2'(0) = \vec{u}_1 \wedge \vec{u}_2 + \vec{v}_1 \wedge \vec{v}_2 - \vec{u}_1 \wedge \vec{v}_2 - \vec{v}_1 \wedge \vec{u}_2 = \vec{\omega}_1 + \vec{\omega}_2,$$
  
with 
$$\begin{cases} \vec{\omega}_1 = \vec{u}_1 \wedge \vec{u}_2, \\ \vec{\omega}_2 = \vec{v}_1 \wedge \vec{v}_2 - \vec{u}_1 \wedge \vec{v}_2 - \vec{v}_1 \wedge \vec{u}_2. \end{cases}$$

Thus,

$$\sin(\alpha_p) = \sin(\gamma_1'(0) \wedge \widehat{\gamma_2'}(0), \vec{\omega}_1) = \frac{\|(\vec{\omega}_1 + \vec{\omega}_2) \wedge \vec{\omega}_1\|}{\|\vec{\omega}_1 + \vec{\omega}_2\| \|\vec{\omega}_1\|} = \frac{\|\vec{\omega}_2 \wedge \vec{\omega}_1\|}{\|\vec{\omega}_1 + \vec{\omega}_2\| \|\vec{\omega}_1\|} \le \frac{\|\vec{\omega}_2\|}{\|\|\vec{\omega}_1\| - \|\vec{\omega}_2\|\|}$$

But we have

$$\|\vec{\omega}_{1}\| = \left\|\vec{e}_{1} \wedge \vec{e}_{2} \frac{pq}{l_{1}} \frac{pr}{l_{2}}\right\| = \left|\sin(\theta_{p})\right| \frac{pq}{l_{1}} \frac{pr}{l_{2}},$$
  
and  $\|\vec{\omega}_{2}\| \leq \frac{\rho_{S}l_{1}}{2} \frac{\rho_{S}l_{2}}{2} + \frac{\rho_{S}l_{1}}{2} + \frac{\rho_{S}l_{2}}{2}.$ 

The assumptions of Lemma 3 lead to

 $Rl_1 \leqslant \eta_\Delta$  and  $Rl_2 \leqslant \eta_\Delta$ .

This yields

$$\|\vec{\omega}_1\| \ge |\sin(\theta_p)|R^2$$
 and  $\|\vec{\omega}_2\| \le \left(\frac{\rho_S \eta_\Delta}{2R}\right)^2 + \frac{\rho_S \eta_\Delta}{R}.$ 

Thanks to the assumption, we have

$$\|\vec{\omega}_1\| - \|\vec{\omega}_2\| \ge \frac{1}{4R} \left( 4 \left| \sin(\theta_p) \right| R^4 - \rho_S^2 \eta_\Delta^2 - 4\rho_S \eta_\Delta R \right) > 0.$$

This immediately implies

$$\sin(\alpha_p) \leqslant \frac{\|\vec{\omega}_2\|}{\|\|\vec{\omega}_1\| - \|\vec{\omega}_2\|\|} \leqslant \frac{\rho_S^2 \eta_\Delta^2 + 4\rho_S \eta_\Delta R}{4|\sin(\theta_p)|R^4 - \rho_S^2 \eta_\Delta^2 - 4\rho_S \eta_\Delta R}.$$

**Proposition 3.** Let S be a smooth compact oriented surface of  $\mathbb{R}^3$ ,  $\Delta$  a triangle closely inscribed in S and p a vertex of  $\Delta$ . If

(1)  $\eta_{\Delta} < r$ ,

(2)  $4|\sin(\theta_p)|(1-\delta\rho_S)^4 - \rho_S^2\eta_{\Delta}^2 - 4\rho_S\eta_{\Delta} > 0$ ,

then the angle  $\alpha_p$  between the normals of S and  $\Delta$  at the point p satisfies

$$\sin(\alpha_p) \leqslant \frac{\rho_S^2 \eta_\Delta^2 + 4\rho_S \eta_\Delta}{4|\sin(\theta_p)|(1 - \delta\rho_S)^4 - \rho_S^2 \eta_\Delta^2 - 4\rho_S \eta_\Delta}$$

where  $\rho_S$  is the maximal curvature of S,  $\eta_{\Delta}$  the height of the triangle  $\Delta$ , r the reach of S,  $\theta_p$  the angle at p of  $\Delta$  and  $\delta$  the Hausdorff distance between  $\Delta$  and  $\xi(\Delta)$ .

**Remark 5.** The interpretation of this result is the following: the angle  $\alpha_p$  is quite small if  $|\sin(\theta_p)|$  is large enough with respect to the product  $\rho_S \eta_{\Delta}$ .

**Proof of Proposition 3.** We use the notations of Lemma 3. We are going to bound the lengths  $l_1$  and  $l_2$  of the two geodesics by using Lemma 2. As  $\eta_{\Delta} < r$ , we have

$$Rl_1 \leq pq$$
 and  $Rl_1 \leq pr$ ,

with  $R = 1 - \delta \rho_S$ . Thanks to Lemma 3, we obtain that

$$\sin(\alpha_p) \leqslant \frac{\rho_S^2 \eta_\Delta^2 + 4\rho_S \eta_\Delta R}{4|\sin(\theta_p)|R^4 - \rho_S^2 \eta_\Delta^2 - 4\rho_S \eta_\Delta R}$$
$$\leqslant \frac{\rho_S^2 \eta_\Delta^2 + 4\rho_S \eta_\Delta}{4|\sin(\theta_p)|R^4 - \rho_S^2 \eta_\Delta^2 - 4\rho_S \eta_\Delta}$$
$$= \frac{\rho_S^2 \eta_\Delta^2 + 4\rho_S \eta_\Delta}{4|\sin(\theta_p)|(1 - \delta\rho_S)^4 - \rho_S^2 \eta_\Delta^2 - 4\rho_S \eta_\Delta}.$$

4.2.3. Comparison of the normals of the smooth surface

**Lemma 4.** Let S be a smooth compact oriented surface of  $\mathbb{R}^3$ , a and b two points of S. Then

$$\sin(\alpha_{ab}) \leqslant \rho_S L_S(a,b),$$

where  $\alpha_{ab}$ , is the angle between the normals of S at a and at b,  $\rho_S$  the maximal curvature of S and  $L_S(a, b)$  the length on S between a and b.

**Proof of Lemma 4.** Using the mean-value theorem we have

$$\left\|N_{a}^{S}-N_{b}^{S}\right\| \leq |DN|_{\infty}L_{S}(a,b) = \rho_{S}L_{S}(a,b).$$

Thus,

$$\sin(\alpha_{a,b}) \leq 2\sin\left(\frac{\alpha_{a,b}}{2}\right) = \left\|N_a^S - N_b^S\right\| \leq \rho_S L_S(a,b). \qquad \Box$$

**Proposition 4.** Let S be a smooth compact oriented surface of  $\mathbb{R}^3$ ,  $\Delta$  a triangle closely inscribed in S, p and s two points on  $\Delta$ .

If  $\delta \rho_S < 1$ , then the angle  $\alpha_{sp}$  between the two normals  $N^S_{\xi(p)}$  and  $N^S_{\xi(s)}$  satisfies

$$\sin(\alpha_{sp}) \leqslant \frac{\rho_S \eta_\Delta}{1 - \delta \rho_S},$$

where  $\eta_{\Delta}$  is the height of  $\Delta$  and  $\delta$  the Hausdorff distance between T and S.

Proof of Proposition 4. Lemma 4 implies

 $\sin(\alpha_{ps}) \leqslant \rho_S L_S(\xi(p), \xi(s)).$ 

 $L_{S}(\xi(p), \xi(s))$  is smaller than the length  $L(\xi([p, s]))$  of the curve  $\xi([p, s])$  which joins  $\xi(p)$  and  $\xi(s)$  on S. Thus,

$$L_{\mathcal{S}}(\xi(p),\xi(s)) \leq L(\xi([p,s])) \leq \sup_{m \in \Delta} |D\xi(m)| ps.$$

Lemma 1 implies

$$\sin(\alpha_{sp}) \leqslant \rho_S \sup_{m \in \Delta} \left| D\xi(m) \right| ps \leqslant \frac{\rho_S \eta_\Delta}{1 - \delta \rho_S}. \qquad \Box$$

## 4.2.4. End of proof of Theorem 3

The proof of this theorem uses Propositions 3 and 4.

Let  $s \in \Delta$  and p be a vertex of  $\Delta$ . The angle  $\alpha_a$  is less than  $\alpha_p + \alpha_{ps}$ . Thus we get

$$\sin(\alpha_s) \leqslant \sin(\alpha_p) + \sin(\alpha_{ps}) \leqslant \tan(\alpha_p) + \sin(\alpha_{ps}) \\ \leqslant \eta_{\Delta} \rho_{\xi(\Delta)} \left( \frac{1}{1 - \delta_{\Delta} \rho_{\xi(\Delta)}} + \frac{\rho_{\xi(\Delta)} \eta_{\Delta} + 4}{4 \operatorname{str}(\Delta) (1 - \delta_{\Delta} \rho_{\xi(\Delta)})^4 - \rho_{\xi(\Delta)}^2 \eta_{\Delta}^2 - 4 \rho_{\xi(\Delta)} \eta_{\Delta}} \right).$$

#### 5. Conclusion and perspectives

The knowledge of a triangulated mesh closely inscribed in a smooth surface gives an approximation of the normals and of the area of the smooth surface if we make assumptions on an upper bound  $\rho_S$  on the smooth surface's curvature, on an upper bound  $r_S$  on its reach and on an upper bound  $\delta$  on its distance from the triangulated mesh.

The upper bound on the error of the normals is better when the number  $\eta_{\Delta} \times \rho_{\xi(\Delta)}$  is small. It implies that when the curvature of the smooth surface is large, we need a lot of points in the surface.

Furthermore, this upper bound is better when the straightness is large, that is to say when triangles are almost right-angled (even if there are small angles).

This paper deals with approximations of invariants of order zero (the area) and of order one (the normals). It is interesting to ask whether we can obtain approximation of invariants of second order (i.e., the curvatures).

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