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Topology of a submanifold and external curvatures

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RIASSUNTO - *Sono date proprietà topologiche dello « spazio normale principale » definito su una sottovarietà di una varietà Riemanniana. Si mostra che le curvature esterne definite in* [1], *hanno significato topologico.*

Introduction.

In [1], J. GRIFONE and the Author defined « external curvatures » of a Riemannian submanifold. Geometrical results were proved about the submanifold, in terms of « external curvatures ». The main purpose of the present paper is to give a topological interpretation of these « external curvatures ». More precisely, if $i\colon M^n \to E^{n+p}$ is an isometric immersion of a Riemannian Manifold M^n in the Euclidean space E^{n+p}, we study the characteristic classes of the normal bundle, and of the subbundles complementary to the principal normal spaces (cf. [1], [2], [3]).

We prove that the integral of certain external curvatures, on M^n, gives a majoration of these characteristic classes. Then, using J. H. WHITE's work ([4]), we deduce immediately a majoration of the self-intersection number of M^n, of the sum of the indices of the intersections of M^n with its principal normal spaces and of M^n and its boundary.

Finally, we give a relation between the CHERN-LASHOF curvature and the external curvatures.

The author owes thanks to Professor J. GRIFONE for many interesting discussions on this subject.

1. Notations and Definitions.

Let M^n be a n-dimensional Riemannian Manifold. We denote by $T M^n$ the tangent bundle of M^n, and $\langle\ ,\ \rangle$ the scalar product on M^n. ∇ is the connexion of Levi-Civita associated to $\langle\ ,\ \rangle$. Let i: $M^n \to E^{n+k}$ be an isometric immersion of M^n into the Euclidian space E^{n+k}, $\langle\ ,\ \rangle$ denotes also the scalar product of E^{n+k}, and ∇' the trivial connexion on E^{n+k}. $T^\perp M^n$ designs the normal bundle on M^n, and $\nabla^\perp$ the normal connexion on $T^\perp M^n$. H designs the second fondamental form. It is well known that:

$$\nabla'_X Y = \nabla_X Y + H(X, Y) \qquad \forall\ X, Y \in T M^n.$$

Let R be the curvature tensor on M^n, and $R^\perp$ the curvature tensor on $T^\perp M^n$. If $(a_1, \dots, a_{n+k})$ is a local frame on M^n, such that $(a_1,\dots,a_n) \in TM^n$ and $(a_{n+1}, \dots, a_{n+k}) \in T^\perp M^n$, we note $\Omega_\beta{}^\alpha$ the curvature-forms defined by

$$R^\perp(X, Y)\, a_\alpha = \sum_{i=1}^{n} \Omega_\alpha{}^\beta (X, Y)\, a_\beta \qquad \forall\ a_\alpha, a_\beta \in T^\perp M^n.$$

The Gauss-Codazzi equations give

$$\Omega_\alpha{}^\beta = \sum_{i=1}^{n} \alpha_\alpha{}^i \wedge \alpha_i{}^\beta, \text{ where } \alpha_\alpha{}^i = \langle \nabla' a_\alpha, a_i \rangle.$$

a. External curvatures of a Riemannian submanifold.

Let i: $M \to M'$ an isometric immersion.

LEMMA 1. *Let $\mathcal{D}$ be a distribution on $T M^\perp$, if $\xi \in \mathcal{D}$ and $X \in T_p M$, $pr_{\mathcal{D}^\perp} \nabla_X{}^\perp \xi$ depends only on ξ_p.*

The proof of lemma 1 is obvious.

DEFINITION 1. *Let $m \in M$.*

Let E_{1_m} be the subspace of $T_m{}^\perp M$ defined by $E_{1_m} = [Im\ \sigma]_m$ (i. e.: the space spanned by $Im\ \sigma$).

E_1 is called « the first principal normal space ».

If dim E_1 is constant on a neighborhood of m, the second principal normal space is the subspace of $T_m{}^\perp M^n$ defined by:

$$E_{2_m} = [\overline{E_2}]_m,$$

where $\overline{E_{2_m}} = \{\eta \in T_m^\perp M / \exists X \in T_m M^n, \ \exists \xi \in E_{1_m}$ *such that* $\eta = pr_{E^\perp_{1_m}} (\nabla_X^\perp \xi)_m\}$.

By induction, we define the i^{th} principal normal space in the following way:

If $\dim E_{i-1}$ *is constant on a neighborhood of* m, $E_{i_m} = [\overline{E_{i_m}}]$, *where* $\overline{E_{i_m}} = \{\eta \in T_m^\perp M / \exists X \in T_m M, \ \exists \xi \in E_{i-1_m}$ *such that* $\eta = (pr_{(\underset{j<i}{\oplus} E_j)^\perp} (\nabla_X^\perp \xi)_m\}$.

Definition 2. *A submanifold M of M' is said to be E_j-niced curved if E_i is a vector subbundle of $T^\perp M^n$, $\forall i \leq j$.*

Definition 3. *Let* $m \in M$.

If $E_{1_m}, E_{2_m}, \dots, E_{i_m}$ *are defined, we call « j^{th} external curvature »* $(j=1, \dots, i)$ *at* $m \in M$, *or «j^{th}-Frenet curvature» at* m, *the scalars* $(k_j)_m^{(M)}$ *defined by:*

$j=1$:
$$(k_1^{(M)})_m = \sup_{\substack{X, Y \in TM^n \\ \|X\| = \|Y\| = 1}} \|\sigma(X, Y)\|$$

$j \geq 2$: *we define first the maps:*

$$\begin{cases} (k_2)_m : (E_1)_m \to \mathbf{R}^+ \\ \eta \hookrightarrow \sup\limits_{\substack{X \in TM^n \\ \|X\|=1}} \|pr_{(E_1)^\perp} \nabla_X^\perp \eta\| \\ \cdots \\ (k_i)_m : (E_{i-1})_m \to \mathbf{R}^+ \\ \eta \hookrightarrow \sup\limits_{\substack{X \in TM^n \\ \|X\|=1}} \|pr_{(\underset{j<i}{\oplus} E_j)^\perp} \nabla_X^\perp \eta\| \end{cases}$$

and

$$\begin{cases} (k_2^{(M)})_m = \sup\limits_{\substack{\eta \in (E_1)_m \\ \|\eta\|=1}} k_2(\eta)_m \\ \cdots\cdots\cdots\cdots \\ (k_j^{(M)})_m = \sup\limits_{\substack{\eta \in (E_{j-1})_m \\ \|\eta\|=1}} k_j(\eta)_m . \end{cases}$$

b. Characteristic classes.

Let $\xi = (E \to M^n)$ be a vector bundle on M^n. We denote by $H^j(\xi, G)$ the j^{th} singular cohomology groupe of ξ, with coefficient in the group G, and $H^j(M^n, G)$ the j^{th} singular cohomology group of M^n, with coefficient in G.

$\omega_j(\xi) \in H^j(M^n, \mathbf{Z}/2)$ $(j=0, 1, \ldots)$ design the STIEFEL-WHITNEY classes of ξ. If $\omega(\xi) = 1 + \omega_1(\xi) + \ldots + \omega_n(\xi)$ is the total Stiefel-Whitney class of ξ, we denote by $\bar{\omega}(\xi) = 1 + \bar{\omega}_1(\xi) + \ldots + \bar{\omega}_n(\xi)$ the inverse of $\omega(\xi)$. If ξ is oriented, with $2k$-dimensional fibers, $e(\xi) \in H^{2k}(H^n, \mathbf{Z})$ designs the Euler class of ξ. Finally, if η is a subbundle of ξ, $\chi(\eta)$ designs the Euler characteristic of η.

2. Euler characteristic of the normal bundle of a submanifold M^n, and of the subbundles complementary to the principal normal spaces.

In this paragraph, we shall prove the three following theorems.

THEOREM 1. *Let $i: M^n \to E^n$ be an isometric immersion of a compact oriented Riemannian manifold M^n of even dimension n, in the Euclidean space E^{2n}. Then, the Euler characteristic of the normal bundle (i. e. the normal characteristic of i), $\chi(T^\perp M^n)$, satisfies:*

$$|\chi(T^\perp M^n)| < \frac{n^{n/2}\, n!}{2^n\, \pi^{n/2} \left(\dfrac{n}{2}\right)!} \int_{M^n} k_1^{(i)n}\, dv.$$

Moreover, if $E_1 \neq T^\perp M^n$ at every point (particularly if $k_2^{(i)}$ is defined and $\neq 0$ at every point), $\chi(T^\perp M^n) = 0$.

THEOREM 2. *Let i: $M^n \to E^{2n+m_j}$ be an isometric immersion of a compact oriented n-dimensional Riemannian manifold M^n in the Euclidean space E^{2n+m_j}.*

Let E_j be the j^{th} principal normal space. We suppose that M^n is E_j-niced-curved, $\dim E_j = m_j$, and E_j oriented. Then:

a. If $j=1$,

$$|\chi(E_1^{\perp})| \leq \frac{m_1^{n/2}\, n!}{2^n \pi^{n/2}\left(\frac{n}{2}\right)!} \int_{M^n} k_2^{(i)\,n}\, dv.$$

The equality holds if and only if $k_2^{(i)}=0$ at every point, i. e. if the substantial codimension of M^n is n_1. In this case $\chi(E_1^{\perp})=0$. Moreover if $E_1 \oplus E_2 \neq T^{\perp} M^n$ at every point (in particular if k_3 is defined and $\neq 0$ at every point), $\chi(E_1^{\perp})=0$.

b. If $j=2$,

$$|\chi(E_2^{\perp})| < \frac{(n+m_2)^{n/2}\, n!}{2^n \pi^{n/2}\left(\frac{n}{2}\right)!} \int_{M^n} \operatorname{Sup}\,[k_1^{(i)}\, k_2^{(i)}\, k_3^{(i)}]^n\, dv.$$

Moreover, if $E_1 \oplus E_2 \oplus E_3 \neq T^{\perp} M^n$ at every point (in particular if $k_4^{(i)}$ is defined and $\neq 0$ at every point), $\chi(E_2^{\perp})=0$.

c. If $j=3$,

$$|\chi(E_3)| < \frac{(n+m_3)^{n/2}\, n!}{2^n \pi^{n/2}\left(\frac{n}{2}\right)!} \int_{M^n} \operatorname{Sup}\,[k_1^{(i)}\, k_3^{(i)}\, k_4^{(i)}]^n\, dv.$$

Moreover, if $\bigoplus_1^4 E_j = T^{\perp} M^n$ at every point (in particular if $k_3^{(i)}$ is defined and $\neq 0$ at every point), $\chi(E_3^{\perp})=0$.

d. If $j \geq 4$,

$$\chi(E_j^{\perp})=0.$$

THEOREM 3. *Let g: $N^{n+1} \to E^{2n+1}$ an isometric immersion of an oriented odd dimensional Riemannian manifold with boundary N^{n+1}*

into the Euclidean space. Let M^n be the boundary of N^{n+1}. We suppose that M^n is compact, oriented. We note $g_{|M^n}=f$.

Then, if $\chi(v^\perp)$ designs the Euler characteristic of the subbundle of the normal bundle, complementary to the bundle spanned by the vector field v, normal to M and tangent to N, we have:

$$|\chi(v^\perp)| \leq \frac{(n+1)^{n/2}\, n!}{2^n\, \pi^{n/2}\left(\frac{n}{2}\right)!} \int_{M^n} k_1^{(g)n}\, dv.$$

The equality holds if and only if $k^{(g)}_{1|M^n}=0$. In this case $\chi(v^\perp)=0$.

Moreover, if $E_1^{(g)}$, the first principal normal space of N satisfies $E_1^{(g)} \neq T^\perp N^{n+1}$ at every point, (in particular, if $k_2^{(g)}$ is defined and $\neq 0$ at every point), $\chi(v^\perp)=0$.

Before the proof of the three theorems, we will give three corollaries of these theorems.

a) Self-Intersection number of a submanifold.

Let us consider $i\colon M^n \to E^{2n}$ an isometric immersion of a n-dimensional manifold M^n into E^{2n}. We suppose that M^n is compact, oriented, and of even dimension n.

Let $I(i,i)=\{(m,m'), m \neq m' \in M^n \times M^n, \text{ such that } i(m)=i(m')\}$, be the set of non trivial intersection points of $i(M^n)$ with $i(M^n)$. Using the THOM Transversality Theorem, we can see that, under a « small deformation of i », these intersections may be made transverse. Then, $I(i,i)$ is finite, since M^n is compact. In this case, WHITNEY, LASHOF and SMALE [5,6] proved that:

$2I(i,i)=\chi(T^\perp M^n)$. Then, using theorem 1,

$$|I(i,i)| < \frac{n^{n/2}\, n!}{2^{n+1}\, \pi^{n/2}} \int_{M^n} k_1^{(i)n}\, dv.$$

We have proved the

COROLLARY 1. *Let $i\colon M^n \to E^{2n}$ be an isometric immersion of a n-dimensional manifold M^n into E^{2n}. We suppose that M^n is compact, oriented, of even dimension n. We also suppose that the self intersections of M^n are transverse. Then,*

$$|I(i, i)| < \frac{n^{n/2}\, n!}{2^{n+1}\, \pi^{n/2}} \int_{M^n} k_1^{(i)n}\, dv.$$

Moreover, if $E_1 \neq T M^n$ *at every point, (in particular if* $k_2^{(i)}$ *is defined and* $\neq 0$ *at every point),* $I(i, i) = 0$.

b) *Intersection number of a submanifold with its principal normal spaces.*

Let us consider $i\colon M^n \to E^{2n+k}$ an isometric immersion of a compact, oriented Riemannian manifold M^n of even dimension into the Euclidean space E^{2n+k}.

Let N be an oriented k-subbundle of the normal bundle.

We can consider the set $I(M^n, N) = \{(m, (p, e)) \in M^n \times N$ such that $i(m) = (p, e)\}$.

Using the THOM Transversality Theorem, we can suppose that the intersections of M^n with N are transverse. Then, in this case $I(M^n, N)$ is finite, for M^n is compact. J. H. WHITE proved that

$$I(M^n, N) = \chi(N^{\perp}), \quad \text{(cf. [4])}.$$

Applying this result, we deduce from theorem 2, the

COROLLARY 2. *Let* $i\colon M^n \to E^{2n+mj}$ *be an isometric immersion of a compact oriented, n-dimensional manifold* M^n *in the Euclidean space* E^{2n+mj}. *Let* Ej *be the* j^{th} *principal normal space. We suppose that* M^n *is* Ej*-niced curved,* $\dim Ej = mj$, Ej *is oriented, and* M^n *and* E_j *are transverse.*

Then

a. If $j=1$,

$$|I(M^n, E_1)| \leq \frac{m_1^{n/2}\, n!}{2^n\, \pi^{n/2} \left(\frac{n}{2}\right)!} \int_{M^n} k_2^{(i)\,n}\, dv.$$

The equality holds if and only if $k_2^{(i)} = 0$ *at every point, i. e. if the substantial codimension of* M^n *is* n_1. *In this case* $\chi(E_1^{\perp}) = 0$. *Moreover if* $E_1 \oplus E_2 \neq T^{\perp} M^n$ *at every point (in particular if* $k_3^{(i)}$ *is defined and* $\neq 0$ *at every point),* $I(M^n, E_1) = 0$.

b. If $j=2$,

$$|I(M^n, E_2)| < \frac{(n+m_2)^{n/2}\, n!}{2^n\, \pi^{n/2} \left(\frac{n}{2}\right)!} \int_{M^n} \mathrm{Sup}\, [k_1^{(i)}\, k_2^{(i)}\, k_3^{(i)}]^n\, dv.$$

Moreover, if $E_1 \oplus E_2 \oplus E_3 \neq T^\perp M^n$ *at every point (in particular if* $k_4^{(i)}$ *is defined and* $\neq 0$ *at every point),* $I(M^n, E_2) = 0$.

c. *If* $j=3$,

$$|I(M^n, E_3)| < \frac{(n+m_3)^{n/2}\, n!}{2^n\, \pi^{n/2} \left(\frac{n}{2}\right)!} \int_{n} \mathrm{Sup}\, [k_1^{(i)}\, k_3^{(i)}\, k_4^{(i)}]^n\, dv.$$

Moreover, if $\bigoplus_1^4 E_j = T^\perp M^n$ *at every point (in particular of* $k_5^{(i)}$ *is defined and* $\neq 0$ *at every point),* $I(M^n, E_3) = 0$.

d. *If* $j \geq 4$, $I(M^n, E_j) = 0$.

Intersection number of a submanifold with its boundary.

Let $g: N^{n+1} \to E^{2n+1}$ an isometric immersion of a oriented-odd dimensional manifold N^{n+1} with oriented compact boundary. We denote M^n the boundary of N^{n+1} and $g = f_{|M}$.

If N^{n+1} and M^n are transverse, that is, the number of non trivial intersections of N^{n+1} and M^n is finite, then, applying a result of J. H. WHITE [4], we obtain

$$I(g, f) = \frac{1}{2}\, \chi(v^\perp),$$

where $\chi(v^\perp)$ is the Euler characteristic of the normal bundle complementary to the bundle spanned by the vector v, which is normal to M and tangent to N, and where $I(g, f)$ is the sum of the indices of the non trivial intersections of $g(N)$ with $f(M)$. Using Theorem 3, we obtain the

COROLLARY 3. *Let* $g: N^{n+1} \to E^{2n+1}$ *be an isometric immersion of an oriented odd-dimensional Riemannian manifold* N^{n+1} *with boundary, into the Euclidean space. Let* M^n *be the boundary of* N^{n+1}*. We suppose that* M^n *is compact, oriented. We note* $g_{|M^n} = f$ *and we suppose that* f *and* *g are transverse. Then*

$$|I\,(g,f)| \leq \frac{(n+1)^{n/2}\, n!}{2^{n+1}\, \pi^{n/2} \left(\frac{n}{2}\right)!} \int_{M^n} k_1{}^{(g)n}\, dv.$$

The equality holds if and only if $k^{(g)}_{1\,M^n}=0$ *every point* $p \in M^n$, *in* E^{2n+1}. *In this case,* $I\,(g,f)=0$.

Moreover, if $E_1{}^{(g)}$, *the first principal normal space of N, satisfies*

$$E^{(g)}_{1_p} = T_p{}^{\perp}\, N^{n+1}\ \forall p \in M^n, \text{ (in particular, if } k^{(g)}_{2_p}$$

is defined and $\neq 0$ *at every point),* $I\,(g,f)=0$.

PROOF OF THEOREM 1. Let $(a_1, \dots, a_{2n})$, be a local frame over M^n, such that $(a_1, \dots, a_n)$ is tangent to M^n, and $(a_{n+1}, \dots, a_{2n})$ is normal to M^n.
We have

$$\chi\,(T^{\perp}\, M^n) = \frac{(-1)^{n/2}}{2^n\, \pi^{n/2} \left(\frac{n}{2}\right)!} \int_{M^n} \varepsilon_{s_1 \dots s_n}\, \Omega^{s_1}_{s_2} \wedge \dots \wedge \Omega^{s_{n-1}}_{s_n}$$

where $\Omega^{s_j}_{s_k} = \sum_{i=1}^{n} \alpha^{s_j}_{i} \wedge \alpha^{i}_{s_k}$, and $\alpha^{s_j}_{i} = \langle \nabla' a_i, a_{s_j} \rangle$, with $s_j \in \{n+1, \dots, 2n\}$.
Since $a_j \in T\, M^n$, $\alpha^{s_j}_{i}(X) = \langle \nabla'_X\, a_i, a_{s_j} \rangle = \langle H\,(X, a_i), a_{s_j} \rangle$. $\forall a_{s_j} \in T^{\perp}\, M^n$, $\forall a_j \in T\, M^n$. Consequently, $\|\alpha^{s_j}_{i}\| \leq k_1{}^{(i)}$.

Then, $\|\Omega^{s_j}_{s_k}\| \leq n\, k_1{}^{(i)2}$, and $\|\Omega^{s_1}_{s_2} \wedge \dots \wedge \Omega^{s_{n-1}}_{s_n}\| \leq [n\, k_1{}^{(i)2}]^{n/2}$. Finally,

$$|\chi\,(T^{\perp}\, M^n)| \leq \frac{n^{n/2}\, n!}{2^n\, \pi^{n/2} \left(\frac{n}{2}\right)!} \int_{M^n} k_1{}^{(i)n}\, dv. \tag{1}$$

Now, we shall prove that the inequality is strict.

If $\chi\,(T^{\perp}\, M^n)$ is null, (1) is an equality, if and only if $k_1{}^{(i)} \equiv 0$. This is impossible, since M^n is compact.

If $\chi\,(T^{\perp}\, M^n)$ is not null, the equality holds if and only if every term $\Omega^{s_1}_{s_2} \wedge \dots \wedge \Omega^{s_{n-1}}_{s_n}$ has a maximal norm, for every local frame over $p \in H$, $(a_1, \dots, a_{2n})$. In particular, we can choose a frame $(a_{n+1}, \dots, a_{2n})$ on the normal bundle and $(a_1, \dots, a_n)$ on the tangent bundle, such that the matrix $\langle H(\ ,\ ), a_{2n}\rangle_p$ is diagonal in the frame $(a_1, \dots, a_n)$. If (1) is an equality, $\Omega^{s_1}_{s_2} \wedge \dots \wedge \Omega^{s_{n-1}}_{s_n}$ (written in this frame), has a maximal norm. This implies: $\|\alpha\, i^s\| = k_1^{(i)}$ $\forall i \in \{1, \dots, n\}$, $s \in \{n+1, \dots, 2n\}$, and every sequence $\{\alpha^{s_1}_{i_1}, \alpha^{i_1}_{s_2}, \dots, \alpha^{s_{n-1}}_{i_{n-1}}, \alpha^{i_{n-1}}_{s_n}\}$ is composed by orthogonal forms, for every $i \in \{1, \dots, n\}$ and $s_k \in \{n+1, \dots, 2n\}$. We have

$$\langle H(a_j, a_k), a_{2n}\rangle = \alpha_j^{2n}(a_k) = 0 \quad \text{if} \quad j \neq k. \quad \text{Then} \quad \alpha_j^{2n}(a_j) = \pm k_1^{(i)}$$

$$H(a_j, a_j) = \alpha_j^{n+1}(a_j)\, a_{n+1} + \dots + \alpha_j^{2n}(a_j)\, a_{2n}.$$

Since $k_1^{(i)} = \sup_{\substack{\|X\|=1 \\ \|Y\|=1}} \|H(X, Y)\|$, and $\alpha_j^{2n}(a_j) = \pm k_1^{(i)}$, we obtain immediately $\alpha^j_{n+k}(a_j) = 0$ if $k \in \{1, \dots, n-1\}$.

If h designs the mean curvature vector field, $h_p = \sum_{j=1}^{n} H(a_j, a_j)_p =$ $= \sum_{j=1}^{n} \sum_{k=1}^{n} \langle H(a_j, a_j), a_{n+k}\rangle\, a_{p^n,h} = q\, k^{(i)}_{1_p}\, a_{2n}$ where $q \in \mathbf{Z}$. Since a_{2n} is arbitrary and h is global, $q=0$ and $h=0$ everywhere, which is impossible for M^n is compact. Then, in every case, $(*)$ is a strict inequality.

Finally, if $E_1 \neq T^\perp M^n$, at every point, we can choose a frame $(a_1, \dots, a_{2n})$ such that $a_{2n} \in E_1^\perp$. In this case, $\alpha^i_{2n}(X) = \langle H(X, a_i), a_{2n}\rangle = 0$ and $\Omega^s_{2n} = 0$ $\forall s \in \{n+1, \dots, 2n-1\}$, and $\Omega^{s_1}_{s_2} \wedge \dots \wedge \Omega^{s_{n-1}}_{s_n} = 0$ $\forall s_1, \dots, s_n \in \{n+1, \dots, 2n\}$. Then $\chi(T^\perp M^n) = 0$.

Theorem 1 is completely proved.

Proof of theorem 2.

a. A majoration of $\chi(E_1^\perp)$.

Let us consider a local frame $(a_1, \dots, a_{2n+m_1})$ such that

$$(a_1, \dots, a_n) \in T M^n, \quad (a_{n+1}, \dots, a_{n+m_1}) \in E_1, \quad (a_{n+m_1+1}, \dots, a_{2n+m_1}) \in E_1^\perp.$$

we hawe:

$$\chi(E_1^\perp) = \frac{(-1)^{n/2}}{2^n \pi^{n/2} \left(\frac{n}{2}\right)!} \int_{M^n} \varepsilon_{s_1 \dots s_n}\, \Omega^{s_1}_{s_2} \wedge \dots \wedge \Omega^{s_{n-1}}_{s_n},$$

where

$$\Omega_t^s = \sum_{j=1}^{n+m_1} \alpha_j^s \wedge \alpha_t^j, \quad \alpha_j^{s_k} = \langle \nabla' a_j, a_{s_k}\rangle, \; s, t, s_k \in \{n+m_1+1, \dots, 2n\}.$$

We remark that $\alpha^s{}_j = \langle \nabla' a_j, a_s \rangle = 0$ if $a_j \in T M^n$, $a_s \in E_1{}^\perp$ and

$$||\alpha_j{}^s|| = ||\langle \nabla^\perp a_j, a_s \rangle|| \leq k_2{}^{(i)} \text{ if } a_j \in E_1.$$

Since $\dim E_1 = m_1$, $||\Omega_t{}^s|| \leq m_1 k_2{}^{(i)2}$ and

$$|\chi(E_1{}^\perp)| \leq \frac{m_1{}^{n/2}\, n!}{2^n \pi^{n/2} \left(\frac{n}{2}\right)!} \int_{M^n} k_2{}^{(i)\,n}\, dv. \tag{2}$$

Now, we shall prove that (2) is a strict inequality if $\chi(E_1{}^\perp) = 0$.

First step. We suppose $n > 2$.

(2) is an equality if and only if every $\Omega^{s_1}_{s_2} \wedge \ldots \wedge \Omega^{s_{n-1}}_{s_n}$ has a maximal norm. We have

$$\nabla_X{}^\perp a_{n+1} = \alpha^{n+m_1+1}_{n+1}(X)\, a_{n+m_1+1} + \ldots + \alpha^{2n+m_1}_{n+1}(X)\, a_{2n+m_1}$$
$$\vdots$$
$$\nabla_X{}^\perp a_{n+j} = \alpha^{n+m_1+1}_{n+j}(X)\, a_{n+m_1+1} + \ldots + \alpha^{2n+m_1}_{n+j}(X)\, a_{2n+m_1}$$
$$\vdots$$
$$\nabla_X{}^\perp a_{n+m_1} = \alpha^{n+m_1+1}_{n+m_1}(X)\, a_{n+m_1+1} + \ldots + \alpha^{2n+m_1}_{n+m_1}(X)\, a_{2n+m_1}$$

with $||\alpha^{n+m_1+k}_{n+1}|| = k_2{}^{(i)}$, $k \in \{1, \ldots, n\}$, and where $\alpha^{n+m_1+1}_{n+1}$ is orthogonal to $\alpha^{n+m_1+2}_{n+1}, \ldots, \alpha^{2n+m_1}_{n+1}, \alpha^{n+m_1+2}_{n+2}, \ldots, \alpha^{2n+m_1}_{n+2}$, and $\alpha^{n+m_1+1}_{n+2}$ is orthogonal to $\alpha^{n+m_1+2}_{n+2}, \ldots, \alpha^{n+m_1+2}_{n+1}, \ldots, \alpha^{2n+m_1}_{n+1}$.

Consequently $\alpha^{n+m_1+1}_{n+1} = \pm \alpha^{n+m_1+1}_{n+2}$. This is impossible: for in this case, $\left\|\langle \nabla^\perp \frac{a_{n+1} \pm a_{n+2}}{\sqrt{2}}, a_{n+m_1+1} \rangle\right\| = \frac{2k_2{}^{(i)}}{\sqrt{2}} > k_2{}^{(i)}$, which is excluded. Finally, if $n > 2$, (2) is a strict inequality when $\chi(E_1{}^\perp) \neq 0$.

Second step. We suppose $n = 2$.

We consider a local frame $(a_1, a_2, a_3, \ldots, a_{2+n_1}, a_{3+n_1}, a_{4+n_1})$ such that $(a_1, a_2) \in T M^n$, $(a_3, \ldots, a_{2+n_1}) \in E_1$, $(a_{3+n_1}, a_{4+n_1}) \in E_1{}^\perp$ (2) is an equality if and only if $\Omega^{3+n_1}_{4+n_1}$ has a maximal norm. We have

$$\nabla_X^{\perp}\, a_3=\alpha_3^{3+n_1}(X)\, a_{3+n_1}+\alpha_3^{4+n_1}(X)\, a_{4+n_1}$$

$$\vdots$$

$$\nabla_X^{\perp}\, a_{2+j}=\alpha_{2+j}^{3+n_1}(X)\, a_{3+n_1}+\alpha_{2+j}^{4+n_1}(X)\, a_{4+n_1}$$

$$\vdots$$

$$\nabla_X^{\perp}\, a_{2+n_1}=\alpha_{2+n_1}^{3+n_1}(X)\, a_{3+n_1}+\alpha_{2+n_1}^{4+n_1}(X)\, a_{4+n_1}\,.$$

If $n_1>2$, since $||\alpha_3^{3+n_1}||=||\alpha_3^{4+n_1}||=\dots=||\alpha_{2n+n_1}^{4+n_1}||=k_2^{(i)}$, and $||\langle\nabla^{\perp}\, a_{2+j}, a_{2+m_1+1}\rangle||\leq k_2^{(i)}$, $||\langle\nabla\, a_{2+j}, a_{2+m_1+2}\rangle||\leq k_2^{(i)}$, it is easy to remark that these conditions implies: $\alpha_3^{3+n_1}\perp\alpha_{2+j}^{3+n_1}$ $\forall j\in\{2,\dots,2+n_1\}$. This is excluded, since $\dim M=2$.

If $n_1=2$, we can write, on a neighborhood of a point $p\in M^n$:

$$(**)\qquad \begin{cases}\nabla_X^{\perp}\, a_3=k_2^{(i)}\,[\pm\langle X,T\rangle\, a_5\pm\langle X,T'\rangle\, a_6]\\ \nabla_X^{\perp}\, a_4=k_2^{(i)}\,[\pm\langle X,T'\rangle\, a_5\pm\langle X,T\rangle\, a_6],\end{cases}$$

where T and T' are orthogonal vectors, dual of $\dfrac{\alpha_3^{5}}{||\alpha_3^{5}||}$ and $\dfrac{\alpha_4^{6}}{||\alpha_4^{6}||}$ and where $(a_1,a_2,a_3,a_4,a_5,a_6)$ is an arbitrary frame.

Conversely, it is easy to prove from $(**)$ that if we take two orthonormal vectors $(U,U')\in T\,M^n$, we can find, for every choice of (a_3,a_4), two vectors $(a_5,a_6)\in E_1^{\perp}$ such that

$$\nabla_X^{\perp}\, a_3=k_2^{(i)}\,[\pm\langle X,U\rangle\, a_5\pm\langle X,U'\rangle\, a_6]$$

$$\nabla_X^{\perp}\, a_4=k_2^{(i)}\,[\pm\langle X,U'\rangle\, a_5\pm\langle X,U\rangle\, a_6].$$

Since the choice of (U,U') is free, we can take U,U' such that the matrix $\langle H(\cdot,\cdot),a_3\rangle_p$ is diagonal.

Thus, in the following, we consider a local frame (T,T',a_3,a_4,a_5,a_6) such that:

(T,T') diagonalize $\langle H(\cdot,\cdot),a_3\rangle_p$

(a_5,a_6) satisfy equations $(**)$.

Now, we shall use the Gauss Codazzi equation:

$$(\tilde{\nabla}_X H)(Y,Z)=(\tilde{\nabla}_Y H)(X,Z),$$

(where $(\tilde{\nabla}_X H)(Y, Z) = \nabla_Y^{\perp}[H(Y, Z)] - H(\nabla_X Y, Z) - [H(Y, \nabla_X Z)]$).
We set $H(X, Y) = h(X, Y)\, a_3 + k(X, Y)\, a_4$.

We obtain:

$$\begin{cases} \pm h(Y, Z) \langle X, T \rangle_p \pm k(Y, Z) \langle X, T' \rangle_p = \\ \qquad \pm h(X, Z) \langle Y, T) \rangle_p \pm k(X, Z) \langle Y, T' \rangle_p \\ \pm h(Y, Z) \langle X, T' \rangle_p \pm k(Y, Z) \langle X, T_p \rangle = \\ \qquad = \pm h(X, Z) \langle Y, T' \rangle_p \pm k(X, Z) \langle Y, T \rangle_p. \end{cases}$$

This implies $k(T, T)_p = k(T', T')_p = 0$, and the mean curvature vector is

$$H(T, T)_p + H(T, T')_p = [h(T, T) + h(T', T')]_p\, a_3 +$$
$$+ [k(T, T) + k(T', T')]_p\, a_4 = [h(T, T) + h(T', T')]_p\, a_3,$$

which is impossible, for the mean curvature vector is global, a_3 is arbitrary, and M^n is not minimal.

b. A majoration of $\chi(E_2^{\perp})$.

Let us consider, in this case, a local frame $(a_1, \dots, a_{2n+m_2})$ such that $(a_1, \dots, a_n) \in T M^n$, $(a_{n+1}, \dots, a_{n+m_2}) \in E_2$, $(a_{n+m_2+1}, \dots, a_{2n+m_2}) \in (E_2 \oplus T M)^{\perp}$. We have

$$\chi(E_2^{\perp}) = \frac{(-1)^{n/2}}{2^n \pi^{n/2} \left(\dfrac{n}{2}\right)!} \int_{M^n} \varepsilon_{s_1 \dots s_n} \Omega^{s_1}_{s_2} \wedge \dots \wedge \Omega^{s_{n-1}}_{s_n},$$

where $\Omega^{s_j}_{s_k} = \sum_{i=1}^{n+m_2} \alpha^{s_j}_i \wedge \alpha^i_{s_k}$, with $\alpha^{s_j}_i = \langle \nabla' a_i, a_{s_j} \rangle$, $s_1, \dots, s_n \in \{n + m_2 + + 1, \dots, 2_n\}$.

We remark that:

$$\|\alpha_j^s\| = \|\langle \nabla^{\perp} a_j, a_s \rangle\| \le k_2^{(i)} \text{ if } a_j \in E_2, a_s \in E_1$$

$$\|\alpha_j^s\| = \|\langle H(a_j, \cdot), a_s \rangle\| \le k_1^{(i)} \text{ if } a_j \in T M^n, a_s \in E_1$$

$$\|\alpha_j^s\| = \|\langle \nabla^{\perp} a_j, a_s \rangle\| \le k_3^{(i)} \text{ if } a_j \in E_2, a_s \in E_3$$

$$\alpha_j^s=0 \text{ if } a_j\in T M^n, a_s\in E_\rho, \rho>2$$

$$\alpha_j^s=0 \text{ if } a_j\in E_2, a_s\in E_\rho, \rho\geq 4.$$

Thus, we obtain:

$$|\chi (E_2^\perp)| \leq \frac{(n+m_2)^{n/2}\, n!}{2^n\, \pi^{n/2}\left(\dfrac{n}{2}\right)!} \int_{M^n} \text{Sup } (k_1^{(i)}\, k_2^{(i)}\, k_3^{(i)})^n\, dv. \tag{3}$$

Now, we shall prove that (3) is a strict inequality:

If $\chi (E_2^\perp)$ is null, (2) is an equality if and only if Sup $(k_1^{(i)}\, k_2^{(i)}\, k_3^{(i)})^n$ is null everywhere. This implies that $k_1^{(i)}\equiv 0$ which is impossible for M^n is compact.

If $\chi (E_2^\perp)$ is not null, the equality holds if and only if every term $\Omega^{s_1}_{s_2}\wedge \dots \wedge \Omega^{s_{n-1}}_{s_n}$ has a maximal norm, for every local frame $(a_1, \dots , a_{2n})$. In particular, we can choose a frame $(a_n, \dots , a_{2n})\in (T M^n \oplus E_2)^\perp$ such that $a_{2n}\in E_1$, $(a_1, \dots , a_n)$ a local frame of $T M^n$, which diagonalize the bilinear form $\langle H (,), a_{2n}\rangle$ and $(a_{n+1}, \dots , a_{n+m_2})$ a local frame of E_2. If (3) is an equality, $\Omega^{s_1}_{s_2}\wedge\dots\wedge \Omega^{s_{n-1}}_{s_n}$ (written in this frame), has a maximal norm. This implies: $||\alpha_i^s|| =k_1^{(i)}$ if $i\in\{1, \dots , n\}$ and $a_s\in E_1$, and every sequence $\{\alpha^{s_1}_{i_1}, \alpha^{i_1}_{s_2}, \dots , \alpha^{s_{n-1}}_{i_{n-1}}, \alpha^{i_{n-1}}_{s_n} \}$ is composed by orthogonal forms, for every $i_k\in\{1, \dots , n+m_2\}$, $s_k\in\{n+m_2+1, \dots , 2_n\}$.

As in *a.*, we deduce that $\alpha_j^{2n} (a_j)=\pm k_1^{(i)}$, and $\alpha^j{}_{n+m_2+k} (a_j)=0$ if $a_j\in T M^n, a_{n+m_2+k}\in E_1$. If h designs the mean curvature vector field, $h=q\, k_1^{(i)}\, a_{2n}$, where $q\in\mathbf{Z}$. We remark that $\dim E_1>1$ (for if $\dim E_1=1$, $\dim E_2\leq 1$ ([1]), which is excluded.

Since a_{2n} is arbitrary in E_1, we conclude that $q=0$, and $h\equiv 0$ which is excluded, for M^n is compact.

Finally, if $E_1 \oplus E_2\neq T^\perp M^n$ at every point (in particular, if $k_3^{(i)}\neq 0$ at every point), we can choose a local frame $(a_1, \dots , a_n, a_{n+1}, \dots , a_{n+m_2},$ $a_{n+m_2+1}, \dots , a_{2n})$ such that $(a_1, \dots , a_n)\in T M^n$, $(a_{n+1}, \dots , a_{n+m_2})\in E_2$, $a_{2n}\in$ $\in (E_1 \oplus E_2 \oplus E_3)$. With such a frame, $\alpha^i{}_{2n}=0$ $\forall i\in\{1, \dots , n+m_2\}$. This implies $\Omega^i{}_{2n}=0$, and $\chi (E_2^\perp)=0$.

c. A majoration of $\chi (E_3^\perp)$.

The proof is exactly similar to b.

d. If $j\geq 4$, $\chi (E_j^\perp)=0$.

Let us consider a local frame $(a_1, \dots, a_n, a_{n+1}, \dots, a_{n+m_j}, \dots, a_{2n+m_j})$ such that $(a_1, \dots, a_n) \in T M^n$, $(a_{n+1}, \dots, a_{n+m_j}) \in E_j$, $(a_{n+m_j+1}, \dots, a_{2n+m_j}) \in E_j^{\perp}$. Obviously, there exists ρ such that $a_\rho \in E_2$. Then, $\alpha_\rho^i(X) = \langle H(X, a_i), a_\rho \rangle = 0$ if $a_i \in T M^n$, and $\alpha_\rho^i(X) = \langle \nabla_X^{\perp} a_i, a_\rho \rangle = 0$ if $a_i \in E_4$. Consequently, $\alpha_\rho^i = 0$ $\forall i \in \{1, \dots, n+m_j\}$. This implies $\Omega_t^s = 0, \Omega_{s_2}^{s_1} \wedge \dots \wedge \Omega_{s_n}^{s_{n-1}} = 0$ $\forall s_1, \dots$ $\dots, s_n \in \{n+m_{j+1}, \dots, 2_{n+m_j}\}$ and $\chi(E_j^{\perp}) = 0$.

Proof of theorem 3. Let us consider a local frame $(a_1, \dots, a_n, a_{n+1}, \dots, a_{2n+1})$ over M^n, such that $(a_1, \dots, a_n) \in T M^n, a_{n+1} \in T N$, $(a_{n+2}, \dots, a_{2n+1}) \in T N^{\perp}$, a_{n+1} is in the direction of the vector v, which is normal to M^n and tangent to N. We have

$$\chi(v^{\perp}) = \frac{(-1)^{n/2}}{2^n \pi^{n/2}\left(\dfrac{n}{2}\right)!} \int_{M^n} \varepsilon_{s_1 \dots s_n} \Omega_{s_2}^{s_1} \wedge \dots \wedge \Omega_{s_n}^{s_{n-1}}$$

where $s_j \in \{n+2, \dots, 2n+1\}$, $\Omega_{s_k}^{s_j} = \sum_{i=1}^{n+1} \alpha_i^{s_j} \wedge \alpha_{s_k}^{i}$, and $\alpha_i^{s_j} = \langle \nabla' a_i, a_{s_j} \rangle$. In an other hand, $\langle \nabla'_X a_i, a_{s_j} \rangle = \langle H^g(a_i, X), a_{s_j} \rangle$, where H^g denotes the second fondamental form associated to g. Then, $\|\alpha_i^{s_j}\| \le k_1^{(g)}$, $\forall i \in \{1, \dots, n+1\}$, $\forall s_j \in \{n+2, \dots, 2n+1\}$. Then

$$|\chi(v^{\perp})| \le \frac{(n+1)^{n/2}\, n!}{2^n \pi^{n/2}\left(\dfrac{n}{2}\right)!} \int_{M^n} k_1^{(g)n}\, dv. \tag{5}$$

Now, we shall prove that (5) is an equality if and only if $k_1^{(g)}{}_{|M^n} = 0$: In fact, let $p \in M^n$, $(a_{n+2}, \dots, a_{2n+1})$ be a local frame, in a neighborhood of p, of the normal bundle $T^{\perp} N$, and $(a_1, \dots, a_{n+1})$ be a local frame, in a neighborhood of p, of the tangent bundle $T N$, such that the bilinear form $\langle H^g(\cdot,\cdot), a_{2n+1} \rangle_p$ is diagonal.

(5) is an equality if and only if every term $\Omega_{s_2}^{s_1} \wedge \dots \wedge \Omega_{s_n}^{s_{n-1}}$ has a maximal norm. In particular, this implies:

$$\alpha^j{}_{2n+1}(a_j) = \pm k_1^{(g)}, \text{ and } \alpha^j{}_{n+i}(a_j) = 0, \quad \forall j \in \{1, \dots, n+1\}.$$

The mean curvature vector field h associated to g is $h = q\, k_1^{(g)} a_{2n+1}$, where $q \in \mathbf{Z}$ is not null, since $\dim N$ is odd. But h is an intrinsec vector field, and a_{2n+1} is arbitrary. Then $k_{1_p}^{(g)} = 0$ $\forall p \in M^n$, (i. e. $H^{(g)}(X, Y) = 0$ $\forall X, Y \in TN$).

Finally, if $E^N_{1_p} \neq T_p^{\perp} N^{n+1}$ $\forall p \in M^n$, we can choose a_{2n+1} in $E_1^{N\perp}$, (where E_1^N designs the first principal normal space of N). In this case $\Omega^s{}_{2n+1} = \sum_{i=1}^{n+1} \alpha_i^s \wedge \alpha^i{}_{2n+1} = 0$ since $\alpha^i{}_{2n+1}(X) = \langle \nabla'_X a_i, a_{2n+1} \rangle = 0$ $\langle H(X, a_i), a_{2n+1} \rangle = 0$.

Then, $\Omega^{s_1}_{s_2} \wedge \ldots \wedge \Omega^{s_n-1}_{s_n} = 0$ and $\chi(v^{\perp}) = 0$.

3. Self - Linking of a Riemannian submanifold.

In [4], J. H. White defined the notion of Self-Linking SL of a submanifold in Euclidean space. We shall prove the

Theorem 4. *Let i: $M^n \to E^{2n+1}$ an isometric immersion of a compact oriented even dimensional Riemannian manifold into the Euclidean spece E^{2n+1}. We suppose that i is everywhere not minimal. Then, if $(E_1 \oplus E_2)_p \neq T_p^{\perp} M_n$ at every point $p \in M^n$, (in particular if $k_3^{(i)}$ is defined and $\neq 0$ at every point), the self linking SL of M^n is null.*

Proof of theorem 4. Let h be the mean curvature vector field. Since i is everywhere not minimal, h is not null everywhere. Using the definition of J. H. White, the Self Linking SL of M^n satisfies:

$SL = \frac{1}{2} \chi(h^{\perp})$, where $\chi(h^{\perp})$ is the Euler characteristic of the complementary (to h) subbundle of the normal bundle. In order to evaluate $\chi(h^{\perp})$, we consider a local frame $(a_1, \ldots, a_{2n+1})$ over M^n, such that $(a_1, \ldots, a_n)$ is tangent to M^n, a_{n+1} is in the direction h, a_{2n+1} is in $(E_1 \oplus E_2)^{\perp}$.

Then,

$$\chi(h^{\perp}) = \frac{(-1)^{n/2}}{2^n \pi^{n/2} \left(\frac{n}{2}\right)!} \int_{M^n} \varepsilon_{s_1 \ldots s_n} \Omega^{s_1}_{s_2} \wedge \ldots \wedge \Omega^{s_n-1}_{s_n} .$$

where $\Omega_t^s = \sum_{i=1}^{n+1} \alpha_i^s \wedge \alpha_t^i$. But every $\Omega^{s_1}_{s_2} \wedge \ldots \wedge \Omega^{s_n-1}_{s_n}$ is a sum of terms which are multiple of α_i^{2n+1}, $i = 1, \ldots, n$. Since $\alpha_i^{2n+1} = \langle \nabla' a_i, a_{2n+1} \rangle = 0$, because $a_{2n+1} \in (E_1 \oplus E_2)^{\perp}$, $\chi(h^{\perp}) = 0$.

4. Euler-class of a Riemannian submanifold.

THEOREM 5. *Let i: $M^n \to E^{n+2k}$ be an isometric immersion of an oriented Riemannian manifold M^n in Euclidean space E^{n+2k}, such that:*

$$\forall p \in M^n, E_{1p} \neq T^{\perp} M_p^n.$$

(In particular, this is satisfied if $k_2^{(i)}$ is defined and $\neq 0$ at every point). Then, $\bar{\omega}_{2k}(T M^n)=0$.

Before the proof of the theorem, we shall give an application.

Let $\mathbf{P}^{2p}$ the oriented real projective space, of dimension $2p$, and S^{2p+1} the sphere of dimension $2q+1$. Since $\omega(S^{2q+1})=1$, $\omega(\mathbf{P}^{2p}\times S^{2q+1})=$ $=\omega(\mathbf{P}^{2p})$. Then, if $\bar{\omega}_{2k}(\mathbf{P}^{2p})\neq 0$, every immersion of $\mathbf{P}^{2p}\times S^{2q+1}$ into $E^{2(p+q+k)+1}$ is such that $E_1=T^{\perp}(\mathbf{P}^{2p}\times S^{2q+1})$ on an open set.

PROOF OF THEOREM 5. Let π: $T^{\perp} M^n \to M^n$ be the projection of $T^{\perp} M^n$ on M^n, and π: $H^{2k}(M^n, \mathbf{Z}) \to H^{2k}(T^{\perp} M^n, \mathbf{Z})$ the canonical isomorphism induced by π, on the $2k^{th}$ cohomology groups.

Let $(a_1, \dots, a_{n+2k})$ be a local frame over M^n, such that $(a_1, \dots, a_n) \in TM^n$, $(a_{n+1}, \dots, a_{n+2k}) \in T^{\perp} M^n$. We denote $\alpha_i^s=\langle \nabla' a_s, a_i\rangle$. With these notations, the Euler-class of the normal bundle $e(T^{\perp} M^n)$, is represented by the closed $2k$-form γ, defined by

$$\pi(\gamma)=\frac{(-1)^k}{2^{2k}\pi^k k!}\Sigma\, \varepsilon_{s_1 \dots s_{2k}}\, \Omega_{s_2}^{s_1}\wedge \dots \wedge \Omega_{s_{2k}}^{s_{2k-1}},$$

where

$$\Omega_t^s=\sum_{i=1}^{n}\alpha_i^s \wedge \alpha_t^i,\quad i\in\{1,\dots,n\},\ s,t\in\{n+1,\dots,n+2k\}$$

$$\alpha_i^s(X)=\langle H(X, a_i), a_\alpha\rangle.$$

If $E_1(=[Im\, H])\neq T^{\perp} M^n$ at every point, (in particular if $k_2^{(i)}\neq 0$ at every point), we can choose $a_{n+2k}\in E_1$. In this case, $\alpha_i^{n+2k}=0$, and every term $\Omega_{s_2}^{s_1}\wedge\dots\wedge\Omega_{s_{2k}}^{s_{2k-1}}$ is null (since it is a multiple of α_i^{n+2k}).

Consequently, $\pi(\gamma)=0$. Since π is an isomorphism, $\gamma=0$, and $e(T^{\perp} M^n)=0$.

Now, let us consider the canonical homomorphism

$$h\colon H^{2k}(M^n) \to H^{2k}(M^n, \mathbf{Z}/2).$$

It is well known ([7]) that $h[e(T^{\perp}M^n)] = \omega_{2k}(T^{\perp}M^n)$, where $\omega_{2k}(T^{\perp}M^n)$ is the $2k^{th}$ Stiefel-Whitney class of $T^{\perp}M^n$. Consequently, $\omega_{2k}(T^{\perp}M^n)=0$.

Using Whiteny duality theorem, we obtain:

$$\omega_{2k}(T^{\perp}M^n)=\bar{\omega}_{2k}(TM^n)=0.$$

5. Chern-Lashof curvature and External curvatures.

We will prove the following

THEOREM. *Let $i: M^n \to E^{n+N}$ be an isometric immersion of an oriented compact manifold M^n into E^{n+N}. Then, if $K(M^n)$ is the Chern-Lashof curvature,*

$$K(M^n) \leq n^n \operatorname{Vol}(S^{N-1}) \int_{M^n} k_1^n \, dv.$$

PROOF OF THE THEOREM. Let $e_{n+N}: \underset{(x,t) \to t}{S(T^{\perp}M^n) \to S^{n+N-1}}$, where $S(T^{\perp}M^n)$ is the bundle of unit normal vectors of M^n, and where S^{n+N-1} is the $n+N-1$ sphere of E^{n+N}, an let $(e_1, \dots, e_n, e_{n+1}, \dots, e_{n+N})$ be a local frame, such that $e_1, \dots, e_n \in TM^n$, and $e_{n+1}, \dots, e_{n+N} \in T^{\perp}M^n$.

It is well known ([8]) that the volume element of $S(T^{\perp}M^n)$ has the following local expression

$$\omega^1{}_{n+N} \wedge \dots \wedge \omega^n{}_{n+N} \wedge \omega^{n+1}_{n+N} \wedge \dots \wedge \omega^{n+N-1}_{n+N},$$

where $\omega^i{}_{n+N} = \langle \nabla' e_{n+N}, e_i \rangle$.

Now, we consider a local frame $(a_1, \dots, a_{n+N})$ such that

$$a_1, \dots, a_n = e_1, \dots, e_n$$

$$a_{n+1}, \dots, a_{n+N} \in T^{\perp}M^n.$$

We set $\alpha_i{}^j = \langle \nabla' a_i, a_j \rangle$, and

$$e_{n+N} = \gamma^{n+1}_{n+N} a_{n+1} + \dots + \gamma^{n+N}_{n+N} a_{n+N}.$$

It is easy to observe that

$$\omega^1{}_{n+N}\wedge\ldots\wedge\omega^n{}_{n+N}=\sum_{i_k=n+1}^{n+N}\gamma^{i_1}_{n+N},\ldots,\gamma^{i_n}_{n+N}\,\alpha^1_{n+i_1}\wedge\ldots\wedge\alpha^n_{n+i_n}\,.$$

On an other hand, dS^{N-1}, the volume element of the unit normal sphere is

$$dS^{N-1}=\omega^{n+N}_{n+N}\wedge\ldots\wedge\omega^{n+N-1}_{n+N}\,.$$

Consequently,

$$K(M^n)=\int_{S(T^\perp M^n)}\omega_{n+N}\wedge\ldots\wedge\omega^{n+N-1}_{n+N}=\int_{S(T^\perp M^n)}(\Sigma\,\gamma^{n+i_n}_{n+N},\ldots,\gamma^{n+i_n}_{n+N}\,\alpha^1_{n+i_1}$$

$$\wedge\ldots\wedge\alpha^n_{n+i_1})=\int_{M^n}\Sigma\,\gamma^{i_1}_{n+N},\ldots,\gamma^{i_n}_{n+N}\alpha^1{}_{n+i_1}\wedge\ldots\wedge\alpha^n{}_{n+i_n}\int_{\text{Fiber}}dS^{N-1}$$

where $(i_1,\ldots,i_n)\in\{1,\ldots,N\}$.

Obviously, $\alpha^k{}_{n+i_k}(X)=\langle H(a_k,X),a_{n+i_k}\rangle$.Thus, $\|\alpha^k{}_{n+i_k}\|\leq k_1{}^{(i)}$. Since $|\gamma^{i_k}_{n+N}|\leq 1$, we obtain $|K(M^n)|\leq n^n(\mathrm{Vol}\,S^{N-1})\cdot\int_{M^n}k_1{}^n\,dv.$

(Compare with [9] p. 220).

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