Discrete Comput Geom 36:393–418 (2006)
DOI: 10.1007/s00454-006-1255-x

Discrete & Computational
Geometry

Unfolding of Surfaces*

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Abstract. This paper deals with the approximation of the unfolding of a smooth globally developable surface (i.e. "isometric" to a domain of $\mathbb{E}^2$) with a triangulation. We prove the following result: let T_n be a sequence of globally developable triangulations which tends to a globally developable smooth surface S in the Hausdorff sense. If the normals of T_n tend to the normals of S, then the shape of the unfolding of T_n tends to the shape of the unfolding of S.

We also provide several examples: first, we show globally developable triangulations whose vertices are close to globally developable smooth surfaces; we also build sequences of globally developable triangulations inscribed on a sphere, with a number of vertices and edges tending to infinity. Finally, we also give an example of a triangulation with strictly negative Gauss curvature at any interior point, inscribed in a smooth surface with a strictly positive Gauss curvature. The Gauss curvature of these triangulations becomes positive (at each interior vertex) only by switching some of their edges.

1. Introduction

Developable surfaces appear naturally in several problems of geometric modeling, such as the modeling of developable strata in structural geology [16], the modeling of clothes [17] and so on. . . . In some of these applications the quality of the shape of the unfolding

* This work was partially supported by the IST Programme of the EU as a Shared-cost RTD (FET Open) Project under Contract No. IST-2000-26473 (ECG—Effective Computational Geometry for Curves and Surfaces), and by Région Rhônes Alpes, Contract 99029744.

is important. It is therefore natural to consider the problem of developable surfaces construction and also to validate the geometric shape of the unfolding of a surface.

A smooth surface is usually said to be *developable* if its Gauss curvature is identically equal to zero. In this paper we say that a smooth surface is *globally developable* if it is globally isometric to a domain of plane $\mathbb{E}^2$. Notice that the *Teorema Egregium of Gauss* implies that a globally developable surface is developable. Similarly a triangulation is said to be *globally developable* if there exists a homeomorphism from the triangulation onto a domain of plane $\mathbb{E}^2$ that preserves distances. Notice that such a triangulation is also developable, in the sense that its discrete Gauss curvature is equal to zero.

We are interested in this article in the following question: if two globally developable surfaces are geometrically close to one another, does it imply that their unfoldings have a "similar shape"? In the main part of this paper we consider a smooth globally developable surface S, and we suppose that there exists a sequence of developable triangulations T_n which tends to S in the Hausdorff sense. We show that if the normals of T_n tend to the normals of S, then the unfolding of T_n tends to the unfolding of S in the Hausdorff sense (up to a motion of the plane).

Notice that the unfolding of a triangulation can be very different from the unfolding of a smooth surface even if both surfaces are very close for the Hausdorff distance. The unfolding of the *half Schwarz lantern* convinces us easily. (This problem appears in many applications, such as geology, where people want to unfold strata under isometric deformations. At a certain scale, the stratum can be considered as a smooth surface, approximated by a triangulation.)

The problem of developable surface reconstruction from scattered sample points $\mathcal{S}$ has already been studied. First, notice that we can always build an infinity of cylindrical triangulations that contain a given sample of points $\mathcal{S}$ (as explained in Section 3). However, these triangulations can be geometrically very far in the Hausdorff sense from the set $\mathcal{S}$. Peternell [13] has recently proposed an algorithm that allows us to build a developable smooth surface close to a set $\mathcal{S}$ of sample points. Thibert et al. [16] have also proposed an algorithm of surface reconstruction which is based on the geometrical properties of developable smooth surfaces (the set of sample points $\mathcal{S}$ belongs to different level curves). Their algorithm allows us to build globally developable triangulations in simple cases (if $\mathcal{S}$ is included in simple developable surfaces, like cones or cylinders). In more complicated situations, they build triangulations that are "almost developable" (in the sense that the Gauss curvature is very small). The problem of developable surface reconstruction from scattered sample points $\mathcal{S}$ is still open and difficult, although it has been solved in particular cases.

We provide in this paper some examples of globally developable triangulations whose vertices are close to globally developable smooth surfaces. Besides the *half Schwarz lantern*, we propose an analogous triangulation (the *Schwarz cone*) whose vertices belong to a cone. We also use the program detailed in [16] to build two simple examples.

We also devote a section to contradicting the mistaken belief that the geometry of a smooth surface is better and better approximated if the triangulation approximation has a bigger and bigger amount of vertices on it. We construct a sequence of globally developable triangulations whose cardinal of vertices and edges goes to infinity, and which are all inscribed on a portion of a sphere of fixed radius. We end this paragraph with an example of a triangulation with a strictly negative Gauss curvature at each interior vertex, incribed on a convex smooth surface, which has the following property: by switching some edges and keeping its vertices fixed, it is still inscribed on the same smooth surface but is now positively curved at each interior vertex.

It is important to notice that our approach is different from the works of Sheffer and de Sturler [14], Desbrun et al. [3] and Lévy et al. [8], who give explicit algorithms to unfold triangulations which are not necessarily globally developable. These authors minimize an energy [3], [8] related to the distortion of the angles of the triangles. However, there is no underlying smooth surface in their work, and then no comparison between smooth and discrete unfoldings is possible. On the contrary, we give a theorem which compares the unfolding of a smooth globally developable surface with the unfolding of a globally developable triangulation.

The notion of the *reach of a smooth surface* is one of the main tools of this paper. It allows us to compare a smooth surface with a triangulation "close to it". It was first introduced by Federer [5]. It is interesting to notice that the *reach* is in fact linked to the (more recent) notions of *medial axis* and *local feature size*, which are used in some problems of reconstructing a surface from scattered sample points. In [18] Wolter gives many interesting results related to the *medial axis* and the *cut locus*.

This paper is organized as follows. Section 2 gives classical and usual definitions. Section 3 gives examples of globally developable triangulations. Section 4 states the main result (Theorem 1), which deals with the approximation of the shape of the unfolding of a smooth surface. Section 5 gives some "bad examples'" concerning the Gauss curvature of smooth surfaces and inscribed triangulations. The other sections sketch the proofs of results: Sections 6 and 7 prove Theorem 1 (Section 6 gives intermediate results of the approximation of the lengths of curves; Section 7 gives a result about plane geometry). Section 8 gives the proof of the existence of the bad examples of Section 5.

2. Definitions

We recall here some classical definitions which concern smooth surfaces, triangulations and the relative position of two surfaces. For more details on smooth surfaces, one may refer to [1], [4] or [15]. For more details on triangulations, one may refer to [2], [5], [6] or [10].

2.1. *Smooth Surfaces*

In the following a smooth surface means a $\mathcal{C}^2$ surface which is regular, oriented, compact with or without boundary. Let S be a smooth surface of the (oriented) euclidean space $\mathbb{E}^3$. Let ∂S denote the boundary of S. S is endowed with the Riemannian structure induced by the standard scalar product of $\mathbb{E}^3$. We denote by da the area form on S and by ds the canonical orientation of ∂S. Let ν be the unit normal vector field (compatible with the orientation of S) and let h be the second fundamental form of S associated with ν. Its determinant at a point p of S is *the Gauss curvature* G_p, its trace is *the mean curvature* H_p. *The maximal curvature of* S *at* p is $\rho_p = \max(|\lambda_p^1|, |\lambda_p^2|)$, where λ_p^1 and λ_p^2 are the principal curvatures of S at p (that is, the eigenvalues of the second fundamental form).

The maximal curvature of S is

$$\rho_S = \sup_{p \in S} \rho_p.$$

We denote by k_p *the geodesic curvature* of ∂S at p.

We need the following:

Proposition 1. *Let S be a smooth compact surface of* $\mathbb{E}^3$. *Then there exists an open set* U_S *of* $\mathbb{E}^3$ *containing S and a continuous map* ξ *from* U_S *onto S satisfying the following: if p belongs to* U_S, *then there exists a unique point* $\xi(p)$ *realizing the distance from* p *to* S (ξ *is nothing but the orthogonal projection onto* S).

A proof of this proposition can be found in [5].

The open set U_S depends locally and globally on the smooth surface S. Locally, it is related to the curvature. Globally, U_S depends on points which can be far from one another on the surface, but close in $\mathbb{E}^3$.

We shall also need the notion of the *reach of a surface*, introduced by Federer in [5].

Definition 1. The reach of a surface S is the largest $r > 0$ for which ξ is defined on the open tubular neighborhood $U_r(S)$ of radius r of S.

Note that the reach r_S of S is smaller than the minimal radius of curvature of S (which is $1/\rho_S$) (see [9] or [18] for more details). Thus, we have

$$\rho_S r_S \leq 1,$$

where ρ_S is the maximal curvature of S.

2.2. *Triangulations*

2.2.1. *Generalities*.

A triangulation T is a (finite and connected) union of triangles of $\mathbb{E}^3$, such that the intersection of two triangles is either empty, or equal to a vertex or equal to an edge.

We denote by $\mathcal{T}_T$ the set of triangles of T and by Δ a generic triangle of T.

- η_Δ denotes the length of the longest edge of Δ, and $\mathcal{A}(\Delta)$ the area of Δ.
- The *area* $\mathcal{A}(T)$ is the sum of the areas of all the triangles of T.

For the following, we need to define a new geometric invariant on the triangulation:

Definition 2. Let Δ be a triangle of a triangulation T.

- The *rightness* of a Δ is the real number

$$rig(\Delta) = \sup_{p \text{ vertex of } \Delta} \sin(\theta_p),$$

where θ_p is the angle at p of Δ.

- The *rightness* of T is

$$rig(T) = \min_{\Delta \in \mathcal{T}_T} rig(\Delta).$$

Remark 1. In particular, if β is any of the three angles of the triangle Δ, we have:

$$\sin \beta \leq rig(\Delta).$$

2.2.2. *Triangulation Close to a Smooth Surface.*

- A triangulation (or a smooth surface) M is *closely near* a smooth surface S if:
  1. M lies in $U_r(S)$, where r is the reach of S,
  2. the restriction of ξ to M is one-to-one (where ξ is the map defined in Proposition 1).
- We say that a triangulation of $\mathbb{E}^3$ is *inscribed* in a smooth surface S if all its vertices belong to S.
- A triangulation T is *closely inscribed* in a smooth surface S if:
  1. T is closely near S,
  2. all the vertices of T belong to S.
- Let T be a triangulation closely near a smooth surface S. Let m be a point lying in the interior of a triangle Δ of T. Let N^{Δ} be the normal line through m to Δ. We put

$$\alpha_m = \langle N^{\Delta}, \nu^S_{\xi(m)} \rangle \in \left[0, \frac{\pi}{2}\right].$$

The real number α_m is defined almost everywhere on T. (If m is a point on an edge or a vertex, one can define α_m by taking the supremum of the angles between the triangles which contain m and the normal $\nu^S_{\xi(m)}$.)

We can define the real number

$$\alpha = \sup_{m \in T} \alpha_m.$$

α is called *the maximal angle between the normals of S and T*.

We now introduce an invariant which relates the triangulation and the smooth surface:

Definition 3. Let T be a triangulation (or just a triangle) closely near a smooth surface S. *The relative curvature of S to T* is the real number defined by

$$\omega_S(T) = \sup_{m \in T \setminus \partial T} \|\xi(m) - m\| \rho_{\xi(m)}.$$

Remark 2. A compact triangulation T closely near a smooth surface S satisfies

$$\omega_S(T) \leq 1.$$

(If $m \in T$, then the two points m and $m - 2\overrightarrow{m\xi(m)}$—which is the orthogonal symmetric of m with respect to the tangent plane of S at $\xi(m)$—belong to $U_r(S)$. This implies that

$\|\xi(m) - m\|$ is strictly less than the radius of curvature $1/\rho(\xi(m))$ of S at the point $\xi(m)$, see [9] or [18] for more details).

Moreover, a triangulation T closely inscribed in a smooth surface S satisfies

$$\omega_S(T) \leq \sup_{\Delta \in \mathcal{T}_T} \eta_\Delta \rho_{\xi(\Delta)}.$$

(In fact, if m belongs to a triangle Δ, then $\|\xi(m) - m\|$ is smaller than the distance from m to any point of S. If s is a vertex of Δ, we have $\|\xi(m) - m\| \leq ms \leq \eta_\Delta$.)

2.2.3. *Gauss Curvature of a Triangulation*. Let T denote a triangulation, p a vertex of T and $\mathcal{T}_T(p)$ the set of triangles of T which contain p as a vertex. Let S_T^o denote the set of interior vertices of T and $S_{\partial T}$ the set of vertices of the boundary ∂T of T.

- We call the *angle at the vertex* p the real:

$$\alpha_T(p) = \sum_{\sigma \in \mathcal{T}_T(p)} \alpha_\sigma(p),$$

  where $\alpha_\sigma(p)$ is the angle at p to the triangle σ.
- The *discrete Gauss curvature at a vertex* $p \in S_T^o$ is

$$G_T(p) = 2\pi - \alpha_T(p).$$

- The *discrete geodesic curvature at a vertex* $p \in S_{\partial T}$ is

$$k(p) = \pi - \alpha_T(p).$$

- The *total interior Gauss curvature* of T is

$$G_{int}(T) = \sum_{p \in S_T^o T} G_T(p) = \sum_{p \in S_T^o T} (2\pi - \alpha_T(p)).$$

- The *total geodesic curvature* of ∂T is

$$\mathcal{K}(\partial T) = \sum_{p \in S_{\partial T}} k(p) = \sum_{p \in S_{\partial T}} (\pi - \alpha_T(p)).$$

2.3. *Developable Surfaces*

A smooth surface (resp. a triangulation) M is usually said to be *developable* if its Gauss curvature is null at each interior point (resp. vertex).

In this paper a surface M is said to be *globally developable* if there exists a homeomorphism $g_M\colon M \to u(M)$ from M onto a domain $u(M) \subset \mathbb{E}^2$ that preserves distances (a domain of the plane $\mathbb{E}^2$ is either an open connected set of $\mathbb{E}^2$ or the closure of an open connected set of $\mathbb{E}^2$). To be more precise, if M is a smooth surface, we suppose that the map g_M is a $\mathcal{C}^2$-diffeomorphism.

Note that the *theorema egregium* of Gauss implies that the Gauss curvature of a *globally developable* smooth surface is identically equal to zero. Similarly, if M is a *globally developable* triangulation, its Gauss curvature is equal to zero.

The surface $u(M)$ is called an *unfolding* of M.

2.4. *Hausdorff Distance between Two Subsets of* $\mathbb{E}^3$

The *Hausdorff distance* between two subsets A and B of $\mathbb{E}^3$ is

$$\delta_{\text{Haus}}(A, B) = \max\left(\sup_{x\in A} d(x, B), \sup_{y\in B} d(A, y)\right).$$

Remark 3. A compact triangulation T closely near a smooth surface S satisfies

$$\omega_S(T) \leq \delta_{\text{Haus}}(T, S)\rho_S.$$

3. Examples of Globally Developable Surfaces

In this section we consider the following problem: given a set $\mathcal{S} = \{p_1, \ldots, p_n\}$ of points of $\mathbb{E}^3$, is it possible to build a globally developable triangulation T which is "close" (for example in the Hausdorff sense) to $\mathcal{S}$? Suppose now that such a globally developable triangulation T exists and that the set of points $\mathcal{S}$ belongs to a globally developable surface S. Then another question arises: does the unfolding of T give a good estimation of the shape of the unfolding of S?

First, notice that it is always possible to build a globally developable triangulation T which contains the points of $\mathcal{S}$. To do so, we consider a plane $\mathcal{P}$, a line $\mathcal{D}$ and the projection pr onto $\mathcal{P}$ and parallel to $\mathcal{D}$. Note that there exists a polygonal curve $\mathcal{C}$ of the plane $\mathcal{P}$ which contains $\{pr(p_1), \ldots, pr(p_n)\}$ and that the cylinder generated by the directrix $\mathcal{C}$ and whose rulings are parallel to $\mathcal{D}$ is piecewise linear and contains all the points of $\mathcal{S}$. We can therefore build a triangulation T (included in this cylinder) which contains $\mathcal{S}$ (see Fig. 1). However, the Hausdorff distance between T and $\mathcal{S}$ can be very large (some points of the triangulation T may be "far" from $\mathcal{S}$). We also notice that we could build another globally developable triangulation $\widetilde{T}$ containing $\mathcal{S}$ (for example by using another projection $\widetilde{pr}$), such that the Hausdorff distance between T and $\widetilde{T}$ is large and such that the unfoldings of T and $\widetilde{T}$ are very different from one another.

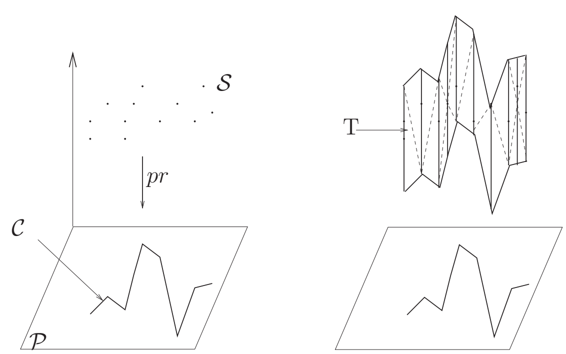


Fig. 1. Developable triangulation containing a set $\mathcal{S}$ of data points.

The problem of developable surface reconstruction has already been studied: Peternell [13] proposed an algorithm that enables us to build a smooth developable surface close to a set of data points. However, his algoritm does not allow us to build globally developable surfaces in the general case (the set of data points has to satisfy a criterion). In [16] Thibert et al. proposed an algorithm of construction of triangulations that mimicks the geometrical properties of developable smooth surfaces. They also proposed an algorithm that "slightly" modifies a triangulation so as to improve its developability. Their algorithms allow us to build globally developable triangulations in some particular cases, but there is no warranty in the general case.

Building a globally developable triangulation T close to a set of data points of $\mathbb{E}^3$ in the Hausdorff sense is still an open problem, which is difficult since it is constrained both locally and globally: locally, the discrete Gauss has to vanish and globally the problem is constrained by the Gauss–Bonnet theorem. In this section, we give several examples of globally developable triangulations:

- In Section 3.1 we describe the *half Schwarz lantern*. This example is well known for the non-convergence of the area. Since it is a globally developable triangulation, it is also a counterexample of the approximation of the shape of the unfolding of a smooth surface.
- In Section 3.2 we define the *Schwarz cone* whose vertices belong to a cone (the construction mimicks the construction of the *half Schwarz lantern*).
- In Section 3.3, using the program described in [16], we build two triangulations close to cones or cylinders.

3.1. *Half Schwarz Lantern*

We consider the following situation: S is a smooth globally developable surface and T is a globally developable triangulation close to S. We aim at knowing whether the unfolding of T gives a good estimation of the shape of the unfolding of S. A nice example of this situation is the famous *Schwarz lantern*. It is a developable triangulation inscribed on a cylinder. It is not simply connected, but we can "cut a piece of it" which is homeomorphic to a topological disc: we consider here a *half Schwarz lantern* (which is inscribed in half a cylinder) (Fig. 2). We illustrate two phenomena:

- In Section 3.1.1 we give two *half Schwarz lanterns* whose unfoldings are very different from one another. Therefore we cannot expect to have a result of convergence without other assumptions.
- In Section 3.1.2 we build two triangulations which have the same vertices but whose unfoldings are very different from one another. That implies that the shape of the unfolding depends on the construction of a triangulation from scattered sample points.

3.1.1. *Comparison of Two Half Schwarz Lanterns.* Let C be a half-cylinder of finite height H and of radius R. It can be parametrized by

$$\forall t \in [0, \pi], \quad \forall u \in [0, H], \qquad f(t, u) = (R\cos(t), R\sin(t), u).$$

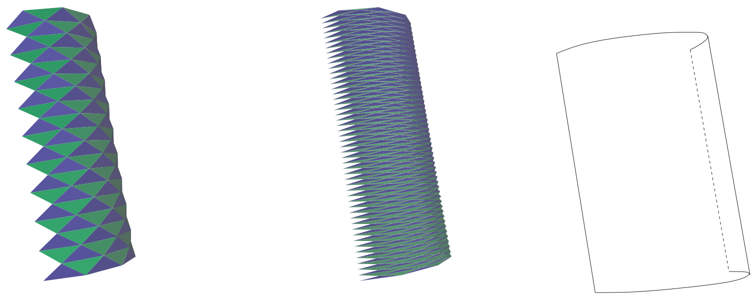


Fig. 2. Examples of *half Schwarz lanterns*.

Let $P(n, N)$ denote the triangulation whose vertices $S_{i,j}$ belong to C and are defined as follows:

$$\forall i \in \{0, \ldots, n-1\}, \qquad S_{i,j} = (R\cos(i\alpha), R\sin(i\alpha), jh) \text{ if } j \text{ is even},$$
$$\forall j \in \{0, \ldots, N\}, \qquad S_{i,j} = (R\cos(i\alpha + \tfrac{\alpha}{2}), R\sin(i\alpha + \tfrac{\alpha}{2}), jh) \text{ if } j \text{ is odd},$$

and whose faces are

$$S_{i,j} S_{i+1,j} S_{i,j+1},$$
$$S_{i,j} S_{i-1,j+1} S_{i,j+1},$$

where $\alpha = \pi/n$ and $h = H/N$.

Those triangulations $P(n, N)$ are called *half Schwarz lanterns*.

Proposition 2. *Half Schwarz lanterns are globally developable.*

Proof of Proposition 2. First, notice that all the triangles of $P(n, N)$ are isometric to one another. Therefore, for every interior vertex p of $P(n, N)$, the angle $\alpha_{P(n,N)}(p)$ at the vertex p is equal to twice the sum of the three angles of one triangle, that is to say 2π. That implies that the Gauss curvature $G_{P(n,N)}(p)$ is equal to zero and that $P(n, N)$ is developable.

Let us prove that $P(n, N)$ is globally developable, i.e. there is no overlap during the unfolding. Let $j \in \{0, \ldots, N-1\}$. Let us consider the set $Strip_j$ of all the triangles of $P(n, N)$ whose vertices have jh or $(j+1)h$ as a third component. Let $j \in \{1, \ldots, n-1\}$. Since all the triangles are isometric to one another, the sum of the angles of the triangles of $Strip_j$ at the vertex $S_{i,j}$ is equal to π. This implies that the vertices $S_{0,j}, \ldots, S_{n,j}$ of the unfolding of $Strip_j$ belong to the same line segment L_j. Similarly the vertices $S_{0,j+1}, \ldots, S_{n,j+1}$ of the unfolding of $Strip_j$ belong to the same line segment L_{j+1}. Note, furthermore, that the two line segments L_j and L_{j+1} are parallel (this is also due to the fact that all the triangles are isometric to one another). Therefore the unfolding of $Strip_j$ is a trapezoid delimited by the two parallel line segments L_j and L_{j+1}.

Now consider two consecutive sets of triangles $Strip_j$ and $Strip_{j+1}$ (Fig. 3). Their unfoldings do not overlap and share the common line segment L_{j+1}. Since the line

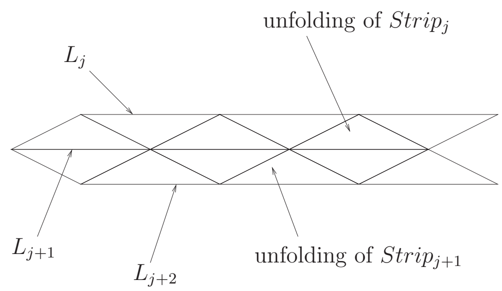


Fig. 3. Proof of the developability of the *half Schwarz lantern*.

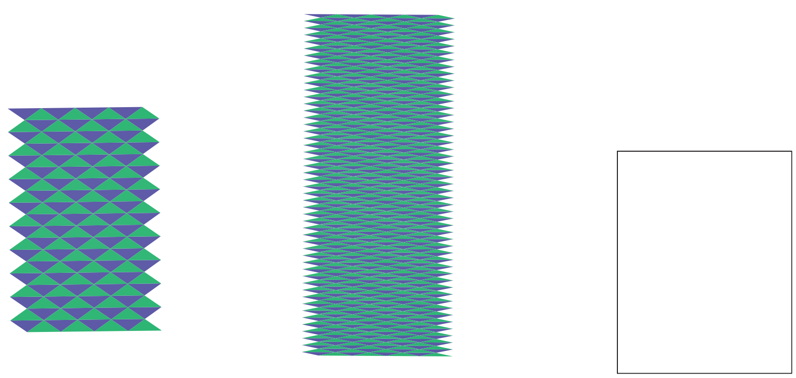
segments $L_0, \ldots, L_{N+1}$ are parallel to each other, the unfolding of $P(n, N)$ is made of a union of trapezoids that do not overlap. $P(n, N)$ is therefore globally isometric to a domain. □

It is well known that we can construct a *half Schwarz lantern* whose area is as large as we wish (see [1] or [11]). That implies that its unfolding can be "very different" from the unfolding of the half-cylinder C.

Note that the boundaries of the two unfolded *half Schwarz lanterns* of Fig. 4 are very different from one another and can be very different from the unfolding of the half-cylinder C. The unfolding of C is a rectangle of height H. The height of the unfolded *half Schwarz lanterns* $P(99, 5)$ is more than $1.5H$. In fact, the height of a *half Schwarz lantern* increases when it is unfolded.

Furthermore, if we consider the problem of the convergence of a sequence of triangulations, we may notice that the height of the unfolding of the *half Schwarz lantern* $P(n, n^3)$ tends to infinity when n tends to infinity.

That is why, without other assumptions, we cannot expect the unfolding of a sequence of triangulations to give us a good approximation of the unfolding of the smooth surface.



(a) Unfolded P(19,5) (b) Unfolded P(99,5) (c) Unfolded half cylinder

Fig. 4. Unfolding of C and of two *half Schwarz lanterns* closely inscribed in C (the scale is the same).

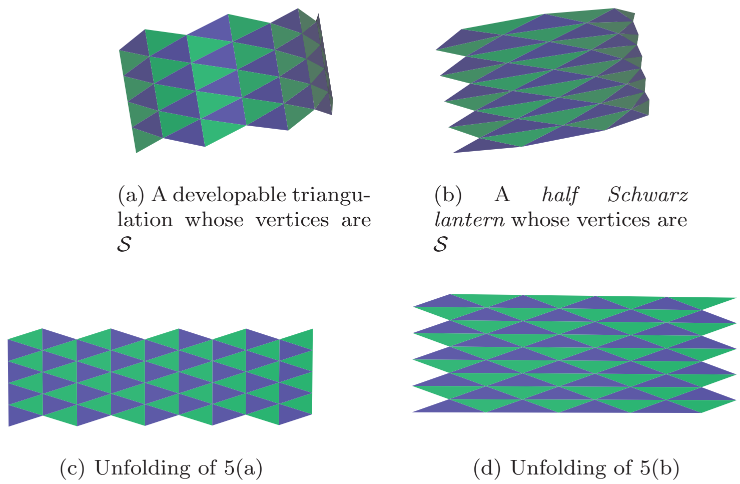


Fig. 5. Unfolding of two triangulations which have the same set of vertices $\mathcal{S}$.

As we will see in Section 4, this is linked to the fact that the normals of $P(n, n^3)$ do not converge to the normals of the half-cylinder C (or that the rightness of $P(n, n^3)$ tends to zero).

3.1.2. *Two Globally Developable Triangulations with the Same Vertices*. We consider a finite family of points $\mathcal{S}$ (which belong to a half-cylinder) and we build two triangulations whose vertices are these points (Fig. 5). The triangulation shown in Fig. 5(b) is a *half Schwarz lantern*. The two unfoldings are different from one another.

Every internal edge of the triangulation shown in Fig. 5(a) is contained in two triangles which form a quadrilateral with the edge as a diagonal. Just notice that if we delete the "vertical" internal diagonals and we replace them by the other diagonals, we get the triangulation shown in Fig. 5(b) (if we do not consider the boundary of the two surfaces).

3.2. *Schwarz Cone*

The *Schwarz cone* is a triangulation (denoted by $C(n, N)$) whose construction is inspired by the *half Schwarz lantern*. The vertices belong to a cone instead of a cylinder.

Detailed Construction. Let $\mathcal{C}$ be a cone whose center is the origin O, whose basis is the circle of equation $x^2 + y^2 + (z-1)^2 = 1$. Let r denote the rotation of axis (Oz) and of angle $2\pi/n$, let h_k denote the homothetic transformation of ratio k and center O, let $\mathcal{P}_h$ be the horizontal plane defined by $z = h$ and let $\mathcal{C}_h$ be the circle $\mathcal{C} \cap \mathcal{P}_h$.

Let $u_1, \ldots, u_n$ be n points regularly positioned on the circle $\mathcal{C}_1$. We consider a circle $\mathcal{C}_{z1}$ (with $0 < z1 < 1$). For every $i \in \{1, \ldots, n\}$, we put $v_i = r(h_{z1}(u_i))$. Let T_1 denote the "slice" composed of the $2n$ triangles $u_i v_i u_{i+1}$ and $v_i u_{i+1} v_{i+1}$ (by convention $u_{n+1} = u_1$ and $v_{n+1} = v_1$).

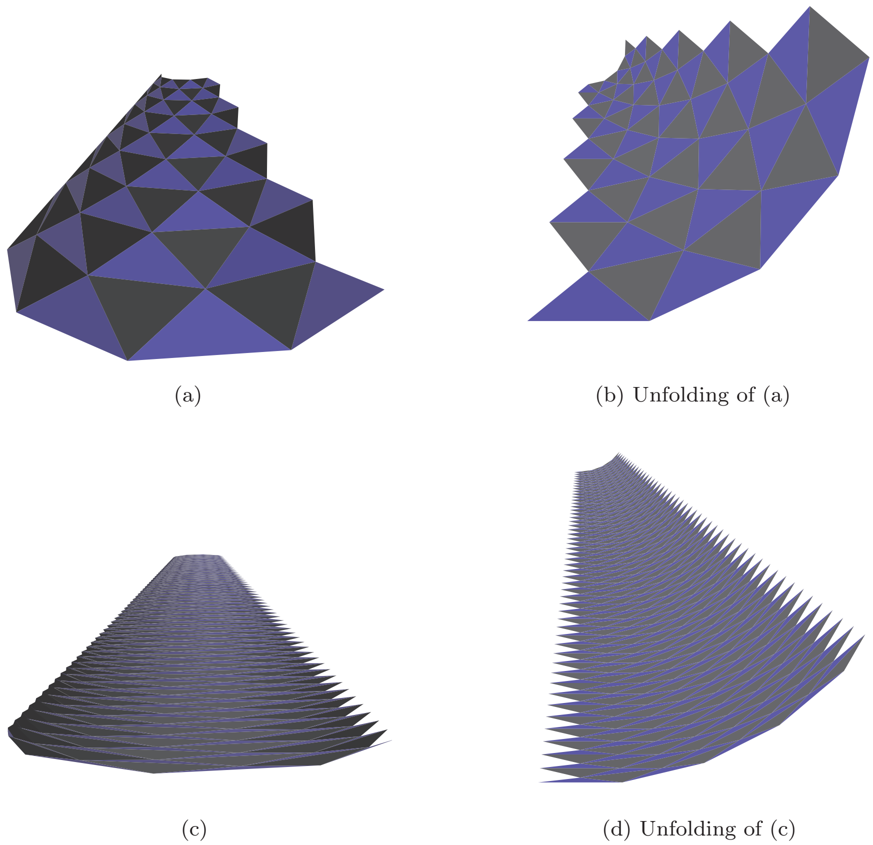


Fig. 6. Examples of Schwarz cone.

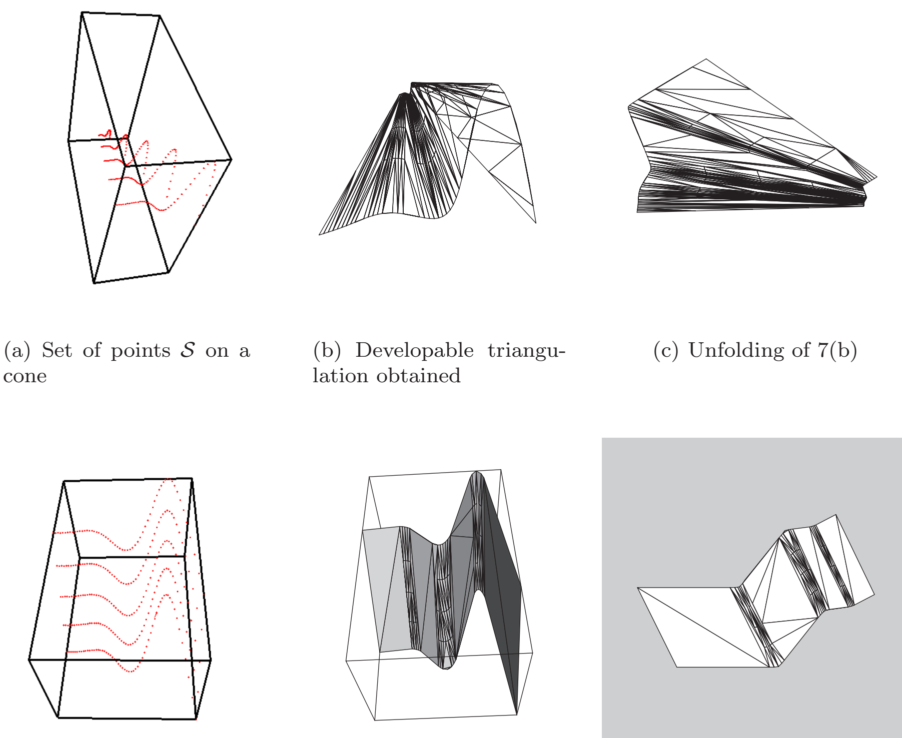
We now define the "slice" $T_2 = r(h_{z1}(T_1))$. The triangles of T_2 are homothetic to the triangles of T_1. This implies that the sum of angles of $T_1 \cup T_2$ at a vertex v_i is equal to 2π. The triangulation $T_1 \cup T_2$ is therefore developable.

Similarly, we define the "slices" $T_3, \ldots, T_N$ by induction in the following way: if the third components of the vertices of T_i are z_i and z_{i-1} (with $z_i < z_{i-1}$), then we put $T_{i+1} = r(h_{ki}(T_i))$ where $ki = z_i/z_{i-1}$.

As far as the *half Schwarz lantern* is concerned, we can show that the normals of $C(n, n^3)$ do not converge to the normals of the cone, and the unfolding of $C(n, n^3)$ does not converge to the unfolding of the cone (Fig. 6). This is also due to the fact that the rightness of $C(n, n^3)$ tends to zero (see Section 4).

3.3. *Cones and Cylinders*

In this part we use the algorithm of surface reconstruction described in [16]. In particular cases this algorithm allows us to build globally developable triangulations that are "close" to a set of data points $\mathcal{S}$ (when $\mathcal{S}$ is included in a set of level curves).



(a) Set of points $\mathcal{S}$ on a cone

(b) Developable triangulation obtained

(c) Unfolding of 7(b)

(d) Set of points $\mathcal{S}$ on a cylinder

(e) Developable triangulation

(f) Unfolding of 7(e)

Fig. 7. Globally developable triangulations built by the program detailed in [16].

We give two examples of globally developable triangulations (Fig. 7) generated by this program (when the set of level curves belongs to a cylinder or a cone). The triangulations obtained are globally developable and we unfold them.

4. Convergence of the Unfolding

The main result of this paper is the following theorem. It states that the unfolding $u(S)$ of a smooth surface S can be approximated by the shape of the unfolding of a triangulation T which is close to S in the Hausdorff sense and whose normals are close to the normals of S.

Theorem 1. *Let S be a smooth compact connected globally developable surface of $\mathbb{E}^3$ and let $(T_n)_{n\geq 0}$ be a sequence of globally developable triangulations closely near S such that*:

1. *the normals of T_n tend uniformly to the normals of S*,
2. *T_n tends to S in the Hausdorff sense*;

then the sequence $(u(T_n))_{n\geq 0}$ *of unfoldings of* $(T_n)_{n\geq 0}$ *tends in the Hausdorff sense to an unfolding* $u(S)$ *of* S.

The second condition may be weakened by asking that the relative curvature $\omega_S(T_n)$ tends to zero when n tends to infinity (in some sense, the Hausdorff distance between T_n and S may be large when the curvature of S is small). Furthermore, we know that the convergence of the normals is implied by a condition on the rightness when the vertices of triangulations belong to S [12]. Therefore we have the following corollary:

Corollary 1. *Let S be a smooth compact connected globally developable surface of* $\mathbb{E}^3$ *and let* $(T_n)_{n\geq 0}$ *be a sequence of globally developable triangulations closely inscribed in S such that*:

- *the rightness of the sequence* $(T_n)_{n\geq 0}$ *is uniformly bounded from below by a strictly positive constant*;
- *the lengths of the edges of* T_n *tend to zero when n tends to infinity*;

then the sequence $(u(T_n))_{n\geq 0}$ *of unfoldings of* $(T_n)_{n\geq 0}$ *tends in the Hausdorff sense to an unfolding* $u(S)$ *of* S.

5. Some Remarks on the Approximation by Developable Triangulations

The crucial assumption in the previous theorem is that both smooth surface and triangulation are developable. The goal of this section is to insist on the mistaken belief that one can get a good approximation of the Gauss curvature of a smooth surface by computing the Gauss curvature of an inscribed triangulation closely inscribed on it, and having a big amount of vertices. *In particular, the fact that the triangulation is developable does not imply at all that the smooth underlying surface has a "weak Gauss curvature" at some points.*

The following theorem gives a family of examples of developable triangulations (the Gauss curvature is thus zero at each interior vertex) closely inscribed in a piece of the sphere $\mathbb{S}^2(r)$ of radius $r > 0$.

Theorem 2. *Let* $n \geq 3$. *There exists* $\alpha_0 \in\,]0, 1]$ *such that for every* $\alpha \in\,]0, \alpha_0]$, *there exists a developable triangulation* T_α^n *satisfying*:

1. T_α^n *is closely inscribed in* S_α^n, *where* S_α^n *is "an open connected portion of sphere* $\mathbb{S}^2(r)$*"*;
2. T_α^n *contains* $(3n + 1)$ *vertices* ($(n + 1)$ *of them are interior*) *and* $4n$ *faces*.

Remark 4. The parameter α depends on the diameter of T_α^n: if α tends to zero, then the diameter of T_α^n tends to zero.

Since the Gauss curvature of S_α^n is $1/r^2$ at every interior point, the smooth surface S_α^n is not developable. However, the triangulation T_α^n, which is closely inscribed in S_α^n, is developable. That implies in the general case that the knowledge of the Gauss curvature of a triangulation closely inscribed in a smooth surface does not give enough information about the Gauss curvature of the smooth surface.

However, the shape of the unfolding of a triangulation T_α^n of Theorem 2 is "unusual": the discrete geodesic curvature at each vertex of the boundary is large and the normals between two adjacent triangles of T_α^n are not close to one another (and not close to the normals of the sphere). This seems to indicate that it is very difficult to build a developable triangulation closely and over a large region of sphere $\mathbb{S}^2(r)$ simultaneously.

This observation is coherent with the results of Fu [6], who showed that the convergence of the curvature measures of a sequence of triangulations which tends to a smooth surface in the Hausdorff sense was implied by a result of convergence of the normals.

Remark 5. The (developable) triangulations T_α^n are homeomorphic to a disc. In fact, there does not exist any compact developable triangulation without boundary closely inscribed in the whole sphere. This is an obvious consequence of the Gauss–Bonnet Theorem (see [4]): it states that the Euler characteristic $\chi(S)$ of a smooth compact surface S (whose boundary ∂S is composed of $C_1, \ldots, C_n$ positively oriented closed curves of class $\mathcal{C}^2$) satisfies

$$2\pi\chi(S) = \int_S G_p\, da(p) + \sum_{i=1}^{n} \int_{C_i} k_p\, ds(p) + \sum_{i=1}^{p} \theta_i,$$

where $\{\theta_1, \ldots, \theta_p\}$ is the set of all external angles of the curves $C_1, \ldots, C_n$.

The discrete analogous result for the Euler characteristic $\chi(T)$ of a triangulation T is the following:

$$2\pi\chi(T) = G_{\text{int}}(T) + \mathcal{K}(\partial T).$$

Since the Euler characteristic of a smooth surface S equals the Euler characteristic any triangulation T closely inscribed in it, one gets

$$\int_S G_p\, da(p) + \sum_{i=1}^{n} \int_{C_i} k_p\, ds(p) + \sum_{i=1}^{p} \theta_i = G_{\text{int}}(T) + \mathcal{K}(\partial T).$$

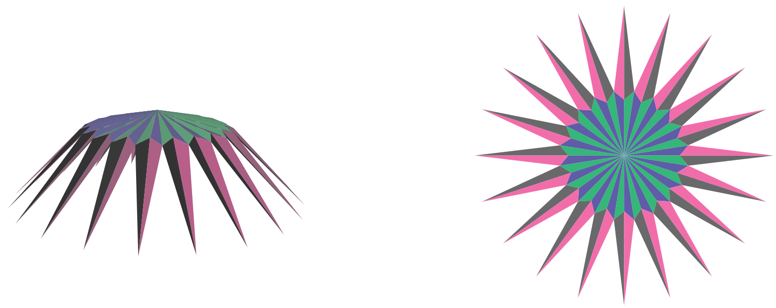
In particular, if S is a sphere, we have

$$G_{\text{int}}(T) = \int_S G_p\, da(p) = 4\pi \neq 0.$$

In Figs. 8–11 we present some of those triangulations T_α^n which are closely inscribed in a piece of sphere $\mathbb{S}^2$ and we unfold them. We use *Geomview* [7] to visualize the examples.

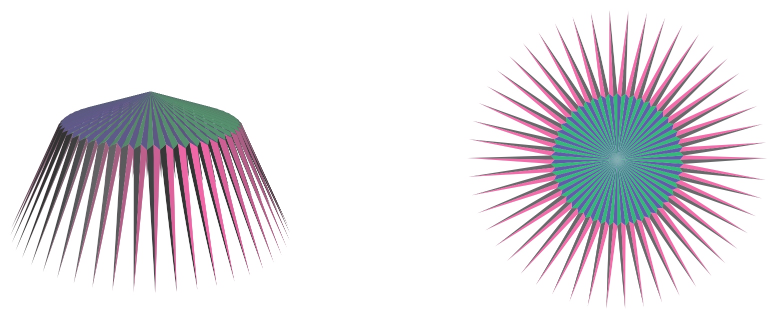
The triangulation of Fig. 12 is still inscribed in sphere $\mathbb{S}^2$ and the discrete Gauss curvature at each interior vertex is strictly negative ($G_T(p) \approx -0.02$ if p is the central vertex and $G_T(p) \approx -0.04$ otherwise). Thus we have a triangulation with strictly negative Gauss curvature inscribed in sphere $\mathbb{S}^2$.

The triangulation of Fig. 13 is developable and its boundary "*is quite regular*" in the sense that the discrete geodesic curvature at each vertex of the boundary is not too large.



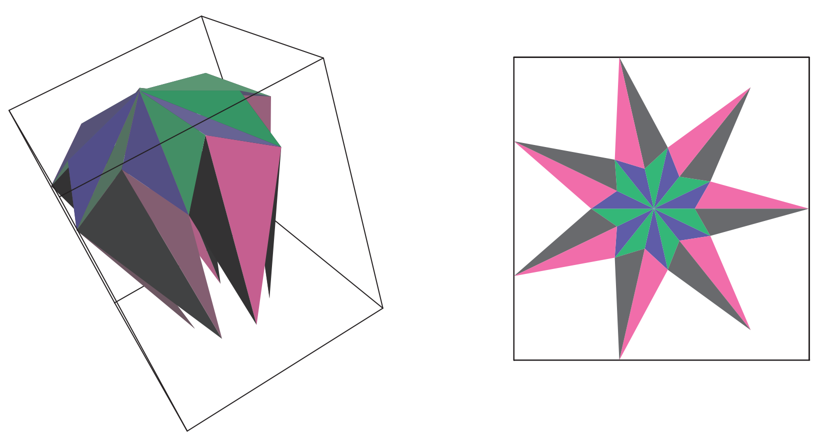
(a) triangulation (b) Unfolded triangulation

Fig. 8. The case $n = 20, \alpha = 0.4$.



(a) triangulation (b) Unfolded triangulation

Fig. 9. The case $n = 50, \alpha = 0.6$.



(a) triangulation (b) Unfolded triangulation

Fig. 10. The case $n = 7, \alpha = 0.6$.

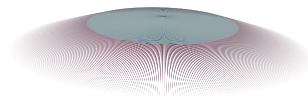


Fig. 11. The case $n = 500$, $\alpha = 0.2$.

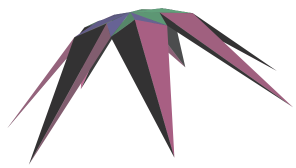
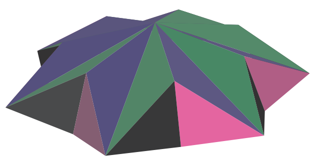
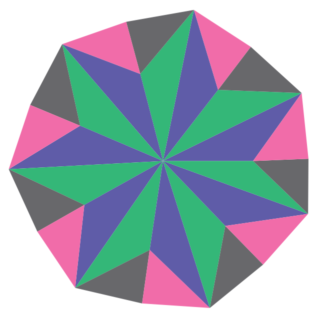


Fig. 12. Triangulation with negative Gauss curvature inscribed in $\mathbb{S}^2$.



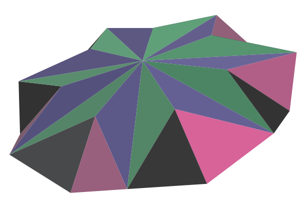
(a) Developable triangulation



(b) Unfolded triangulation

Fig. 13. Developable triangulation inscribed in a smooth surface with strictly positive Gauss curvature.

(a) Smooth surface with Gauss curvature larger than 0.5



(b) triangulation

Fig. 14. Triangulation of strictly negative Gauss curvature inscribed in a smooth surface with strictly positive Gauss curvature.

This triangulation is not inscribed in a sphere, but in a smooth surface of revolution, whose Gauss curvature is strictly positive at each interior point.

The triangulation of Fig. 14 is not developable. More precisely, the discrete Gauss curvature at each interior vertex is strictly negative (in fact $G_T(p) \leq -0.02$ at each interior vertex p). However, this triangulation is closely inscribed in a smooth surface of revolution, whose Gauss curvature is strictly positive at each interior point.

Thus we have a triangulation with strictly negative Gauss curvature inscribed in a smooth surface with strictly positive Gauss curvature.

6. Comparison of Shortest Paths

This section gives some intermediate results which are used in Section 7 to prove Theorem 1. The main result of this section is Proposition 3. Roughly speaking, it states that a triangulation closely near a smooth surface, whose normals are close enough to the normals of the smooth surface and which is close enough to the smooth surface, is "*almost isometric*" to it.

Proposition 3. *Let S be a smooth compact connected surface of $\mathbb{E}^3$ and let T be a triangulation closely near S. Then for every (locally lipschitz) curve $\mathcal{C}$ of T,*

$$\frac{\cos(\alpha)}{1+\omega_S(T)}\, L(\mathcal{C}) \leq L(\xi(\mathcal{C})) \leq \frac{1}{1-\omega_S(T)}\, L(\mathcal{C}),$$

where $L(\mathcal{C})$ is the length of the curve $\mathcal{C}$, $L(\xi(\mathcal{C}))$ is the length of the curve $\xi(\mathcal{C})$, $\omega_S(T)$ is the relative curvature of S to T and α is the maximal angle between the normals of S and T.

In the following we denote by $\|\cdot\|$ the euclidean norm. For every pair of points $u \in \mathbb{E}^3$ and $v \in \mathbb{E}^3$, we denote by $uv = \|u - v\|$ the euclidean distance between u and v. For every compact surface M, we denote by $\mathcal{C}_M(m_1, m_2)$ a shortest path of M joining two points m_1 and m_2 of M and by $d_M(m_1, m_2)$ the distance between m_1 and m_2 on M, i.e. the length of $\mathcal{C}_M(m_1, m_2)$.

Proposition 3 directly implies the following corollary which compares the shortest paths of S and T.

Corollary 2. *Let S be a smooth compact connected surface of $\mathbb{E}^3$ and let T be a triangulation closely near S. Then for every pair of points $a \in T$ and $b \in T$,*

$$\frac{\cos(\alpha)}{1+\omega_S(T)}\, d_T(a,b) \leq d_S(\xi(a),\xi(b)) \leq \frac{1}{1-\omega_S(T)}\, d_T(a,b).$$

If the two surfaces S and T are globally developable, we have the same inequalities with the two unfoldings. Therefore we have:

Corollary 3. *Let S be a smooth compact connected and globally developable surface of $\mathbb{E}^3$ and let T be a globally developable triangulation of $\mathbb{E}^3$ closely near S. Then there exists a homeomorphism $f\colon u(S) \to u(T)$ from an unfolding $u(S)$ of S onto an* *unfolding $u(T)$ of T which satisfies for every pair of points $a \in u(T)$ and $b \in u(T)$,*

$$\frac{\cos(\alpha)}{1+\omega_S(T)}\, d_{u(T)}(a,b) \le d_{u(S)}(f(a), f(b)) \le \frac{1}{1-\omega_S(T)}\, d_{u(T)}(a,b).$$

Corollary 3 leads us to introduce the following definition:

Definition 4. Let $\varepsilon \in [0, 1[$ and let U and V be two compact connected domains in $\mathbb{E}^2$. A homeomorphism $f: U \to V$ is an ε-isometry if it satisfies for every pair of points a and b in U,

$$(1-\varepsilon) d_U(a,b) \le d_V(f(a), f(b)) \le (1+\varepsilon) d_U(a,b).$$

Note that the map of Corollary 3 is an ε-isometry with $\varepsilon = (1+\omega_S(T))/\cos(\alpha) - 1$.

Remark 6. Just notice that if T and S are totally geodesic (that is, included in two planes and the angle between the normals is constant), then Proposition 3 leads to equalities ($\omega_S(T) = 0$). Indeed, if the curve $\mathcal{C} = [a, b]$ is parallel to the two surfaces T and S, then we have

$$L(\xi(\mathcal{C})) = d_S(\xi(a), \xi(b)) = \frac{1}{1-\omega_S(T)} d_T(a,b) = \frac{1}{1-\omega_S(T)} L(\mathcal{C}).$$

Now, if we take two points c and d in T such that $\tilde{\mathcal{C}} = [c, d]$ is orthogonal to $[a, b]$, then we have

$$L(\xi(\tilde{\mathcal{C}})) = d_S(\xi(c), \xi(d)) = \frac{\cos(\alpha)}{1+\omega_S(T)} d_T(c,d) = \frac{\cos(\alpha)}{1+\omega_S(T)} L(\tilde{\mathcal{C}}).$$

6.1. *Proof of Proposition* 3

The curve $\mathcal{C}$ is parametrized by a locally lipschitz map $\gamma: [a, b] \to \mathcal{C}$. We can suppose that γ is unit speed. The curve $\xi(\mathcal{C})$ is then parametrized by the map $\xi \circ \gamma$. By Rademacher's theorem [10], the map γ is differentiable almost everywhere. Therefore the Coarea Formula [10] implies that the lengths of these two curves (Fig. 15) (with the Lebesgues measure) are

$$L(\mathcal{C}) = \int_a^b \|\gamma'(t)\|\, dt \quad \text{and} \quad L(\xi(\mathcal{C})) = \int_a^b \|D\xi(\gamma(t)) \circ \gamma'(t)\|\, dt. \qquad (1)$$

The estimation of $L(\xi(\mathcal{C}))$ is based on the differential of the application ξ and uses the following proposition:

Proposition 4. *Let S be a smooth compact oriented surface of $\mathbb{E}^3$ and U_S a neighborhood of S where the map ξ is defined. For every $m \in U_S$, if $\xi(m) \in S \backslash \partial S$ then ξ is differentiable at m. If $\|\xi(m) - m\| \rho_{\xi(m)} < 1$, then for every $X \in T_m U_S$ we have*

$$\frac{\|pr_{|T_{\xi(m)}M}(X)\|}{1 + \|\xi(m) - m\|\rho_{\xi(m)}} \le \|D\xi(m)X\| \le \frac{\|X\|}{1 - \|\xi(m) - m\|\rho_{\xi(m)}},$$

where $pr_{|T_{\xi(m)}S}$ is the orthogonal projection onto $T_{\xi(m)}S$.

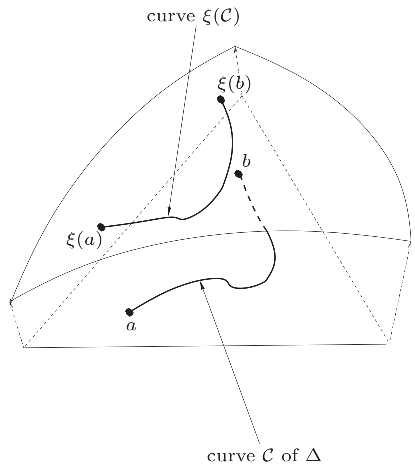


Fig. 15. $L(\mathcal{C})$ is the length of the curve $\mathcal{C}$ and $L(\xi(\mathcal{C}))$ is the length of the curve $\xi(\mathcal{C})$.

A proof of this proposition can be found in [12]. This implies that ξ is continuously differentiable at every point $m \in T$. This also implies that the boundary of S is piecewise continuously differentiable.

Remark 7. If the two principal curvatures of S at the the point $\xi(m)$ are opposite ($\lambda_1(\xi(m)) = -\lambda_2(\xi(m))$), then we have the equalities

$$|D\xi(m)|_\infty = \frac{1}{1 - \|\xi(m) - m\|\rho_{\xi(m)}}$$

and

$$\|D\xi(m)X\| = \frac{1}{1 + \|\xi(m) - m\|\rho_{\xi(m)}} \|pr_{|T_{\xi(m)}M}(X)\|.$$

The proof of Proposition 3 follows directly from (1) and the following lemma:

Lemma 1. *For almost every t, we have*

$$\frac{\cos(\alpha)}{1 + \omega_S(T)} \leq \|D\xi(\gamma(t)) \circ \gamma'(t)\| \leq \frac{1}{1 - \omega_S(T)}.$$

Proof of Lemma 1. We put $m = \gamma(t)$ and $X = \gamma'(t)$. Thanks to Proposition 4, we get

$$\|D\xi(m)X\| \leq \frac{1}{1 - \omega_S(T)},$$

and

$$\|D\xi(m)X\| \geq \frac{\|pr_{|T_{\xi(m)}S}(X)\|}{1 + \omega_S(T)}.$$

For almost every t, $X = \gamma'(t)$ belongs to a triangle Δ of T. Let α_m denote the angle between a normal to the triangle Δ and the normal of S at the point $\xi(m)$. We have

$$\frac{\| pr_{|T_{\xi(m)}S}(X) \|}{\| X \|} \geq \cos(\alpha_m) \geq \cos(\alpha).$$

Thus

$$\| D\xi(m)X \| \geq \frac{\cos(\alpha)}{1 + \omega_S(T)}. \qquad \square$$

6.2. *Proof of Corollaries* 2 *and* 3

Proof of Corollary 2. By using Proposition 3 with the curve $\mathcal{C} = \mathcal{C}_T(a, b)$, we have

$$\begin{aligned} d_S(\xi(a), \xi(b)) \; &\leq \; L(\xi(\mathcal{C}_T(a, b))) \leq \frac{1}{1 - \omega_S(T)} \, L(\mathcal{C}_T(a, b)) \\ &= \; \frac{1}{1 - \omega_S(T)} \, d_T(a, b). \end{aligned}$$

By using Proposition 3 with the curve $\mathcal{C} = \xi^{-1}(\mathcal{C}_S(a, b))$, we have

$$\frac{\cos(\alpha)}{1 + \omega_S(T)} \, d_T(a, b) \leq \frac{\cos(\alpha)}{1 + \omega_S(T)} L(\mathcal{C}) \leq L(\xi(\mathcal{C})) = d_S(\xi(a), \xi(b)). \qquad \square$$

Proof of Corollary 3. Let g_T denote an isometry between T and $u(T)$ and let g_S denote an isometry between S and $u(S)$. We define the application $f = g_T{}_o \xi_{|T}^{-1}{}_o g_S^{-1}$. Since g_T and g_S are two isometries, the result follows directly from Corollary 2. $\square$

7. ε-Isometry of the Plane

The main result of this section is Proposition 5 which is a statement about plane geometry. It states that two compact domains that are "almost isometric" have "almost the same shape". This proposition is then directly used to prove the main result of this paper (i.e. Theorem 1).

Proposition 5. *Let U, $V_i \subset \mathbb{E}^2$ be compact connected domains and let $f_i \colon U \to V_i$ be a sequence of maps*, *where f_i is an ε_i-isometry and $\varepsilon_i \to 0$* (*when $i \to \infty$*). *Then there exist motions d_i such that $d_i(V_i)$ tend to U in the Hausdorff sense* (*when $i \to \infty$*).

Proof of Proposition 5. We will show that there exist motions d_i such that $d_i \circ f_i$ converge uniformly to the identity.

Fix a point $o \in U$ and let $r > 0$ be smaller than the distance from o to the boundary of U. Translating if necessary, we may assume that $o \in V_i$ and that o is a fixed point of all the f_i. Then o lies at a distance of at least $(1 - \varepsilon_i)r > r/2$ from the boundary of V_i. Let $p \in B(o, r/4)$. Rotating if necessary, we may assume that the point $f_i(p)$ belongs to the half-line-segment $[o, p)$. Then $\|o - f_i(p)\| \leq (1+\varepsilon_i)\|o - p\| < r/2$ and $f_i(p) \subset U \cap V_i$. Let us consider another point $q \in B(o, r/4)$ such that the two lines (op) and (oq) are orthogonal. Up to a symmetry if necessary, we may assume that q and $f_i(q)$ are located on the same side of the line (op).

Since f_i is an ε_i-isometry, we have for every point x and y of U,

$$\|f_i(x) - f_i(y)\| \leq d_{V_i}(f_i(x), f_i(y)) \leq (1+\varepsilon_i) d_U(x, y).$$

The set $\{f_i\colon (U, d_U) \to (\mathbb{E}^2, \|\cdot\|)\}$ is therefore equicontinuous. Furthermore, since $\bigcup_i Vi$ is bounded, the Arzela–Ascoli theorem implies that there exists a subsequence (f_i') converging uniformly to some map $g\colon U \to \mathbb{E}^2$. (We now denote the uniform norm by $\|\cdot\|_\infty$.)

We show that the restriction of g to $B(o, r/4)$ is the identity. The fact that $B(o, r/2) \subset U \cap V_i$ implies that d_U and d_{V_i} are the Euclidean distance on $B(o, r/2)$. Then we have for every i,

$$\|g(p) - p\| \leq \|g(p) - f_i'(p)\| + \|f_i'(p) - p\| \leq \|g - f_i'\|_\infty + \varepsilon_i \frac{r}{4}.$$

The convergence of (f_i') to g implies that $g(p) = p$. Furthermore, for every point x and y of $B(o, r/4)$,

$$\begin{aligned}\|g(x) - g(y)\| &\leq \|g(x) - f_i'(x)\| + \|f_i'(x) - f_i'(y)\| + \|f_i'(y) - g(y)\| \\ &\leq 2\|g - f_i'\|_\infty + (1+\varepsilon_i')\|x - y\|.\end{aligned}$$

Then $\|g(x) - g(y)\| \leq \|x - y\|$. Similarly we have $\|g(x) - g(y)\| \geq \|x - y\|$. The map g is therefore an isometry on $B(o, r/4)$. Since that isometry is fixing the two points o and p, it is either the identity or the symmetry of axis (op). The convergence of $f_i'(q)$ to $g(q)$ implies that $g(q)$ and q lie on the same half-plane limited by the line (op), thus the restriction of g to $B(o, r/4)$ cannot be the symmetry of axis (op). As a consequence, it is the identity.

We show that g is the identity on U. Similarly as above, we can prove that for every interior point x of U, there exists $r_x > 0$ such that the restriction of g to $\mathcal{V}_x = B(x, r_x) \subset U$ is an isometry. Let x_0 be an interior point of U. Since the interior of U is connected, there exists a compact curve $\mathcal{C}$ included in the interior of U that joins o and x_0. By compactness, there exist open sets $\mathcal{V}_1, \ldots, \mathcal{V}_n$ of U such that g is an isometry on each $\mathcal{V}_i$ and $\mathcal{C} \subset \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_n$. This sequence $\mathcal{V}_1, \ldots, \mathcal{V}_n$ can be ordered such that $o \in \mathcal{V}_1$ and $\mathcal{V}_i \cap \mathcal{V}_{i+1} \neq \emptyset$. We then prove by induction that the restriction of g to $\mathcal{C}$ is the identity and that $g(x_0) = x_0$.

The map g is then the identity on the interior of U. The uniform convergence of the sequence (f_i') to g implies that the map g is continuous and then g is the identity on U.

Since the limit g is the same regardless of the chosen subsequence of (f_i), it follows that the entire sequence (f_i) converges to the identity. □

Combined with Corollary 3, this proposition implies Theorem 1 as follows:

Proof of Theorem 1. Let $n > 0$. Let $\varepsilon_n = (1 + \omega_S(T_n))/\cos(\alpha_n) - 1$, where α_n is the maximal angle between the normals of S and T_n. The assumptions of Theorem 1 imply that ε_n tends to zero when n tends to infinity. Using Corollary 3, we know that there exists an ε_n-isometry f_n between two unfoldings $u(S)$ and $u(T_n)$. We conclude by using Proposition 5. □

8. Proof of Theorem 2

Let $\alpha \in]0, 1]$. We denote by z_0 the point of $\mathbb{S}^2$ of coordinates $(0, 0, 1)$. We define points $z_1^\alpha, \ldots, z_n^\alpha$ on sphere $\mathbb{S}^2$ by

$$\forall i \in \{1, \ldots, n+1\}, \qquad z_i^\alpha = \left(\alpha \cos \frac{\pi(2i-3)}{n}, \alpha \sin \frac{\pi(2i-3)}{n}, \sqrt{1-\alpha^2}\right).$$

Note that $z_{n+1}^\alpha = z_1^\alpha$.

Step 1. We are going to build points $w_1^\alpha, \ldots, w_n^\alpha$ on $\mathbb{S}^2$ so as to get

$$\forall i \in \{1, \ldots, n\}, \qquad \widehat{z_i^\alpha z_0 w_i^\alpha} = \frac{\pi}{n} \quad \text{and} \quad \widehat{w_i^\alpha z_0 z_{i+1}^\alpha} = \frac{\pi}{n}.$$

Thus, if we define T_n^α as being the triangulation

$$\text{whose vertices are } \begin{cases} z_i^\alpha & \text{for} \quad 0 \le i \le n, \\ w_i^\alpha & \text{for} \quad 1 \le i \le n, \end{cases}$$

$$\text{and whose faces are } \begin{cases} z_i^\alpha z_0 w_i^\alpha & \text{for} \quad 1 \le i \le n, \\ w_i^\alpha z_0 z_{i+1}^\alpha & \text{for} \quad 1 \le i \le n, \end{cases}$$

we get the following property:

$$\alpha_{T_n^\alpha}(z_0) = \sum_{i=1}^{n} (\widehat{z_i^\alpha z_0 w_i^\alpha} + \widehat{w_i^\alpha z_0 z_{i+1}^\alpha}) = 2n\frac{\pi}{n} = 2\pi.$$

Let us build the point w_1^α. Let $(x, 0, z)$ be the coordinates of w_1^α. We have to solve the following equation:

$$(E) = \begin{cases} w_1^\alpha \in \mathbb{S}^2, \\ x \ge 0, \\ \widehat{w_1^\alpha z_0 z_2^\alpha} = \dfrac{\pi}{n}. \end{cases}$$

Let

$$a = \alpha \cos \frac{\pi}{n}, \qquad b = 1 - \sqrt{1-\alpha^2}, \qquad c = 2 \cos \frac{\pi}{n} \sqrt{1 - \sqrt{1-\alpha^2}}$$

and

$$\begin{cases} z_1 = \dfrac{(a^2-b^2)(c^2-a^2-b^2)+2abc\sqrt{2a^2+2b^2-c^2}}{(a^2+b^2)^2}, \\ z_2 = \dfrac{(a^2-b^2)(c^2-a^2-b^2)-2abc\sqrt{2a^2+2b^2-c^2}}{(a^2+b^2)^2}. \end{cases}$$

A simple calculation leads to

$$w_1^\alpha \text{ solution of } (E) \quad \Leftrightarrow \quad \begin{cases} x = \sqrt{1-z^2}, \\ z = z_1 \text{ or } z = z_2. \end{cases}$$

There are two solutions. We take $z = z_2$, which is linked to the farthest point from z_0. Let us construct points $w_2^\alpha, \dots, w_n^\alpha$. For every $i \in \{2, \dots, n\}$ let w_i^α be the point of coordinates

$$\left(x_{w_1}^\alpha \cos \frac{2\pi(i-1)}{n}, x_{w_1}^\alpha \sin \frac{2\pi(i-1)}{n}, z_{w_1^\alpha} \right).$$

Note that if r is the rotation of angle $2\pi/n$ and of axis (Oz_0), we get

$$\forall i \in \{1, \dots, n-1\}, \qquad \begin{array}{l} r^i(z_1^\alpha z_0 w_1^\alpha) = z_{i+1}^\alpha z_0 w_{i+1}^\alpha, \\ r^i(w_1^\alpha z_0 z_2^\alpha) = w_{i+1}^\alpha z_0 z_{i+2}^\alpha. \end{array}$$

Step 2. We are going to build new points $u_1^\alpha, \dots, u_n^\alpha$ on sphere $\mathbb{S}^2$ satisfying

$$\forall i \in \{1, \dots, n\}, \qquad \widehat{w_i^\alpha z_{i+1}^\alpha u_i^\alpha} = \pi - \widehat{w_i^\alpha z_{i+1}^\alpha z_0} \quad \text{and } \widehat{u_i^\alpha z_{i+1}^\alpha w_{i+1}^\alpha} = \pi - \widehat{w_{i+1=}^\alpha z_{i+1}^\alpha z_0}.$$

Thus, by adding to the triangulation T_n^α,

$$\text{the points } u_i \qquad \text{for} \quad 1 \le i \le n,$$

$$\text{and the faces } \begin{cases} w_i^\alpha z_{i+1}^\alpha u_i^\alpha & \text{for} \quad 1 \le i \le n, \\ u_i^\alpha z_{i+1}^\alpha w_{i+1}^\alpha & \text{for} \quad 1 \le i \le n \text{ (with } z_{n+1}^\alpha = z_1^\alpha), \end{cases}$$

we obtain

$$\forall i \in \{1, \dots, n\}, \qquad \alpha_{T_\alpha^n}(z_i^\alpha) = 2\pi.$$

Let us construct the point u_1^α. Let P denote the plane determined by the points O, z_0 and z_2. We want to show that

$$\exists u_1^\alpha \in P \cap \mathbb{S}^2, \qquad \widehat{w_1^\alpha z_2^\alpha u_1^\alpha} = \pi - \widehat{z_0 z_2^\alpha w_1^\alpha}.$$

We define the application β by

$$\beta \colon \quad \mathbb{S}^2 \cap P \to \mathbb{E}$$
$$z \mapsto \widehat{w_1^\alpha z_2^\alpha z}.$$

There exists $\tilde{z} \in P \cap \mathbb{S}^2$ close to z_2^α, such that the triangulation K whose vertices are z_0, z_2^α, w_1^α, w_2^α and $\tilde{z}$ and whose faces are $z_0 z_2^\alpha w_1^\alpha$, $z_0 z_2^\alpha w_2^\alpha$, $w_1^\alpha z_2^\alpha z$ and $w_2^\alpha z_2^\alpha z$ is a subset of the boundary of a strictly convex set of $\mathbb{E}^3$. Thus

$$\alpha_K(z_2) < 2\pi.$$

Since

$$\alpha_K(z_2) = 2\beta(\tilde{z}) + 2\widehat{z_0 z_2^\alpha w_1^\alpha},$$

we get

$$\beta(\tilde{z}) < \pi - \widehat{w_1^\alpha z_2^\alpha z_0}.$$

Let $\tilde{\tilde{z}}$ denote the point of coordinates $(0, 0, -1)$.

$$\begin{aligned} \beta(\tilde{\tilde{z}}) > \pi - \widehat{w_1^\alpha z_2^\alpha z_0} \\ &\Leftrightarrow \quad \cos(\widehat{w_1^\alpha z_2^\alpha \tilde{\tilde{z}}}) < -\cos(\widehat{w_1^\alpha z_2^\alpha z_0}) \\ &\Leftrightarrow \quad -\frac{\cos(\widehat{w_1^\alpha z_2^\alpha \tilde{\tilde{z}}})}{\cos(\widehat{w_1^\alpha z_2^\alpha z_0})} < 1. \end{aligned}$$

Since

$$\lim_{\alpha \to 0} -\frac{\cos(\widehat{w_1^\alpha z_2^\alpha \tilde{\tilde{z}}})}{\cos(\widehat{w_1^\alpha z_2^\alpha z_0})} = 0,$$

we have

$$\exists \alpha_0 \in \,]0, 1], \quad \forall \alpha \in \,]0, \alpha_0], \qquad \beta(\tilde{\tilde{z}}) > \pi - \widehat{w_1^\alpha z_2^\alpha z_0}.$$

Since β is continuous, we get

$$\exists u_1^\alpha \in D \cap \mathbb{S}^2, \qquad \widehat{w_1^\alpha z_2^\alpha u_1^\alpha} = \beta(u_1^\alpha) = \pi - \widehat{z_0 z_2^\alpha w_1^\alpha}.$$

Furthermore, we know that the abscissa and the ordinate of u_1^α are positive. Thanks to the symmetry with respect to plane P, we get

$$\widehat{u_1^\alpha z_2^\alpha w_2^\alpha} = \pi - \widehat{w_2^\alpha z_2^\alpha z_0^\alpha}.$$

We know that for $\alpha \in \,]0, \alpha_0]$, u_1^α is well defined. We construct the points $u_2^\alpha, \ldots, u_n^\alpha$: if r always denotes the rotation of angle $2\pi/n$ and of axis (Oz_0), we define those points by

$$\forall i \in \{1, \ldots, n\}, \qquad r^i(u_1^\alpha) = u_{i+1}^\alpha.$$

We clearly have

$$\forall i \in \{1, \ldots, n\}, \qquad \begin{array}{ll} r^i(w_1^\alpha u_1^\alpha z_2^\alpha) = w_{i+1}^\alpha u_{i+1}^\alpha z_{i+2}^\alpha & \text{with} \quad z_{i+2^\alpha} = z_2^\alpha, \\ r^i(u_1^\alpha z_2^\alpha w_2^\alpha) = u_{i+1}^\alpha z_{i+2}^\alpha w_{i+2}^\alpha & \text{with} \quad w_{i+2}^\alpha = w_2^\alpha. \end{array}$$

The triangulation T_n^α satisfies the conditions of Theorem 2.

Acknowledgements

The authors thank the referee for many interesting remarks and for the simplification of the proof of Proposition 5.

9. Conclusion and Perspective

The fact that two surfaces are close to each other in the Hausdorff sense does not imply that we can compare their Gauss curvature. In particular, the fact of having a globally developable triangulation closely inscribed in a smooth surface does not allow us to conclude on the "unfoldness" of the smooth surface.

However, in the case in which both surfaces are globally developable, the unfolding of the triangulation gives a "good approximation" of the unfolding of the smooth surface if the normals of both surfaces are quite close and if both surfaces are quite close in the Hausdorff sense.

We present several examples of globally developable triangulations. However, the construction of a globally developable triangulation "close" to a set of points is still an open and difficult problem (although some results exist in particular cases).

References

1. M. Berger, B. Gostiaux, *Géométrie différentielle*: *variétés*, *courbes et surfaces*, Second edition, Presses Universitaires de France, Paris, 1992.
2. J. Cheeger, W. Müller, R. Schrader, On the curvature of piecewise flat spaces, *Commun*. *Math*. *Phys*. **92** (1984), 405–454.
3. M. Desbrun, M. Meyer, P. Alliez, Intrinsic parameterizations of surfaces meshes, *Proc*. *Eurographics*, pages 209–218, 2002.
4. M. P. Do Carmo, *Differential Geometry of Curves and Surfaces* (translated from the Portuguese), Prentice-Hall, Englewood Cliffs, NJ, 1976.
5. H. Federer, Curvature measures, *Trans*. *Amer. Math*. *Soc*. **93** (1959), 418–491.
6. J. Fu, Convergence of curvatures in secant approximations, *J. Differential Geom*. **37** (1993), 177–190.
7. S. Levy, T. Munzner, M. Phillips, Geomview. http://www.geomview.org/.
8. B. Lévy, S. Petitjean, N. Ray, J. Maillot, Least square conformal maps for automatic texture atlas generation, *Proc*. *Siggraph*, pages 362–371, 2002.
9. J. Milnor, *Morse Theory*, Princeton University Press, Princeton, NJ, 1963.
10. F. Morgan, *Geometric Measure Theory*, Academic Press, New York, 1987.
11. J.M. Morvan, B. Thibert, On the approximation of a smooth surface with a triangulated mesh, *Comput*. *Geom*. **23**(3) (2002), 337–352.
12. J.M. Morvan, B. Thibert, On the approximation of the normal vector field and the area of a smooth surface with the normals of a triangulated mesh, *Discrete Comput*. *Geom*. **32** (2004), 383–400.
13. M. Peternell, Developable surface fitting to point clouds, *Comput*. *Aided Geom*. *Design* **21** (2004), 785–803.
14. A. Sheffer, E. de Sturler, Parameterization of faceted surfaces for meshing using angle based flattening, *Engrg*. *Comput*. **17**(3) (2001), 326–337.
15. M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. III second edition, Publish or Perish, Wilmington, DE, 1979.
16. B. Thibert, J.P. Gratier, J.M. Morvan, A direct method for modelling and unfolding developable surfaces and its application to the Ventura Basin (California), *J. Structural Geol*. **27**(2) (2005), 303–331.
17. E. Turquin, M.P. Cani, J. Hughes, Sketching garments for virtual characters, *Proc*. *Eurographics Workshop on Sketch-Based Interfaces and Modeling*, pages 175–182, 2004.
18. F. E. Wolter, Cut Locus and Medial Axis in Global Shape Interrogation and Representation, MIT Design Laboratory Memorandum 92-2 and MIT Sea Grant Report, 1992.

Received October 31, 2002, *and in revised form December* 20, 2004, *and October* 18, 2005.
Online publication July 31, 2006.