Automatic sequences fulfill the Sarnak conjecture

Clemens Müllner

23. Sept 2016

Möbius function

The Möbius function is defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is squarefree and} \\ & k \text{ is the number of prime factors} \\ 0 & \text{otherwise} \end{cases}$$

A sequence $\mathbf{u}$ is **orthogonal to the Möbius function** $\mu(n)$ if

$$\sum_{n \leq N} \mu(n) u_n = o\Big(\sum_{n \leq N} |u_n|\Big) \qquad (N \to \infty).$$

Old Heuristic - Mobius Randomness Law

Any "reasonably defined" bounded sequence independent of μ is orthogonal to μ.

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Orthogonality to μ

Results

- Constant sequences $\Leftrightarrow$ PNT
- Periodic sequences $\Leftrightarrow$ PNT in arithmetic Progressions
- Quasiperiodic sequences $f(n) = F(\alpha n \bmod 1)$ - Davenport
- Nilsequences - Green and Tao
- Horocycle Flows - Bourgain, Sarnak and Ziegler
- Bounded depth circuits - Green
- Some special examples/classes of automatic sequences

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Definition

A dynamical system is said to be determinist, if its topological entropy is 0.

Conjecture (Sarnak conjecture, 2010)

Every bounded complex sequence $\mathbf{u} = (u_n)_{n>0}$ that is obtained by a deterministic dynamical system is orthogonal to the Möbius function $\mu(n)$.

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Let $0 \leq a_1 < a_2 < \ldots < a_t$ and $k_1, k_2, \ldots, k_t$ in $\{1, 2\}$ not all even, then as $N \to \infty$

$$\sum_{n \leq N} \mu^{k_1}(n + a_1)\mu^{k_2}(n + a_2) \cdots \mu^{k_t}(n + a_t) = o(N).$$

Theorem (Sarnak)

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Sarnak Conjecture

Dynamical System (X, T) related to $\mathbf{u}$

$\mathbf{u} = (u_n)_{n\geq 0} \ldots$ bounded complex sequence

$T\mathbf{u} = (u_{n+1})_{n\geq 0} \ldots$ shift operator

$X = \overline{\{T^k(\mathbf{u}) : k \geq 0\}}$

We say that $\mathbf{u}$ satisfies the **Sarnak conjecture** if all sequences $\mathbf{a} = (a_n)_{n\geq 0} \in X$ are orthogonal to $\mu(n)$.

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Definition

Let E be a finite set and σ a k-uniform morphism such that $\sigma(E) \subseteq E^k$. Then if $\mathbf{w}$ is a fixed point of σ, i.e. $\sigma(\mathbf{w}) = \mathbf{w}$, then $\mathbf{w}$ is a k-automatic sequence.

Example (Thue-Morse)

$E = \{0, 1\}$
$\sigma(0) = 01$
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Deterministic Finite Automata

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \ldots, k-1\}, \delta, q_0, \tau)$$

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$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

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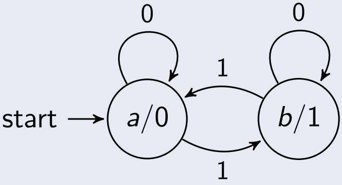

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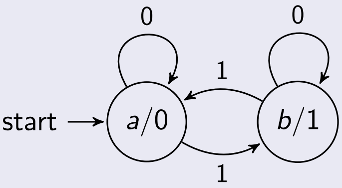

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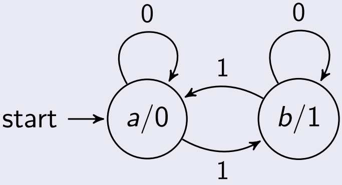

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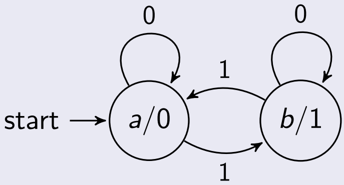

Properties of Automatic Sequences

- For every automatic sequence $\mathbf{u}$ there exists the logarithmic density

$$logdens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{log(N)} \sum_{1\leq n\leq N} \frac{1}{n}\mathbf{1}_{[u_n=a]}.$$

- The subword complexity p_k of an automatic sequence is (at most) linear.The dynamical system (X, T) related to an automatic sequence has zero topological entropy.
- Every subsequence $(u_{an+b})_{n\geq 0}$ along an arithmetic progression of an automatic sequence $(u_n)_{n\geq 0}$ is again automatic.
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Theorem 1 (M., 2016)

Every automatic sequence $(a_n)_{n\geq 0}$ fulfills the Sarnak Conjecture

Theorem 2 (M., 2016)

Let $A = (Q', \Sigma, \delta', q'_0, \tau)$ be a strongly connected DFAO such that $\Sigma = \{0, \ldots, k-1\}$ and $\delta'(q'_0, 0) = q'_0$. Then the frequencies of the letters for the prime-subsequence $(a_p)_{p\in\mathcal{P}}$ exist, i.e.

$$dens_{\mathcal{P}}(\mathbf{u}, \alpha) = \lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{1\leq p\leq N} \mathbf{1}_{[u_p=\alpha]}.$$

Remark: All block-additive (i.e. digital) functions are covered by Theorem 2 and they are "usually" uniformly distributed.

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Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

Example

$\mathbf{w}_0 = 010.$

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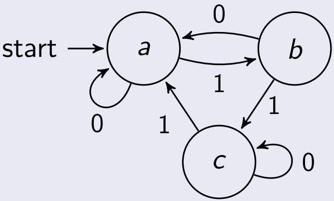

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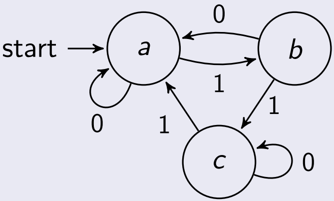

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Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ be generated by a synchronizing automaton. Then for every α the density

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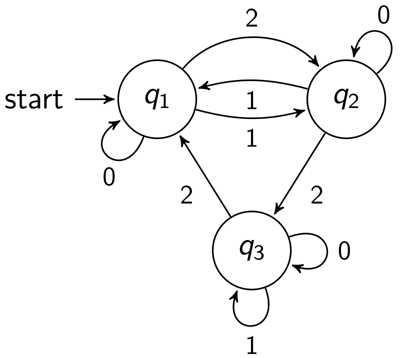
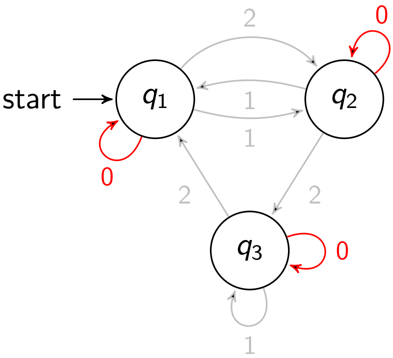
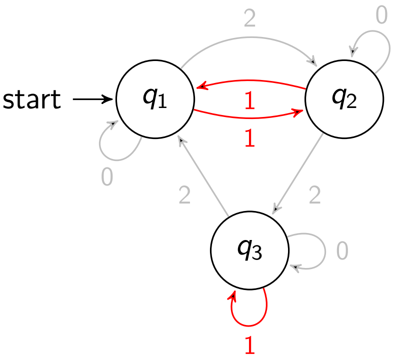
Let $\mathbf{u} = (u_n)n > 0$ be generated by a synchronizing automaton. Then $\mathbf{u} = (u_n)_{n>0}$ is orthogonal to the Möbius function $\mu(n)$.

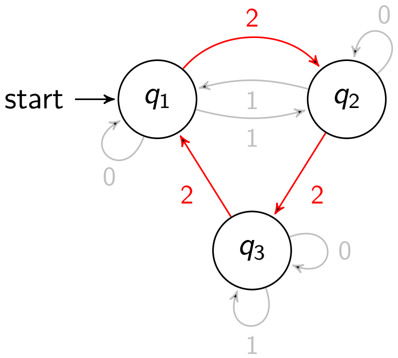

2
0
start
q_1
1
q_2
1
0
2
2
q_3
0
1



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



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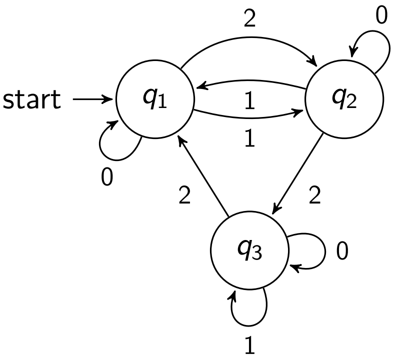

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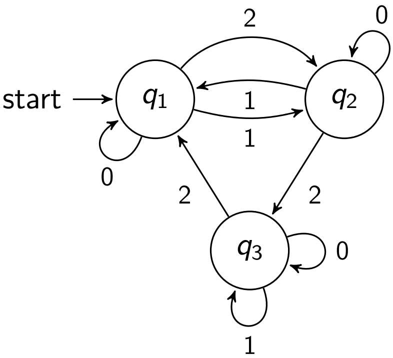
$11 = (102)_3$:

$$M_2 \circ M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



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$$T(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

$$u(n) = f(T(n)\mathbf{e}_1) \qquad \mathbf{e}_1 = (1 \quad 0 \quad 0)^T$$

Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

Remark:

If the matrix $M = M_0 + \ldots + M_{k-1}$ is primitive then the densities

$$dens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N} \mathbf{1}_{[u_n=a]}$$

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Results for Invertible Automata

Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is generated by an invertible automaton.

Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

$\mathbf{u}$ is orthogonal to $\mu(n)$.

Theorem[Drmota]

The frequency of each letter of the subsequence $(u_p)_{p\in\mathcal{P}}$ exists.

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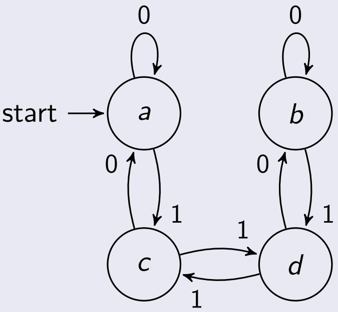

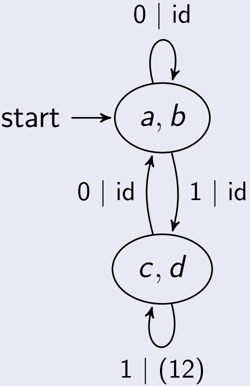

Theorem [Mauduit + Rivat, Tao]

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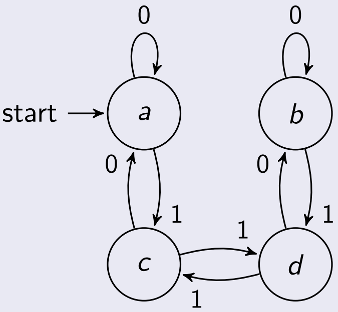

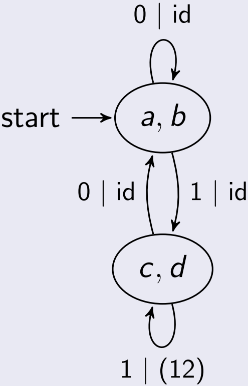

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Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q_0')$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
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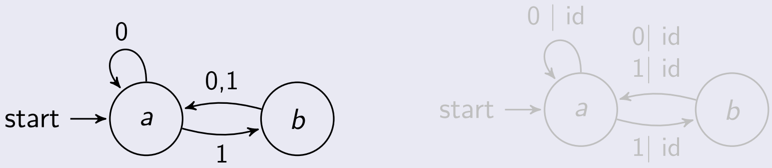

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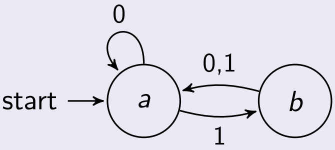

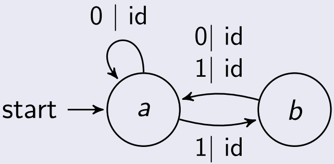

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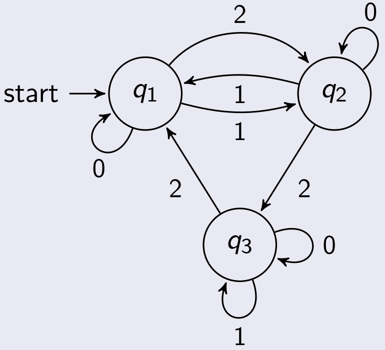

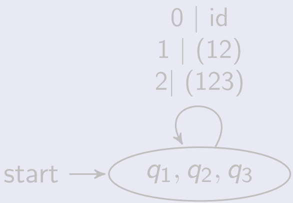

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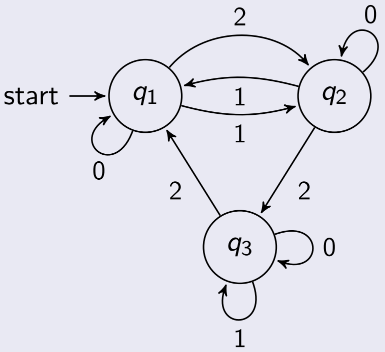

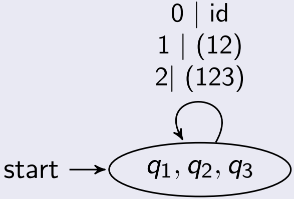

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For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

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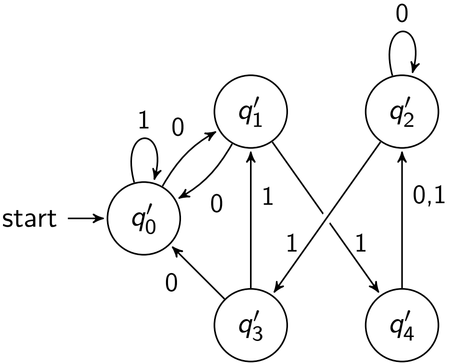

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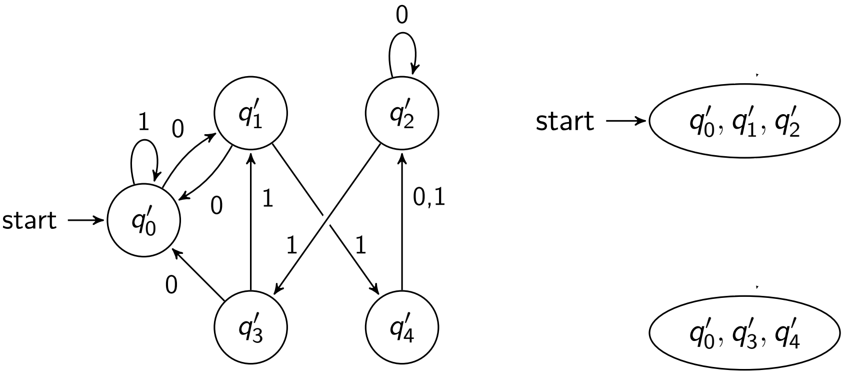

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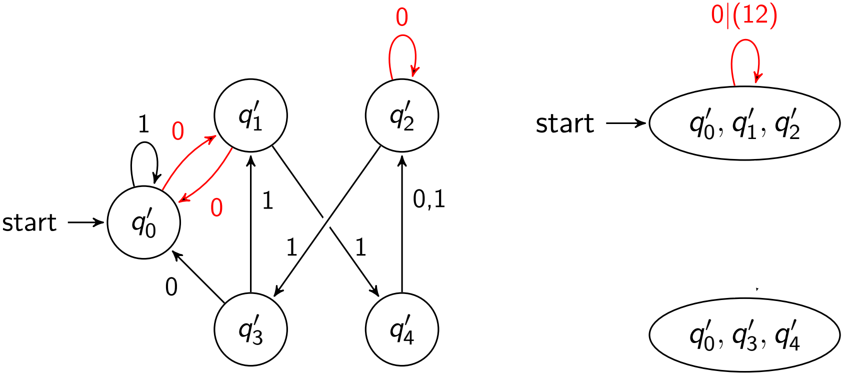

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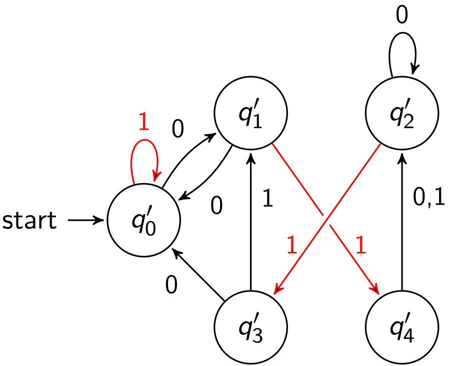

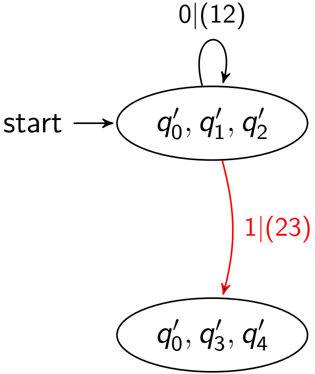

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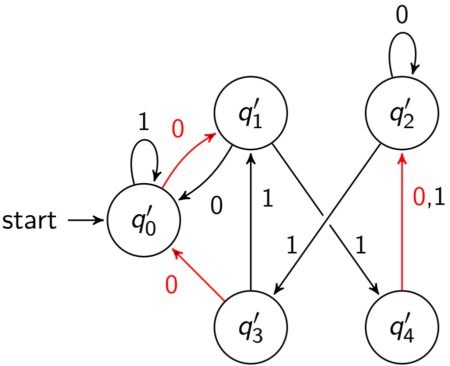

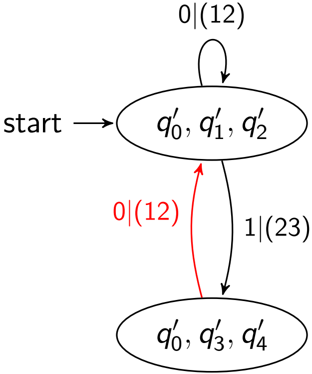

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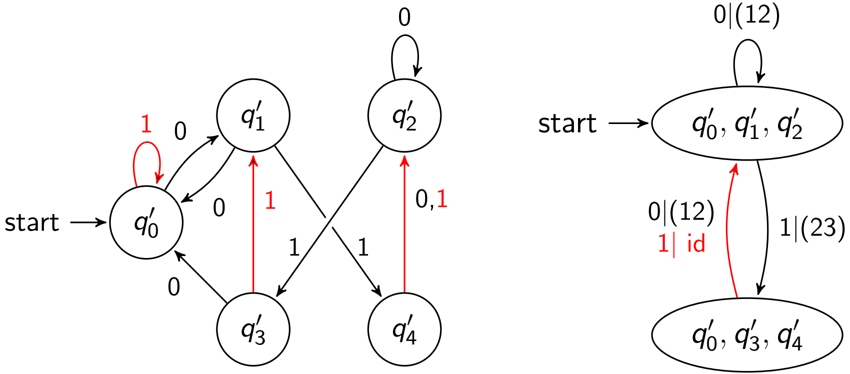

Definition

Denote by

$$T(q, w_1 \dots w_r) := \lambda(q, w_1) \circ \lambda(\delta(q, w_1), w_2) \circ \dots \\ \circ \lambda(\delta(q, w_1 \dots w_{r-1}), w_r).$$

Lemma

Let A be a strongly connected automaton and $\mathcal{T}_A$ a naturally induced transducer. Then,

$$\delta'(q'_0, \mathbf{w}) = \pi_1(T(q_0, \mathbf{w}) \cdot \delta(q_0, \mathbf{w}))$$

holds for all $\mathbf{w} \in \Sigma^*$.

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Are some naturally induced transducers better than others?

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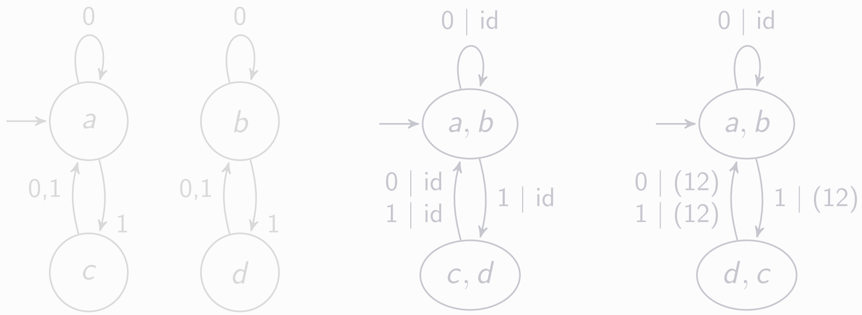

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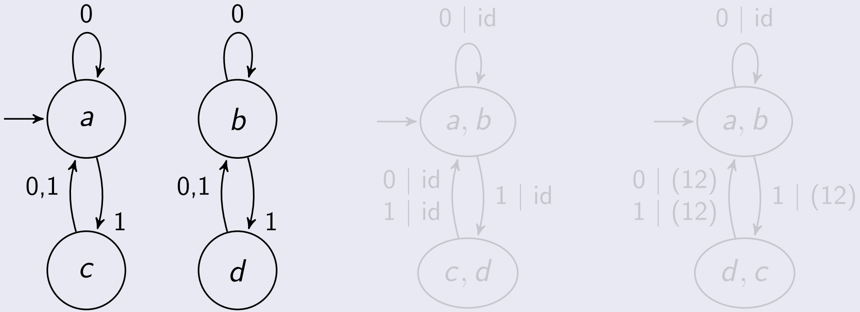

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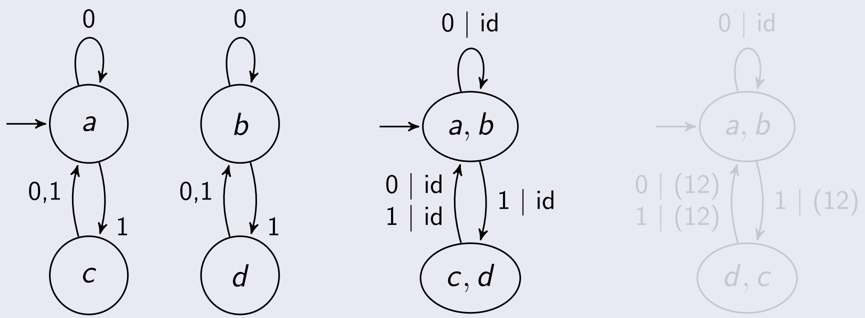

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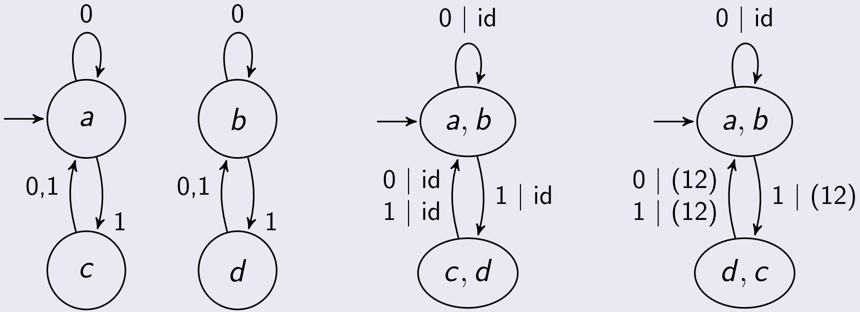

All elements of Δ appear as values of $T(q_0, .)$ for „good“ naturally induced transducer.

Do all elements of Δ appear simultaneously as values of $T(q_0, \mathbf{w})$ for $\mathbf{w} \in \Sigma^n$ for a single n, where n is large?

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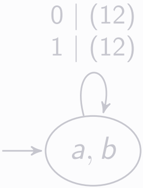

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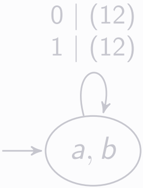

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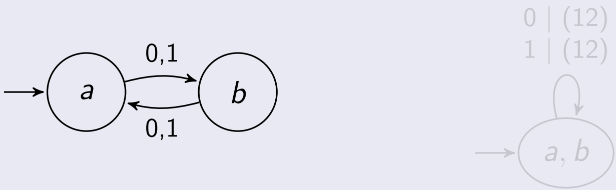

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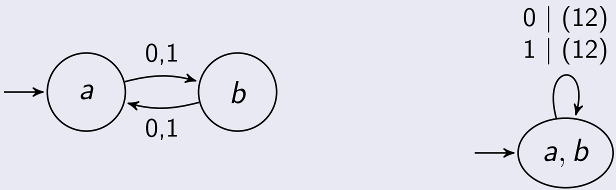

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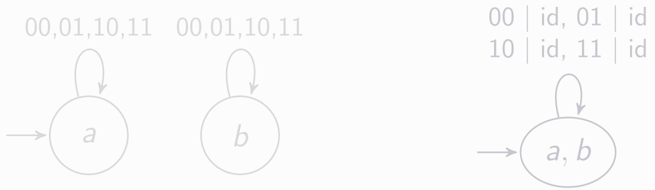

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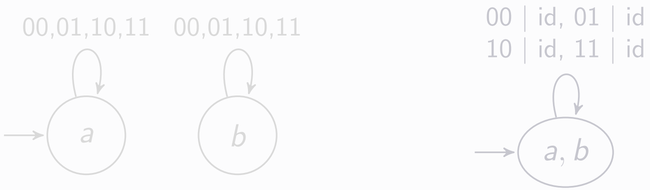

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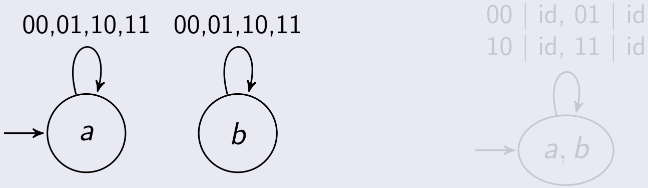

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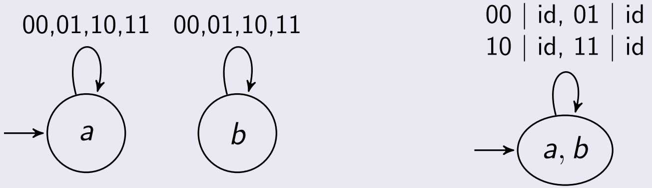

Continuous functions from a compact group to $\mathbb{C}$

Definition (Representation)

Let G be a finite group and $k \in \mathbb{N}$. A **Representation** of rank k is a continuous homomorphism $D : G \to \mathbb{C}^{k \times k}$.

Lemma

Let f be a continuous function from G to $\mathbb{C}$. There exists $r \in \mathbb{N}$ and unitary, irreducible representations $D^{(\ell)} = (d_{i,j}^{(\ell)})_{i,j<k_\ell}$ along with $c_\ell \in \mathbb{C}$ such that

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Lemma

Suppose that

$$\sum_{\substack{n<N\\ \cdots}} D(T(n))\mu(n) = o(N)$$

holds for all irreducible unitary representations of G. Then $\mathbf{u} = (u_n)_{n\geq 0}$ is orthogonal to $\mu(n)$.

We follow the method of Mauduit and Rivat that they use for studying the Rudin-Shapiro sequence.

(Adopted) Definition

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Suppose that $D \circ T$ has the Fourier property. Then we have for any real θ

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Primes vs all natural Numbers

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