Éléments totalement commutatifs et chemins du plan

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GT Combi, LIX, 10 Décembre 2012

Fully commutative elements

(W, S) Coxeter group W given by Coxeter matrix $(m_{st})_{s,t\in S}$.

Relations: $\begin{cases} s^2 = 1 \\ \underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}} \end{cases}$ Braid relations

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Matsumoto property : Given two reduced decompositions of w, there is a sequence of braid relations which can be applied to transform one into the other.

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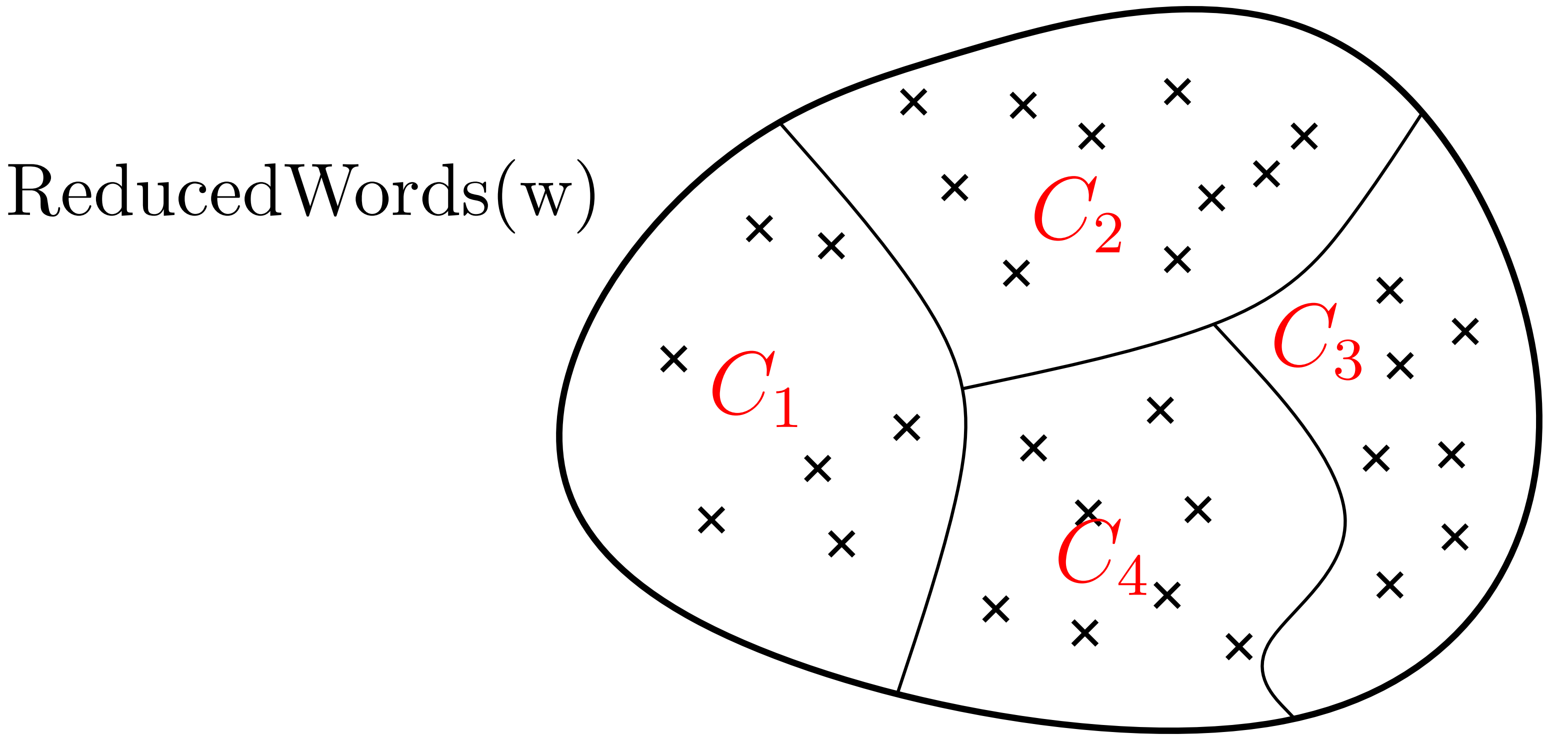

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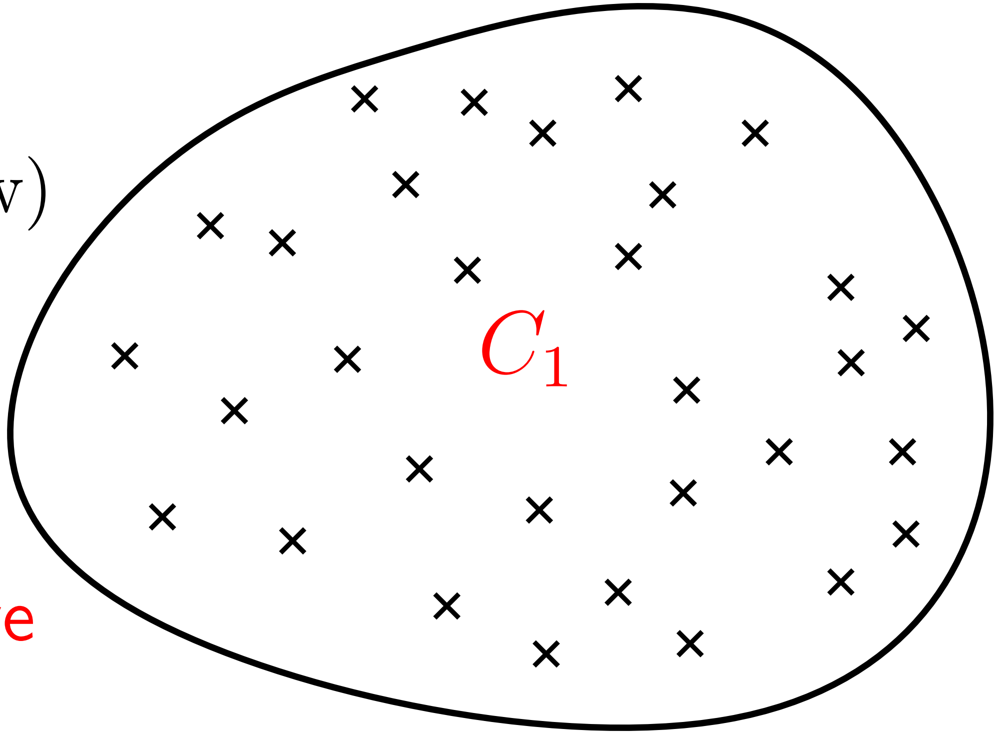
$\mathrm{ReducedWords(w)}$

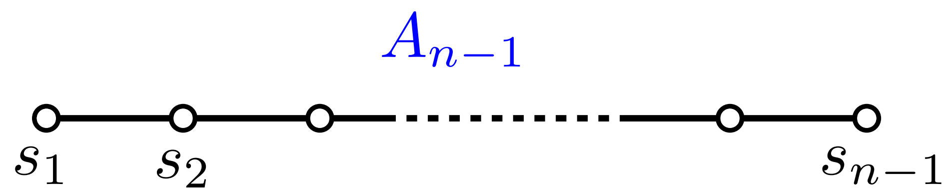


w fully commutative

Type A_{n-1} $\to$ The symmetric group S_n

Consider $S = \{s_1, \ldots, s_{n-1}\}$, with relations $s_i^2 = 1$ and

$$\begin{cases} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i s_j = s_j s_i, \quad |j - i| > 1 \end{cases}$$



$\boxed{\vartheta : s_i \mapsto (i, i+1) \text{ is an isomorphism with } S_n.}$

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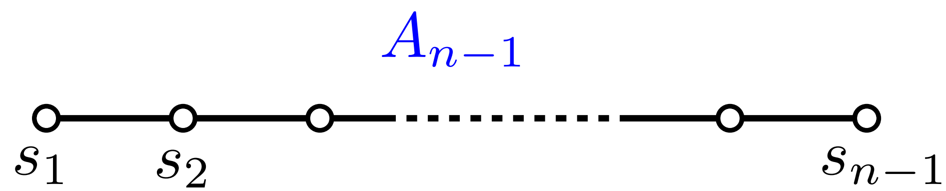

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Theorem [Billey,Jockush,Stanley '93]

w is fully commutative $\Leftrightarrow$ $\vartheta(w)$ is 321-avoiding.

One can use this to show that FC elements in type A_{n-1} are counted by Catalan numbers, i.e. $|S_n^{FC}| = \dfrac{1}{n+1}\dbinom{2n}{n}$.

Previous work

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• [Fan '95] studies FC elements in the special case where $m_{st} \leq 3$ (*the simply laced case*).

• [Graham '95] shows that FC elements in any Coxeter group W naturally index a basis of the (generalized) Temperley-Lieb algebra of W.

• Subsequent works [Greene,Shi,Cellini,Papi] relate FC elements (and some related elements) to Kazhdan-Lusztig polynomials.

Outline

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Today I will focus on types A and $\widetilde{A}$, corresponding to the finite and affine symmetric groups. The idea is to encode the FC elements in these cases by certain lattice paths.

1. FC elements and Heaps

Characterization of FC elements

In general, how can one recognize a FC element ? The following is one step in this direction.

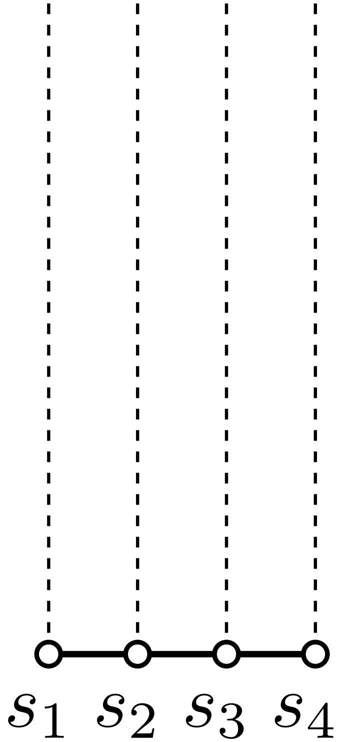
Theorem[Stembridge] A reduced word represents a FC element if and only no element of its commutation class contains a factor $\underbrace{sts\cdots}_{m_{st}}$ for a $m_{st} \geq 3$.

(Proof: when two words are related by a braid relation with $m_{st} \geq 3$, they do not belong to the same commutation class.)

How to tell if a commutation class verifies the property above ?
$\Rightarrow$ Use theory of heaps, which are posets which encode commutation classes.

Example of heaps in $A_4 (= S_5)$

$s_1 s_3 s_4 s_1 s_2 s_3$



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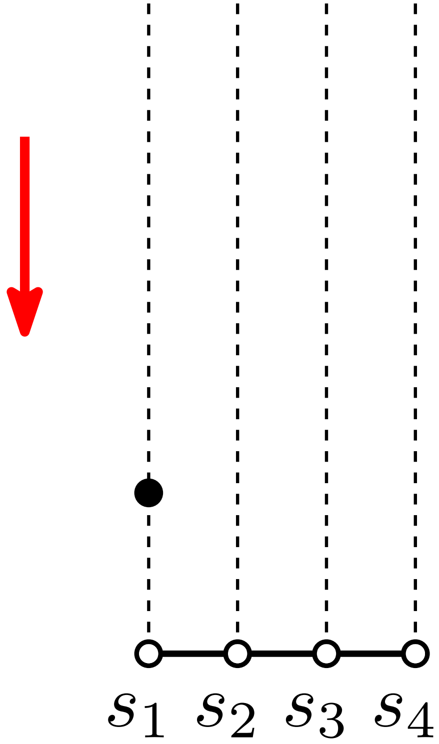

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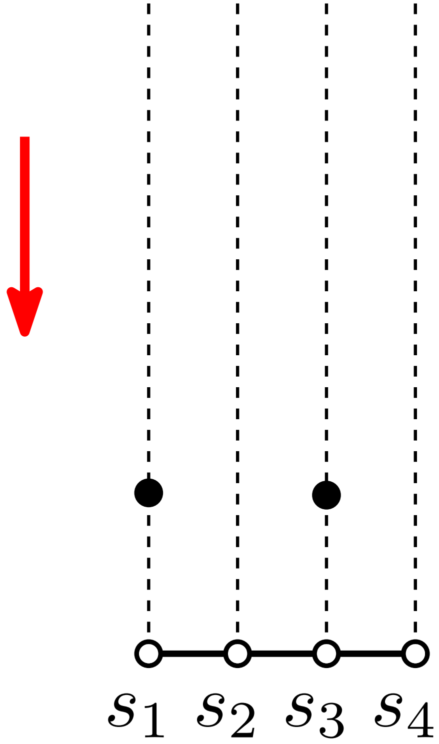

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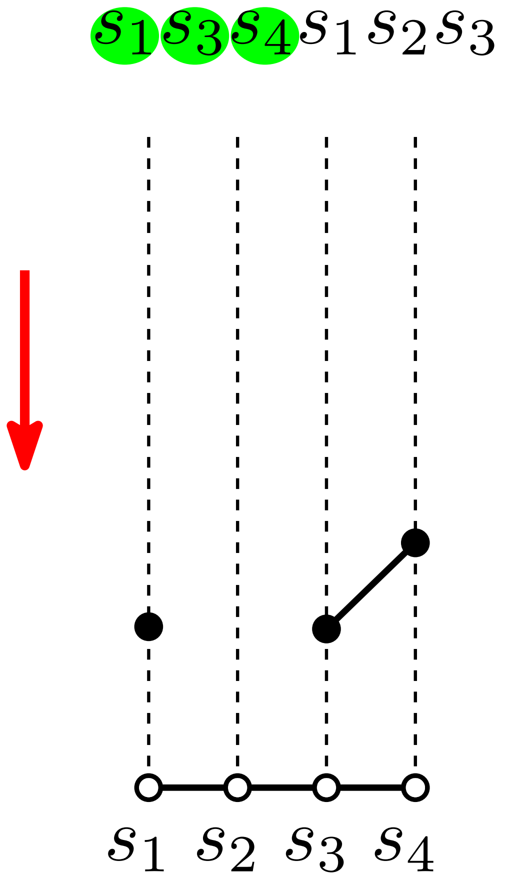

Vertex stays above if corresponding generators do not commute.

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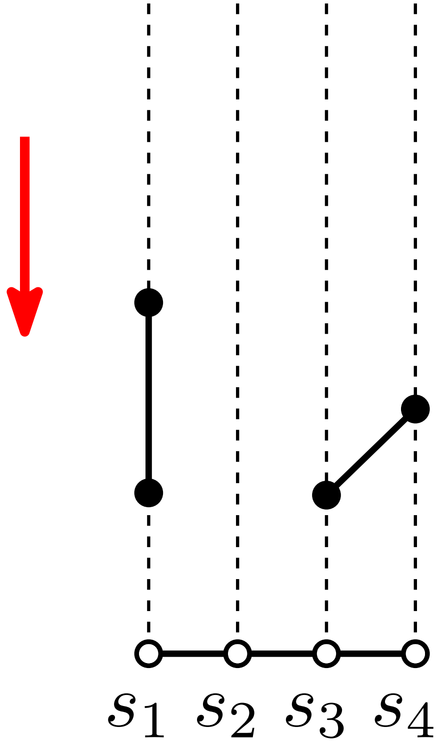

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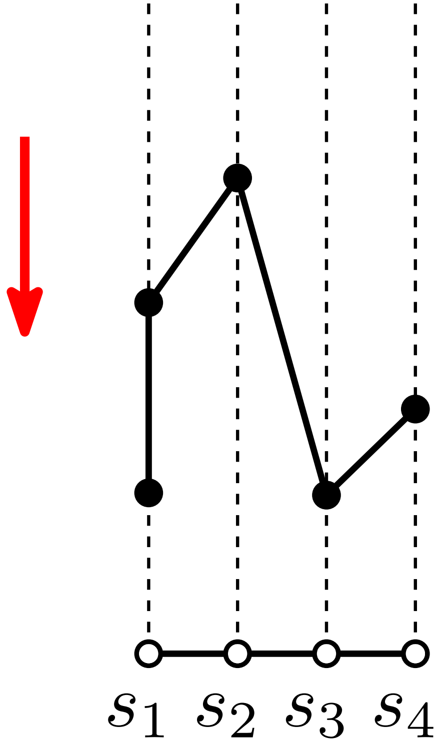

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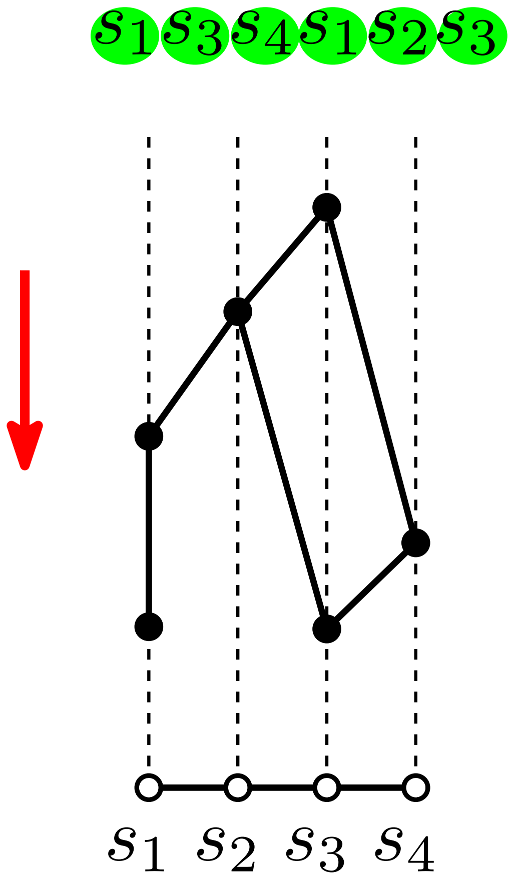

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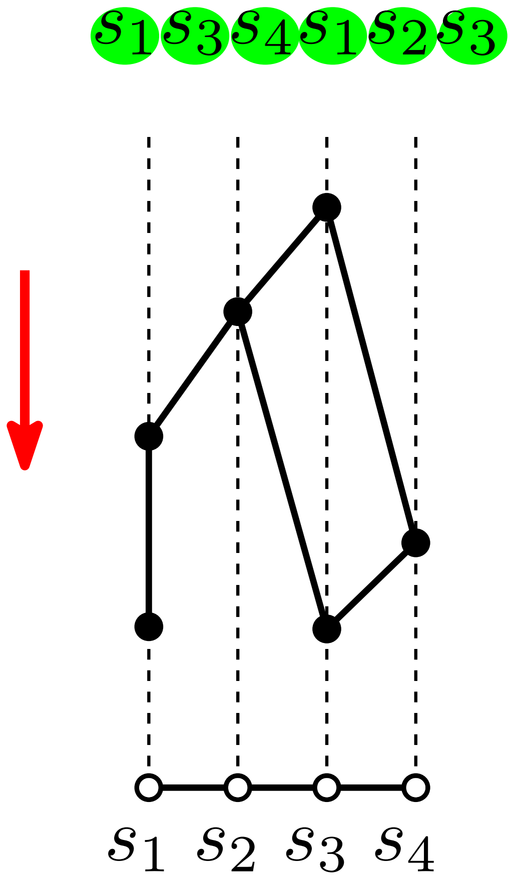
Heap of a word $=$ poset H labeled by generators s_i of W.

Linear extensions of H $\Leftrightarrow$ Words of the commutation class.



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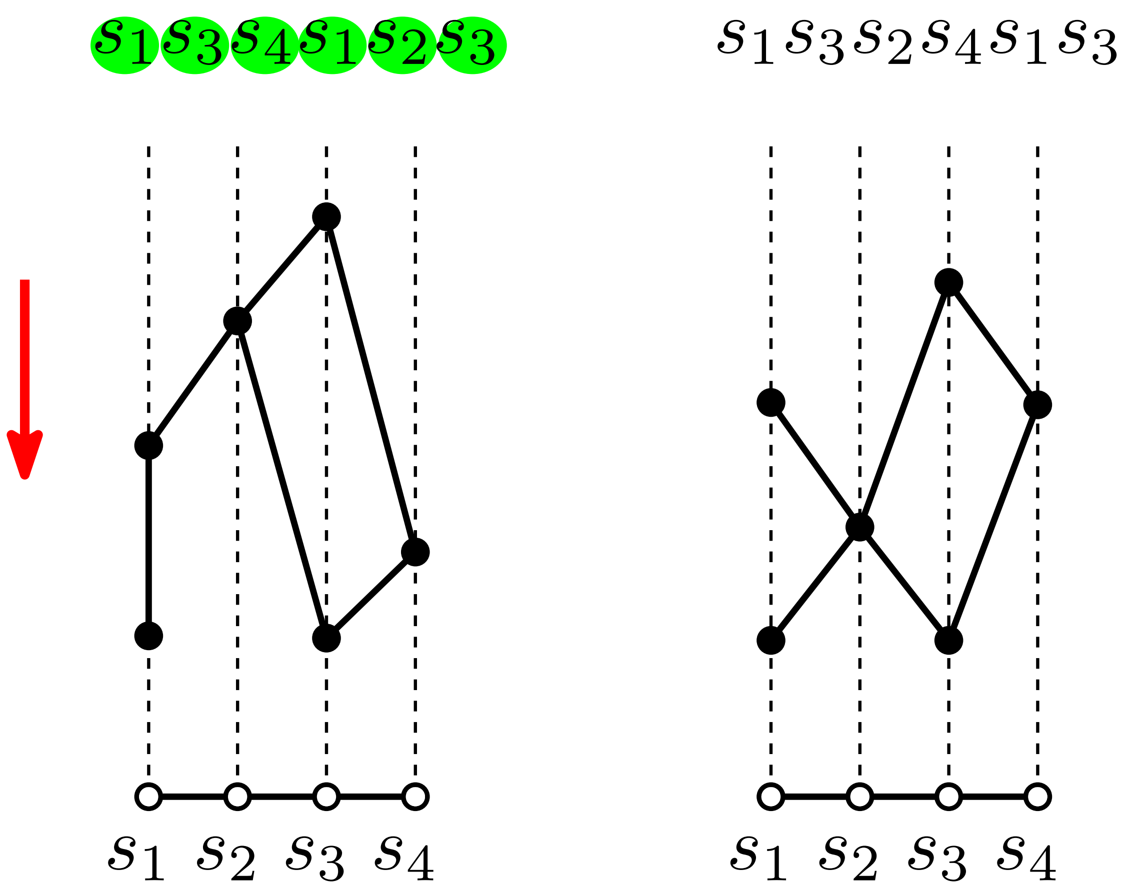

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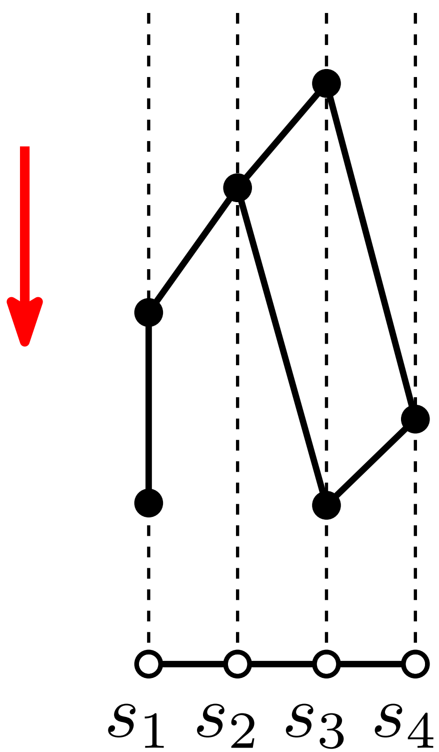

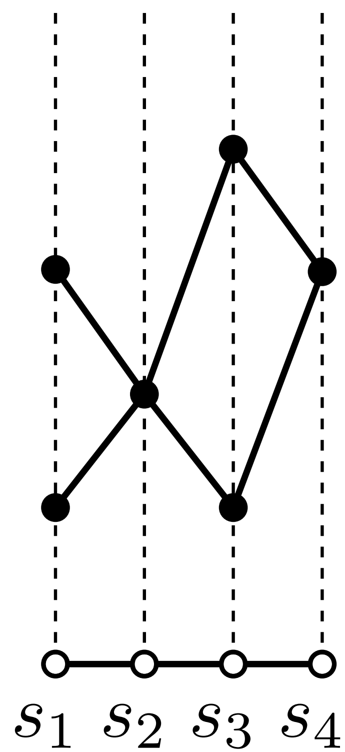

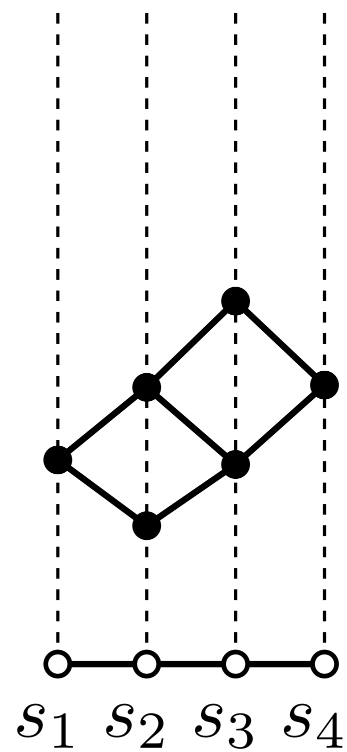

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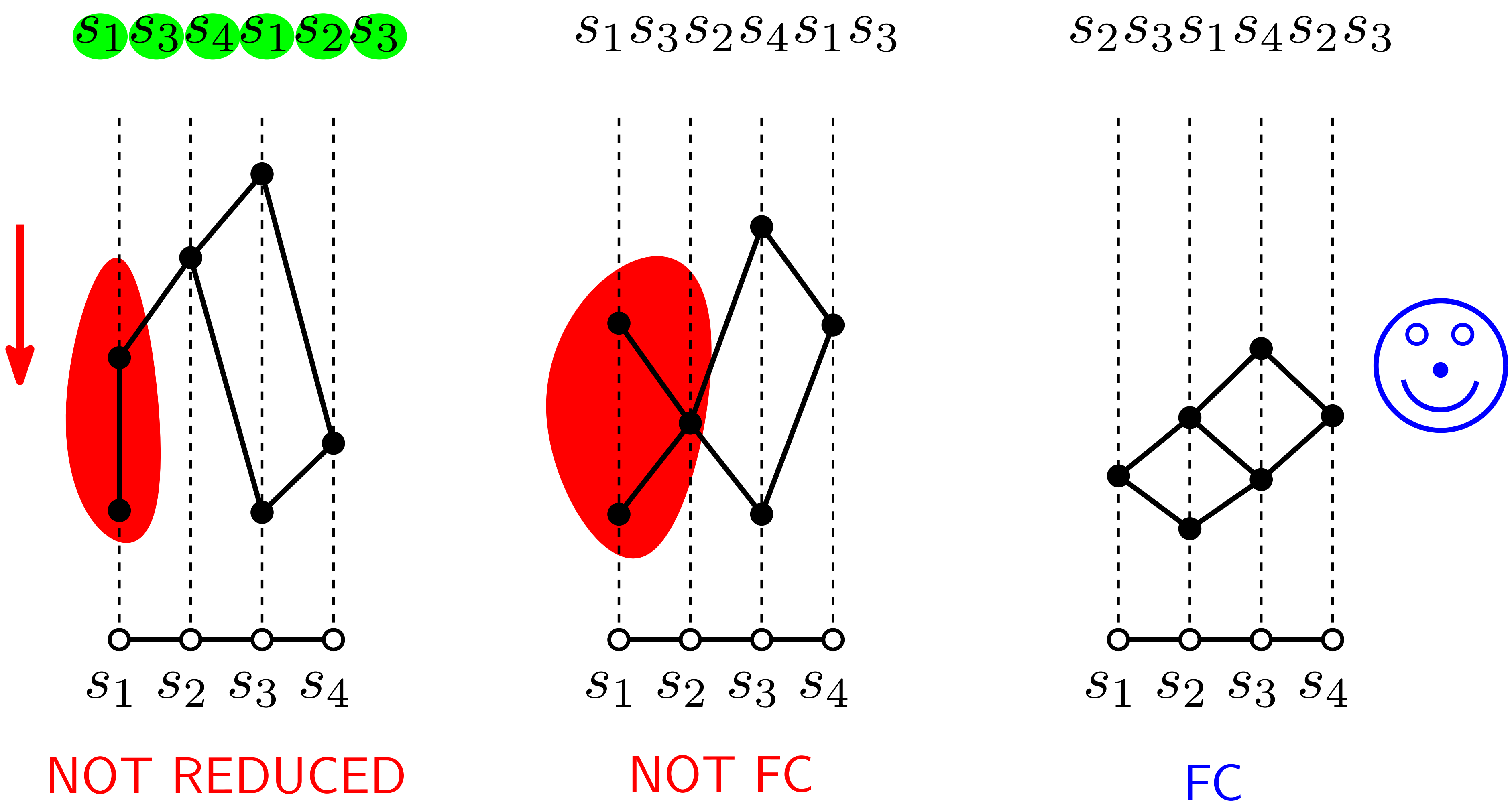

Characterization of heaps

Proposition[Stembridge '95] Heaps H of FC reduced words are characterized by:
(a) No covering relation $i \prec j$ in H such that $s_i = s_j$.
(b) No convex chain $i_1 \prec \cdots \prec i_{m_{st}}$ in H such that $s_{i_1} = s_{i_3} = \cdots = s$ and $s_{i_2} = s_{i_4} = \cdots = t$ where $3 \leq m_{st} < \infty$.

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(the only elements x satisfying $i_1 \leq x \leq i_{m_{st}}$ are the elements i_j of the chain.)

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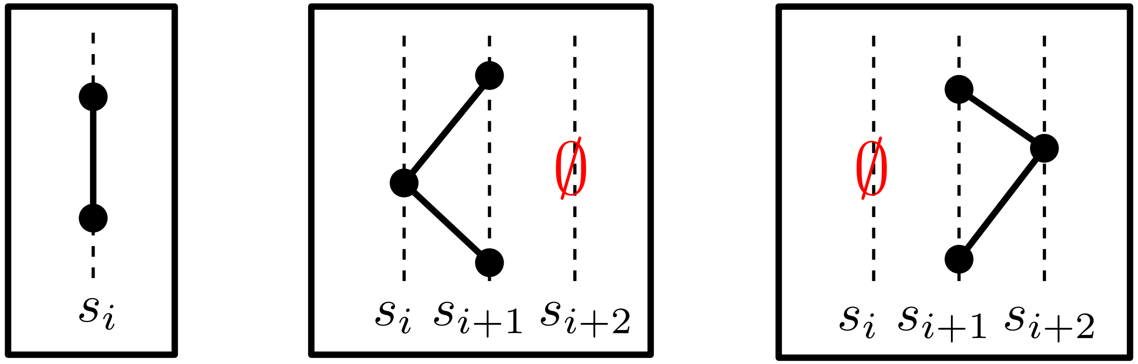
FC element w	$\longleftrightarrow$	Heap H satisfying (a) and (b)
Length $\ell(w)$	$\longleftrightarrow$	Number of elements $\lvert H \rvert$

In type A and $\widetilde{A}$, we will see that the FC heaps above are particularly nice.

1. Type A

Type A

FC heaps avoid precisely



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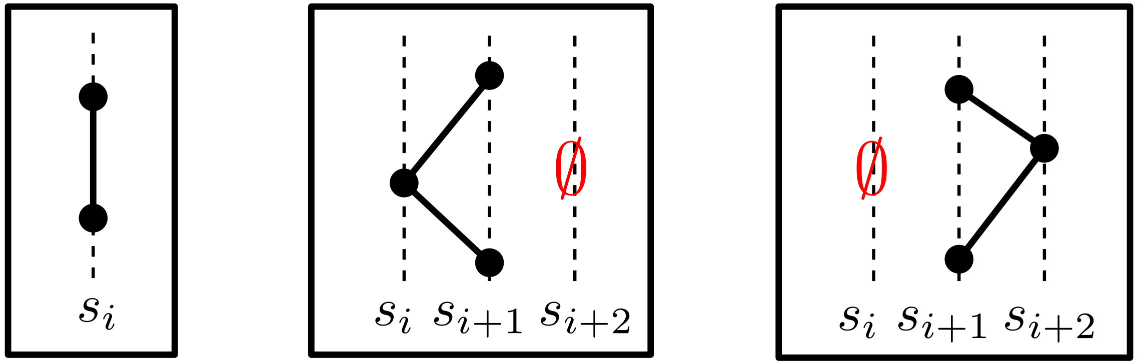

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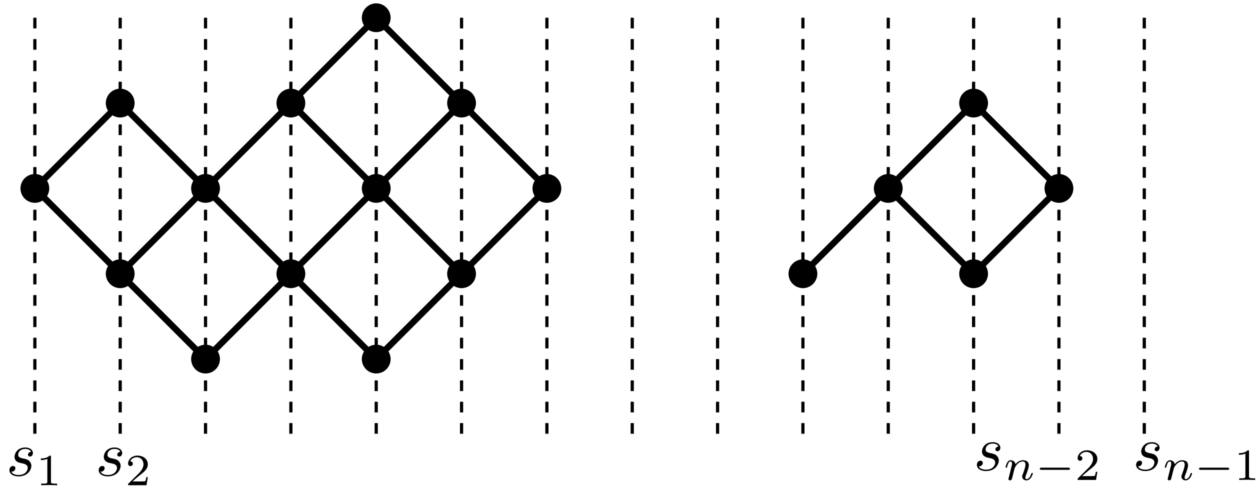

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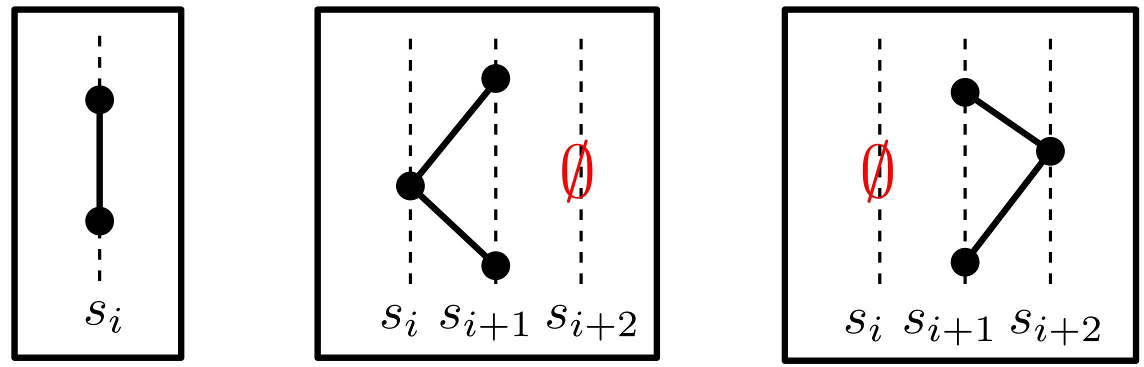

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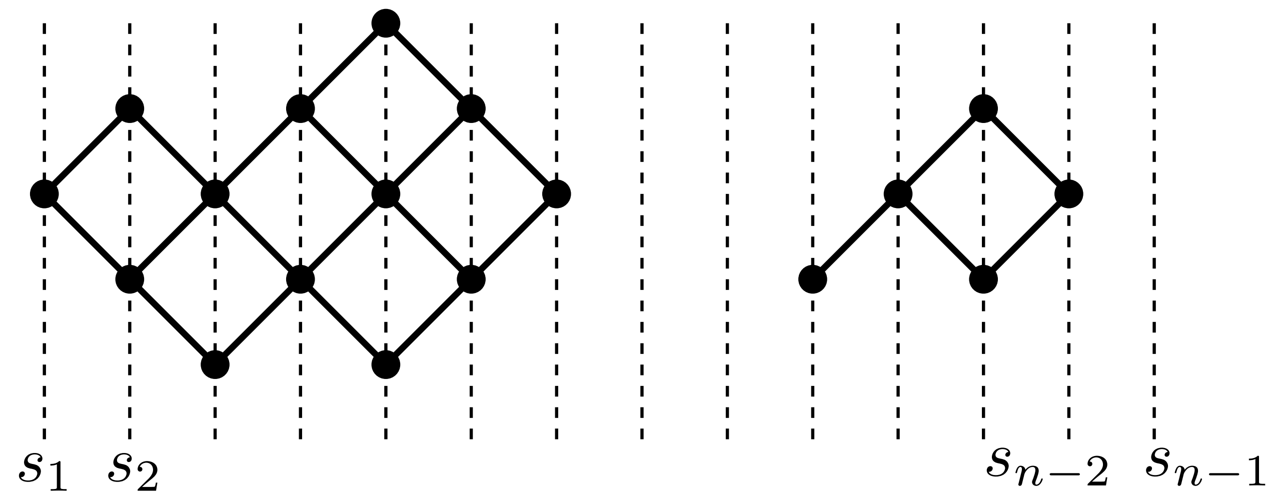
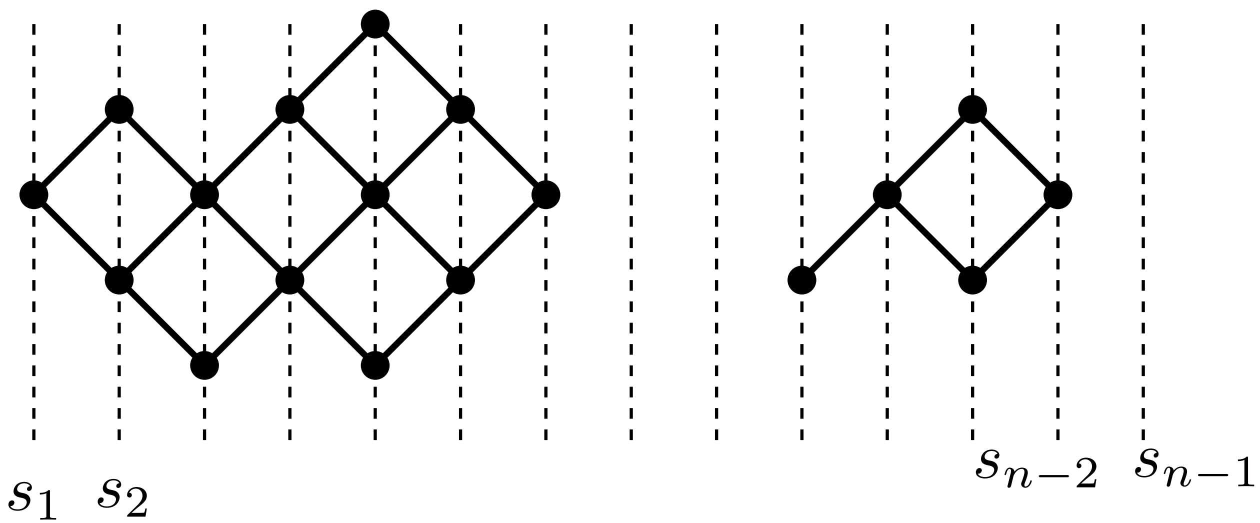


Proposition Heaps of type A are characterized by:
(i) At most one occurrence of s_1 (*resp.* s_{n-1}).
(ii) Elements with labels s_i, s_{i+1} form an alternating chain.

Type A: Bijection

FC Heap



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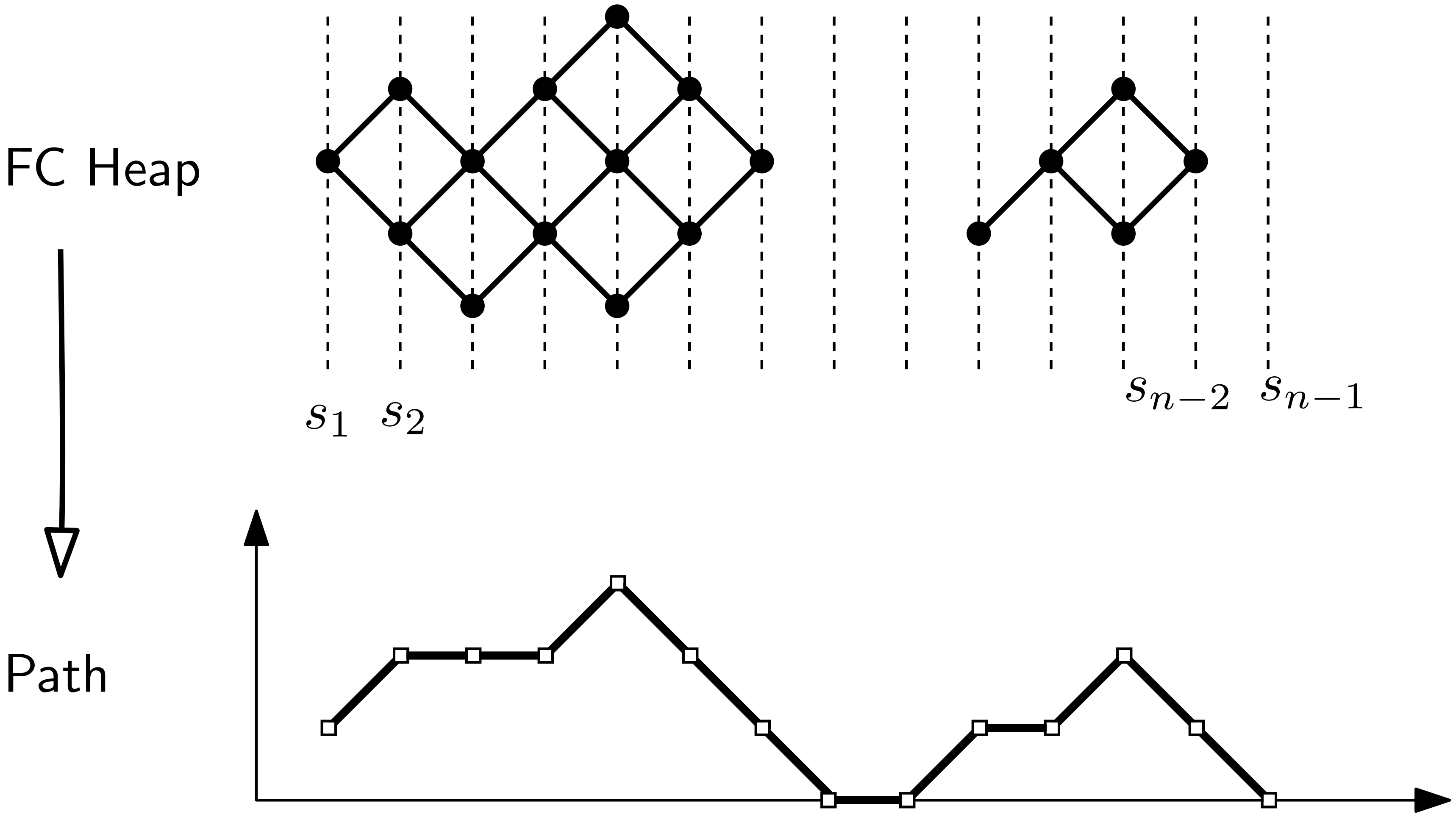

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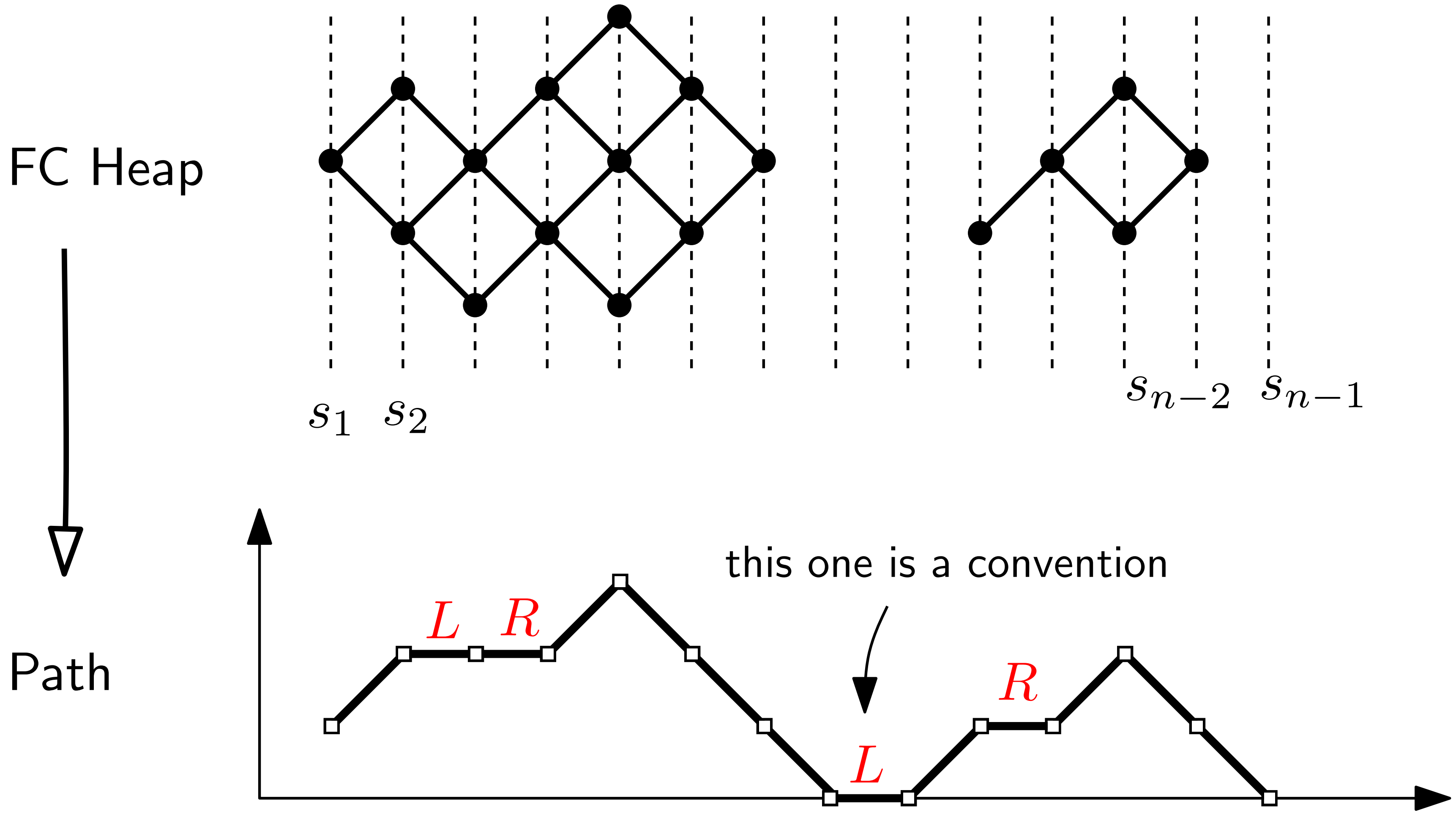

Extra information needed !

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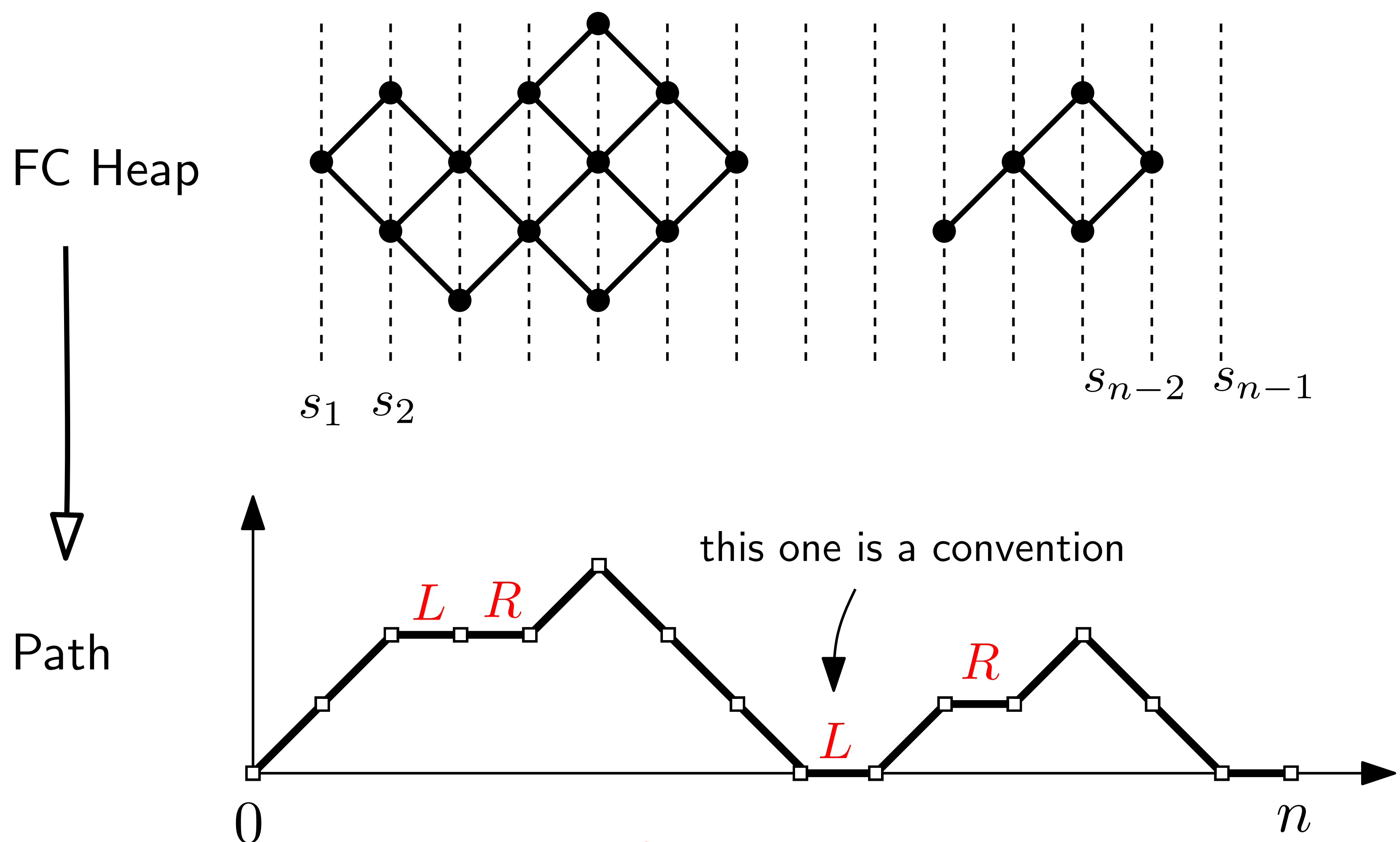

Extra information needed !

To finish, add initial and final steps to the path.

Type A: Bijection

Theorem [BJN '12, known before?]
This is a bijection between FC heaps of type A_{n-1} and Motzkin paths of length n with horizontal steps at height $h > 0$ (*resp.* $h = 0$) labeled L or R (*resp.* labeled L).

Size of the heap $\Leftrightarrow$ **Area** of the path

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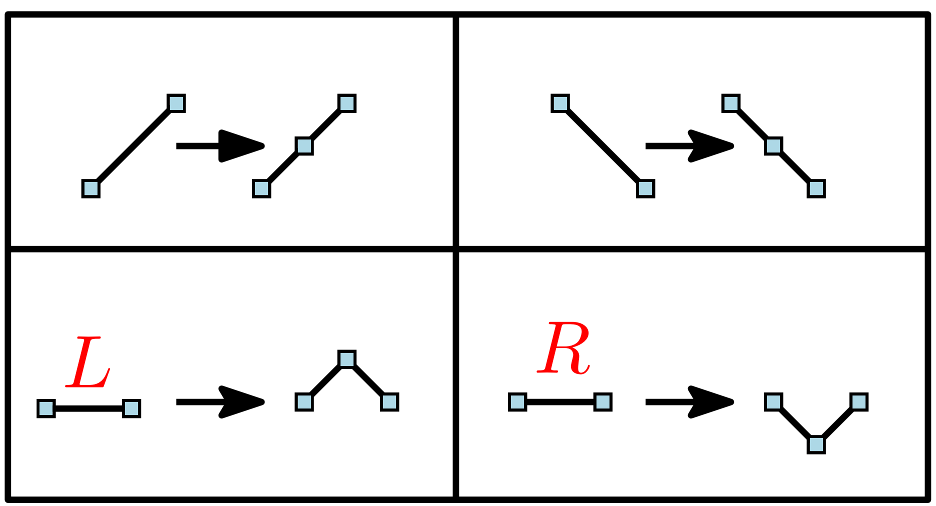
Remark



transforms these paths into Dyck paths $\Rightarrow$ Catalan numbers!

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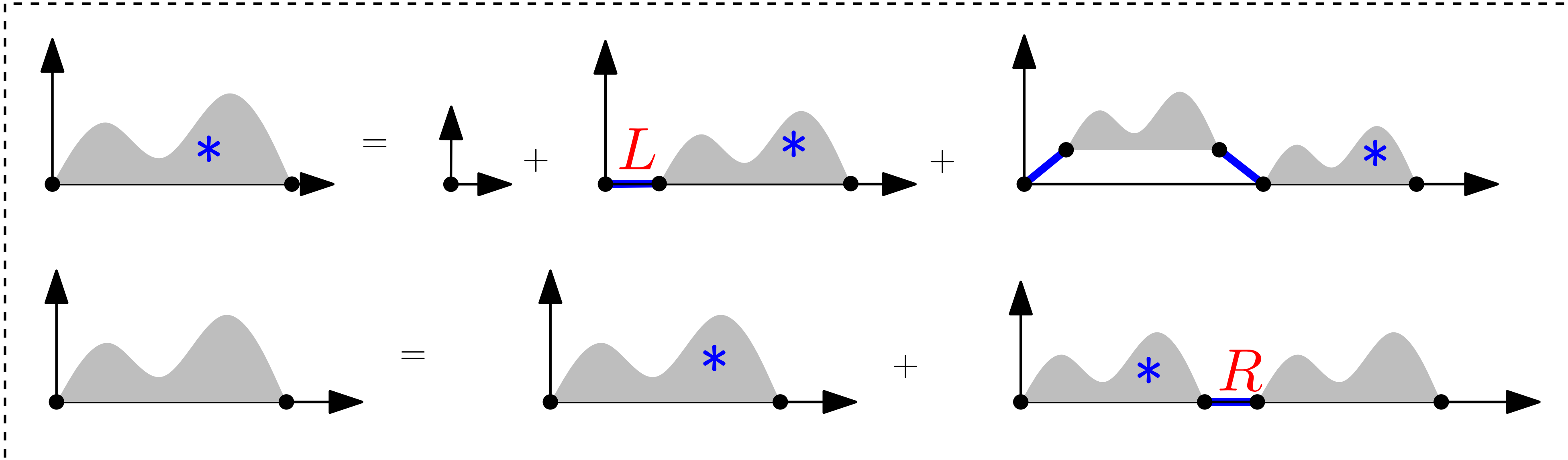
$\rightarrow$ Use recursive decompositions



(* indicates that horizontal steps at height $h = 0$ must have label L)

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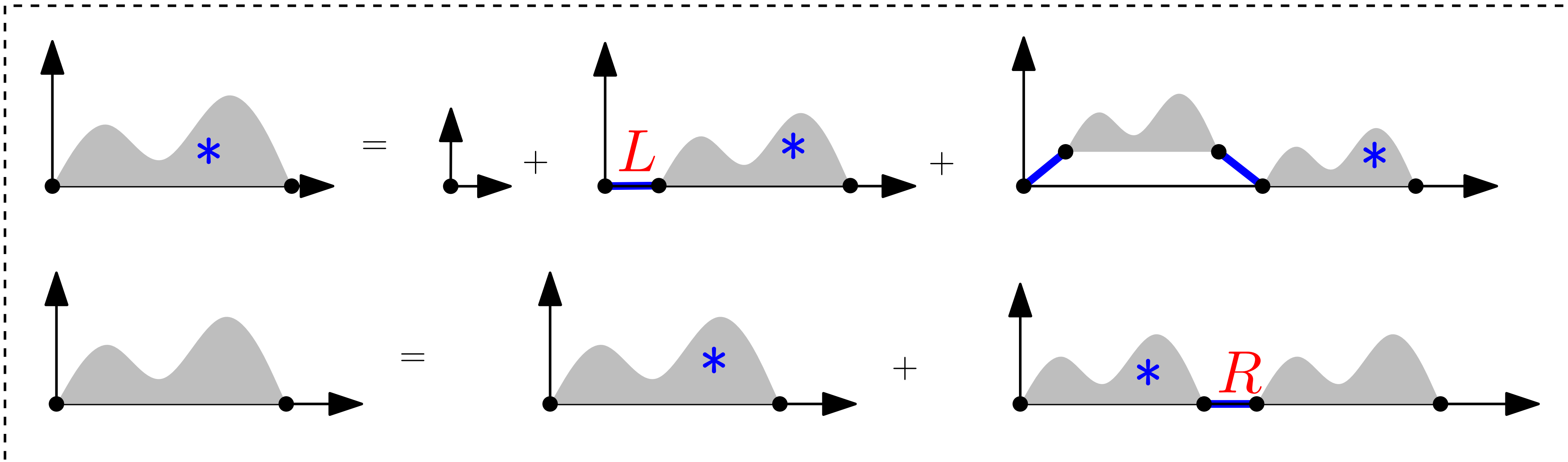


Write the functional equations, and eliminate to get

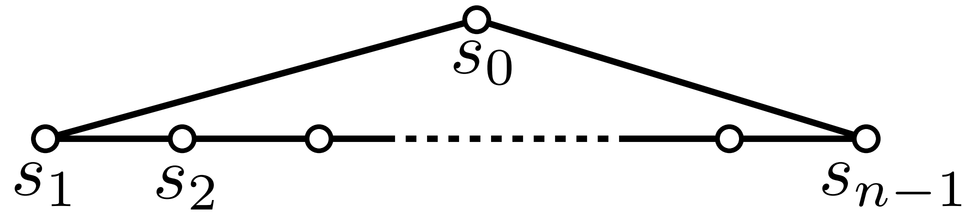
Theorem Define $A^{FC}(x) = \sum_{n\geq 1} A^{FC}_{n-1}(q)x^n$. Then

$$A^{FC}(x) = x + xA^{FC}(x) + qxA^{FC}(x)(A^{FC}(qx)+1).$$

2. Type $\widetilde{A}$

Affine permutations

$\widetilde{A}_{n-1}$



One can represent this group as the set of permutations σ of $\mathbb{Z}$ satisfying $\sigma(i+n) = \sigma(i) + n$, and $\sum_{i=1}^{n} \sigma(i) = \sum_{i=1}^{n} i$.

$$\ldots, 17, -12, \,|\, -14, -1, 17, -8, \,|\, \underset{\sigma(1)}{-\mathbf{10}}, \underset{\sigma(2)}{\mathbf{3}}, \underset{\sigma(3)}{\mathbf{21}}, \underset{\sigma(4)}{-\mathbf{4}}, \,|\, -6, 7, 25, 0, \,|\, -2, 11, 29, 4, \ldots$$

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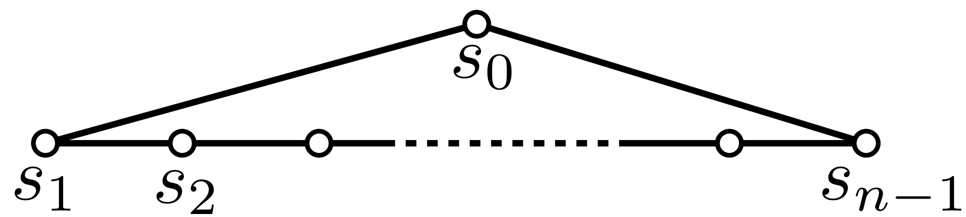

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Theorem [Green '01] Fully commutative elements of type $\widetilde{A}_{n-1}$ correspond to 321-avoiding permutations.

For instance the permutation above is not FC.

Hanusa and Jones used this representation to enumerate FC elements in type $\widetilde{A}$.

Generating functions

They computed the generating functions $f_n(q) = \widetilde{A}^{FC}_{n-1}(q)$; here are the first ones

$$f_3(q) = 1 + 3q + \mathbf{6q^2} + \mathbf{6q^3} + \mathbf{6q^4} + \cdots$$

$$f_4(q) = 1 + 4q + 10q^2 + \mathbf{16q^3} + \mathbf{18q^4} + \mathbf{16q^5} + \mathbf{18q^6} + \cdots$$

$$\begin{aligned} f_5(q) = {} & 1 + 5q + 15q^2 + 30q^3 + 45q^4 \\ & + \mathbf{50q^5} + \mathbf{50q^6} + \mathbf{50q^7} + \mathbf{50q^8} + \mathbf{50q^9} + \cdots \end{aligned}$$

$$\begin{aligned} f_6(q) = {} & 1 + 6q + 21q^2 + 50q^3 + 90q^4 + 126q^5 + 146q^6 \\ & + \mathbf{150q^7} + \mathbf{156q^8} + \mathbf{152q^9} + \mathbf{156q^{10}} + \mathbf{150q^{11}} + \mathbf{158q^{12}} \\ & + \mathbf{150q^{13}} + \mathbf{156q^{14}} + \mathbf{152q^{15}} + \mathbf{156q^{16}} + \mathbf{150q^{17}} + \mathbf{158q^{18}} \\ & + \cdots \end{aligned}$$

Periodicity n in the coefficients ?

Periodicity

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Moreover they can prove that one has periodicity
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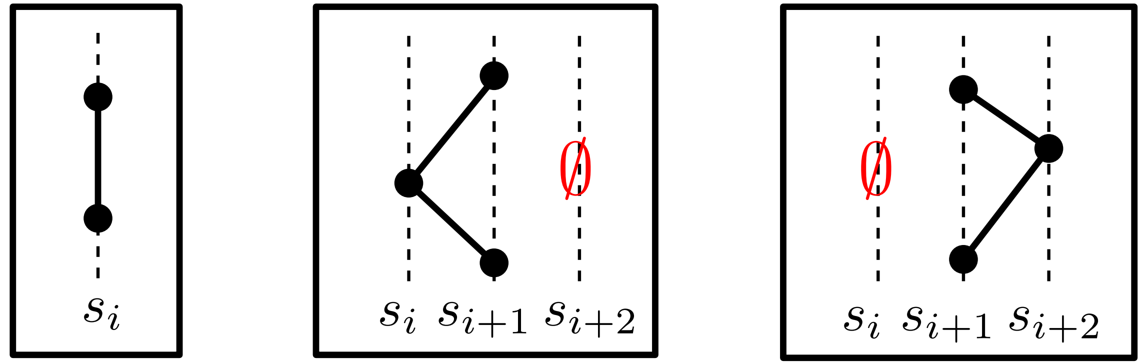
• We will prove this conjecture using heaps/paths.

In the process, we will get much simpler rules to compute the generating functions $\widetilde{A}_{n-1}^{FC}(q)$.

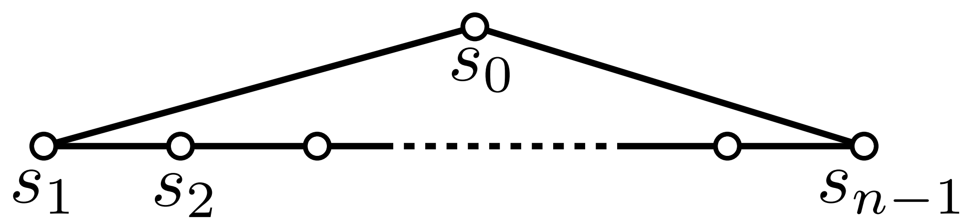
FC elements in type $\widetilde{A}$

FC heap satisfy the same local conditions as in finite type A.

$\rightarrow$ The heaps must avoid



Difference: the cyclic shape of the Coxeter diagram



$\rightarrow$ The labels above must be taken with index modulo n; the heaps must be thought of as "drawn on a cylinder".

Heaps become Paths

We can form a path as before from a heap: because of the cyclic diagram, our paths will start and end at the same height.

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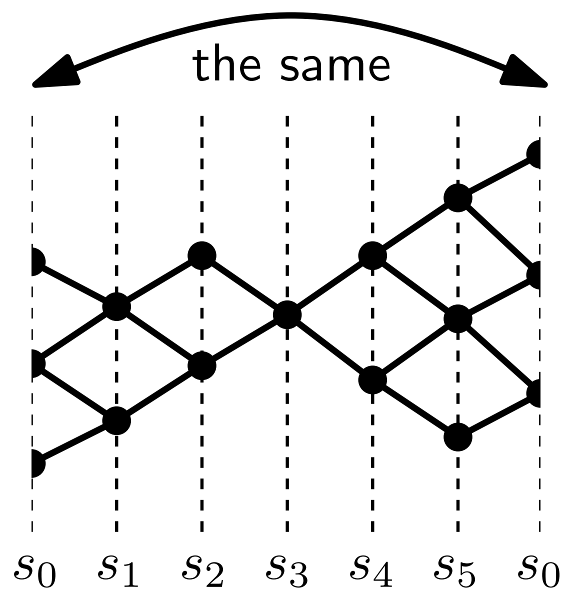
Example:

Heap



Note that there is just one minimal element

Heaps become Paths

We can form a path as before from a heap: because of the cyclic diagram, our paths will start and end at the same height.

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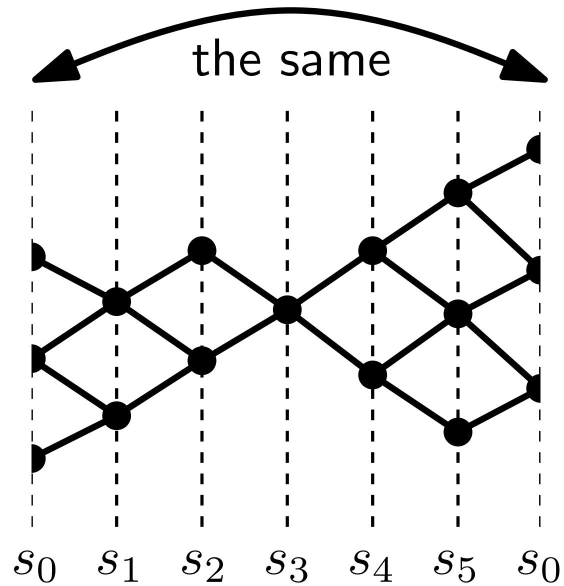
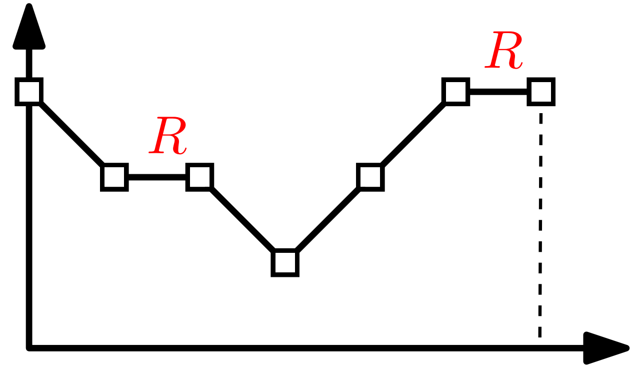


Note that there is just one minimal element

Path



The area does not take into account the final height.

Bijection

Starting from an FC element in $\widetilde{A}_{n-1}$, we thus obtain a path in $\mathcal{O}_n^*$, the set of length n paths with starting and ending point at the same height.

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Indeed such paths clearly cannot correspond to FC elements.

Corollary $\widetilde{A}_{n-1}^{FC}(q) = \mathcal{O}_n^*(q) - \dfrac{2q^n}{1-q^n}$

Periodicity revisited

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$\to k = 1 + \lceil (n-1)/2 \rceil \lfloor (n+1)/2 \rfloor$ is optimal.

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This proves the conjecture of Hanusa and Jones.

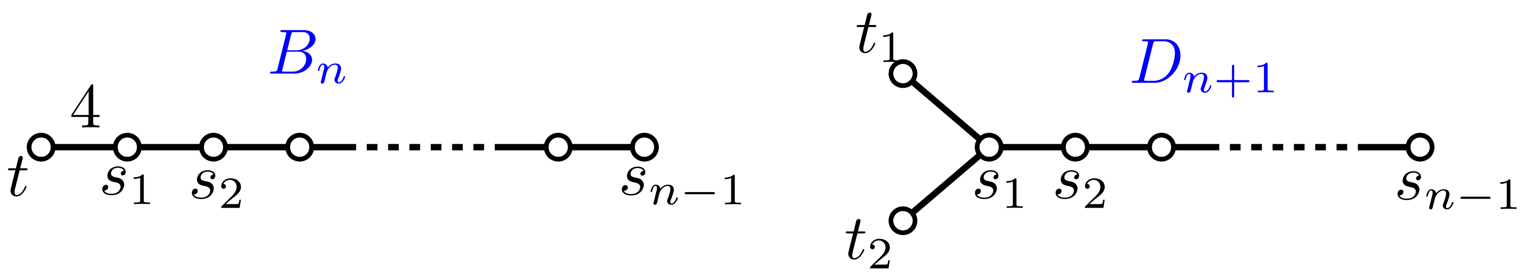
• We still have to compute the generating function $\mathcal{O}_n^*(q)$.

I will leave it to you as an (interesting) exercise in generating functions (maybe you have a better solution than ours).

3. Other finite and affine Coxeter groups

Other finite types

- The remaining "classical types"



were also enumerated by Stembridge
$\rightarrow$ we can reinterpret his proof in terms of paths and give the length generating polynomials in these cases also.

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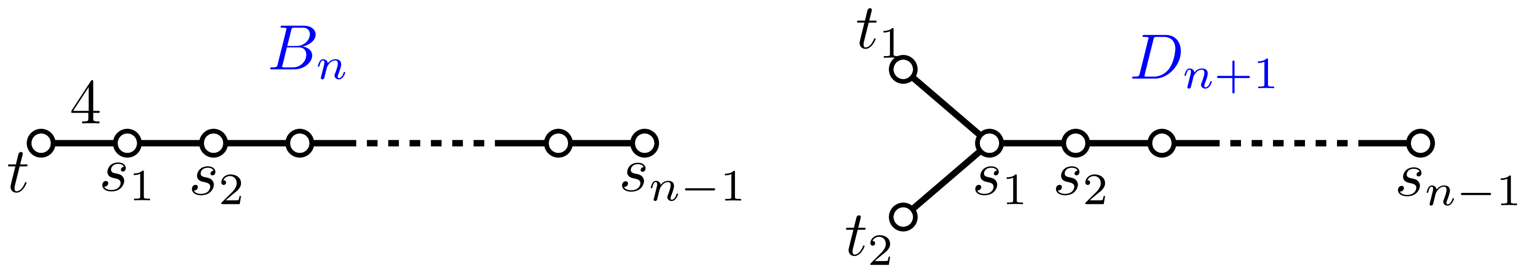

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- Exceptional types $I_2(m), H_3, H_4, F_4, E_6, E_7$, and E_8
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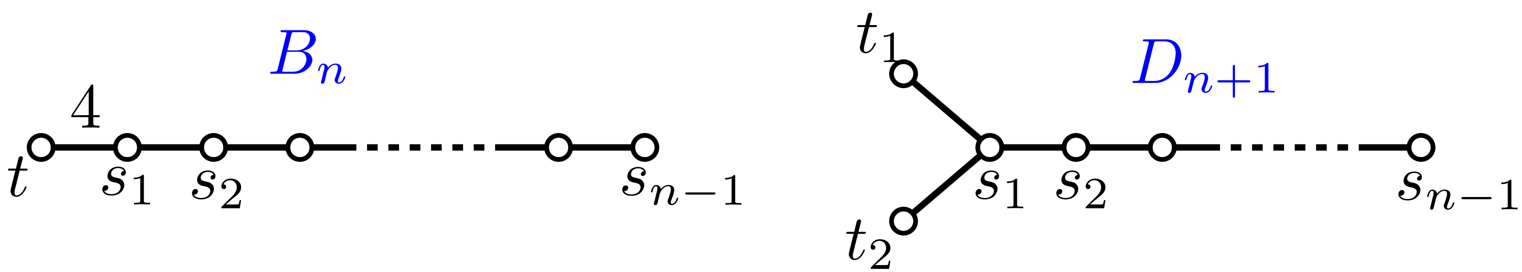

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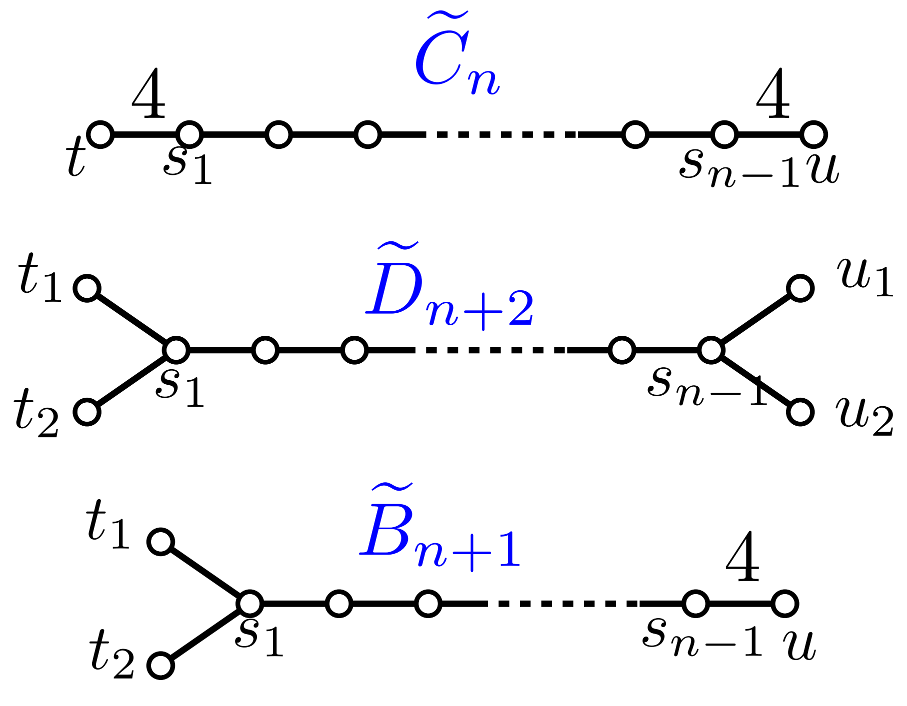
$$E_8^{FC}(q) = 15q^{29}+30q^{28}+43q^{27}+56q^{26}+69q^{25}+83q^{24}+113q^{23}+143q^{22}+171q^{21}+205q^{20} + 259q^{19} + 319q^{18} + 387q^{17} + 457q^{16} + 527q^{15} + 609q^{14} + 701q^{13} + 794q^{12} + 867q^{11} + 924q^{10} + 936q^{9} + 897q^{8} + 796q^{7} + 631q^{6} + 427q^{5} + 238q^{4} + 105q^{3} + 35q^{2} + 8q + 1.$$

Other affine types

There are 3 classical types



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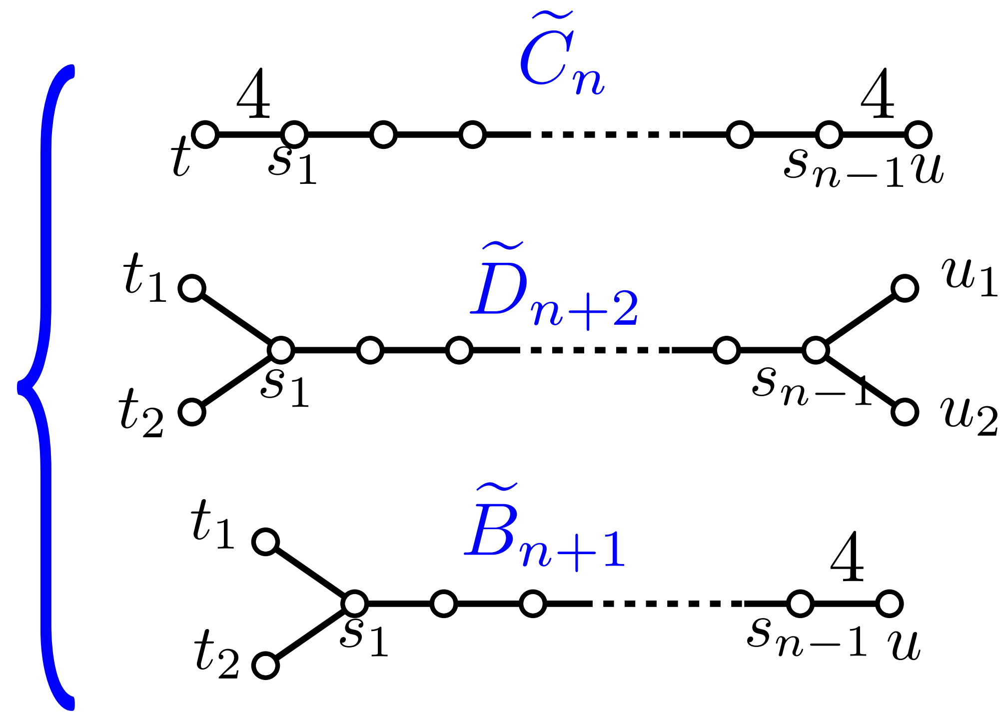
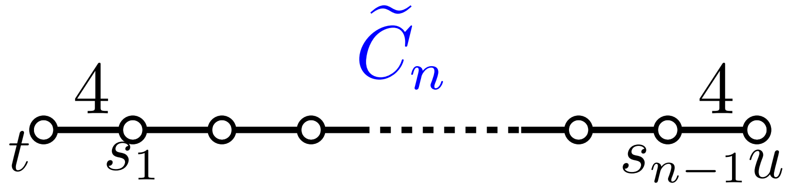


Theorem [BJN '12]
For each irreducible affine group W, the sequence of coefficients of $W^{FC}(q)$ is ultimately periodic, with period recorded in the following table.

Affine Type	$\widetilde{A}_{n-1}$	$\widetilde{C}_n$	$\widetilde{B}_{n+1}$	$\widetilde{D}_{n+2}$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{G}_2$	$\widetilde{F}_4, \widetilde{E}_8$
Periodicity	n	$n+1$	$(n+1)(2n+1)$	$n+1$	4	9	5	1

Type $\widetilde{C}$



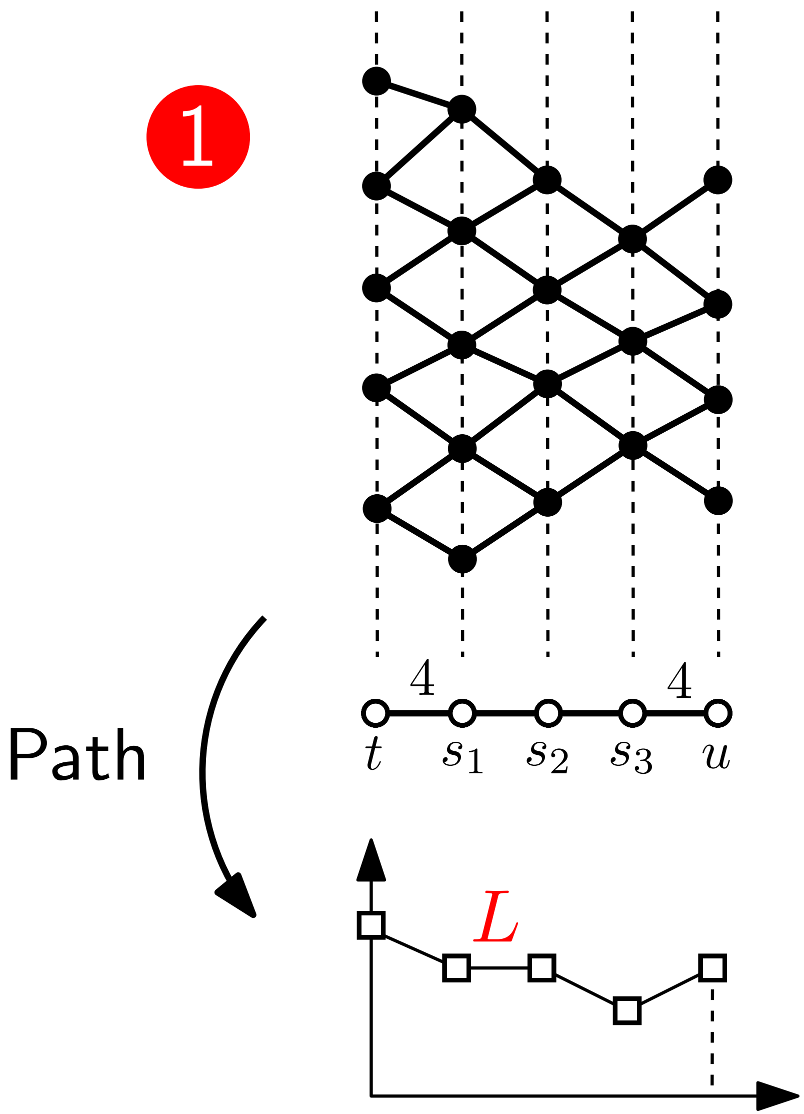
$$\begin{aligned}\widetilde{C}_4^{FC}(q) =& 1 + 5q + 14q^2 + 29q^3 + 47q^4 + 64q^5 + 76q^6 + 81q^7 \\ & + 80q^8 + 75q^9 + 68q^{10} + 63q^{11} + 61q^{12} \\ & \mathbf{+59q^{13} + 59q^{14} + 60q^{15} + 59q^{16} + 59q^{17}} \\ & \mathbf{+59q^{18} + 59q^{19} + 60q^{20} + 59q^{21} + 59q^{22}} \\ & \mathbf{+59q^{23} + 59q^{24} + 60q^{25} + 59q^{26} + 59q^{27}} \\ & + \cdots\end{aligned}$$

We obtain here also certain heaps corresponding to paths, but there are in addition infinitely many exceptional FC heaps, certain “zigzag heaps”.

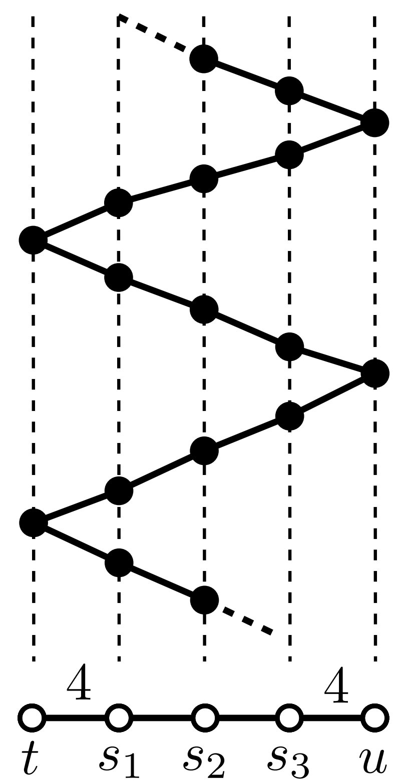
Type $\widetilde{C}$

Two families of paths survive for large enough length:

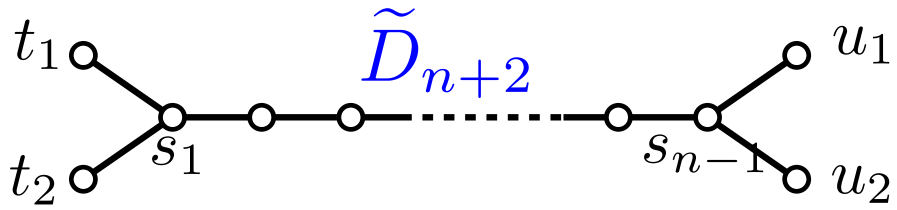
1



2 Finite factors of



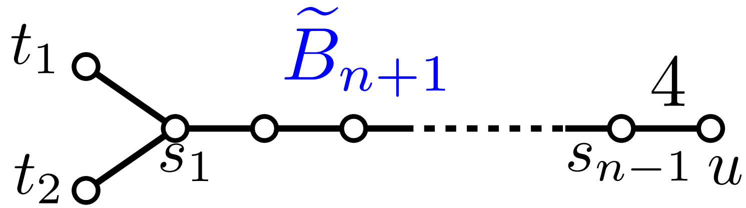
Type $\widetilde{D}$



$$\widetilde{D}_4(q) = 1 + 5q + 14q^2 + 28q^3 + 39q^4 + 44q^5 + 45q^6 + 34q^7 + 30q^8 + 36q^9 + 30q^{10} + 30q^{11} + 36q^{12} + 30q^{13} + 30q^{14} + 36q^{15} + 30q^{16} + 30q^{17} + 36q^{18} + 30q^{19} + 30q^{20} + 36q^{21} + 30q^{22} + 30q^{23} + 36q^{24} + 30q^{25} + 30q^{26} + 36q^{27} + 30q^{28} + 30q^{29} + 36q^{30} + 30q^{31} + 30q^{32} + 36q^{33} + 30q^{34} + 30q^{35} + 36q^{36} + 30q^{37} + 30q^{38} + \cdots.$$

Here the minimal period is 3, while the period predicted by the theorem is 6.

Type $\widetilde{B}$



$$\widetilde{B}_3^{FC}(q) = 1 + 4q + 9q^2 + 15q^3 + 19q^4 + 21q^5 + 21q^6 + 18q^7 + 17q^8 + 19q^9 + 18q^{10} + 17q^{11} + 19q^{12} + 17q^{13} + 17q^{14} + 20q^{15} + 17q^{16} + 17q^{17} + 19q^{18} + 17q^{19} + 18q^{20} + 19q^{21} + 17q^{22} + 17q^{23} + 19q^{24} + 18q^{25} + 17q^{26} + 19q^{27} + 17q^{28} + 17q^{29} + 20q^{30} + 17q^{31} + 17q^{32} + 19q^{33} + 17q^{34} + 18q^{35} + 19q^{36} + 17q^{37} + 17q^{38} + 19q^{39} + 18q^{40} + 17q^{41} + 19q^{42} + 17q^{43} + 17q^{44} + 20q^{45} + 17q^{46} + 17q^{47} + 19q^{48} + 17q^{49} + 18q^{50} + 19q^{51} + 17q^{52} + 17q^{53} + 19q^{54} + 18q^{55} + 17q^{56} + 19q^{57} + 17q^{58} + 17q^{59} + 20q^{60} + 17q^{61} + 17q^{62} + 19q^{63} + 17q^{64} + 18q^{65} + 19q^{66} + 17q^{67} + 17q^{68} + 19q^{69} + 18q^{70} + 17q^{71} + 19q^{72} + 17q^{73} + 17q^{74} + 20q^{75} + 17q^{76} + \cdots$$

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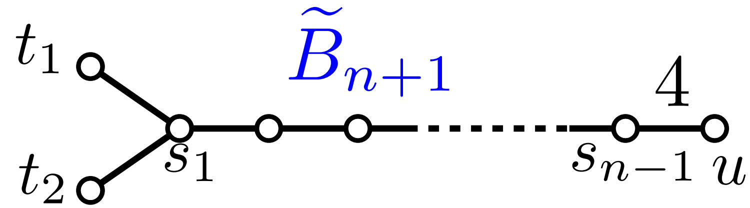

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The period is 15 in this case, corresponding to $(n+1)(2n+1)$ for $n = 2$.

Further questions

• All of this work can be easily restricted to deal with FC involutions.

• Other statistics to consider, e.g. descent numbers.

• Formulas for our generating functions ? (and not just functional equations/recurrences).

• (Affine case) Repartition of the alcoves corresponding to FC elements.

• Classification: for which Coxeter groups W is it true that $W^{FC}(q)$ has periodic coefficients ?

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THANK YOU

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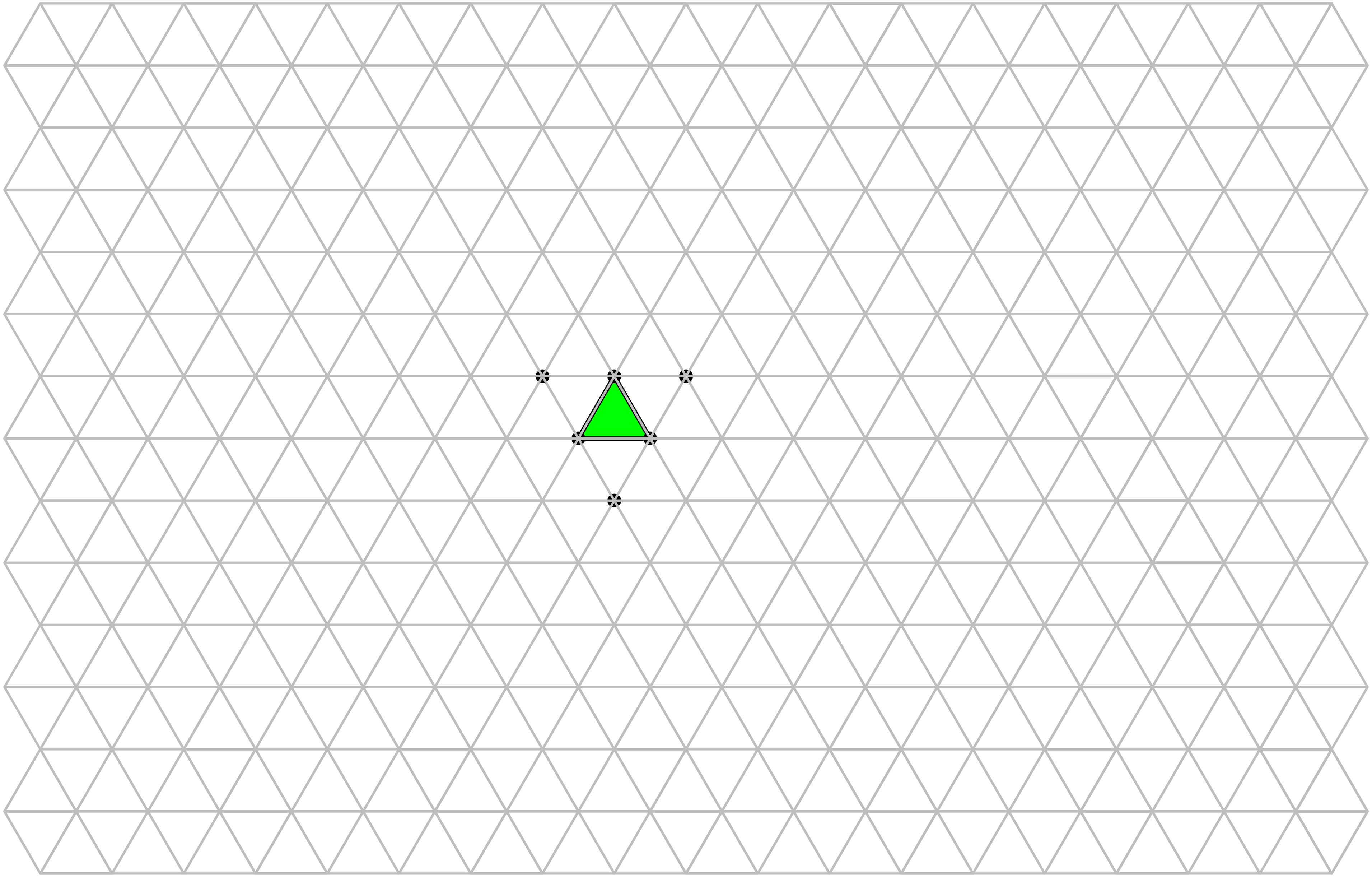

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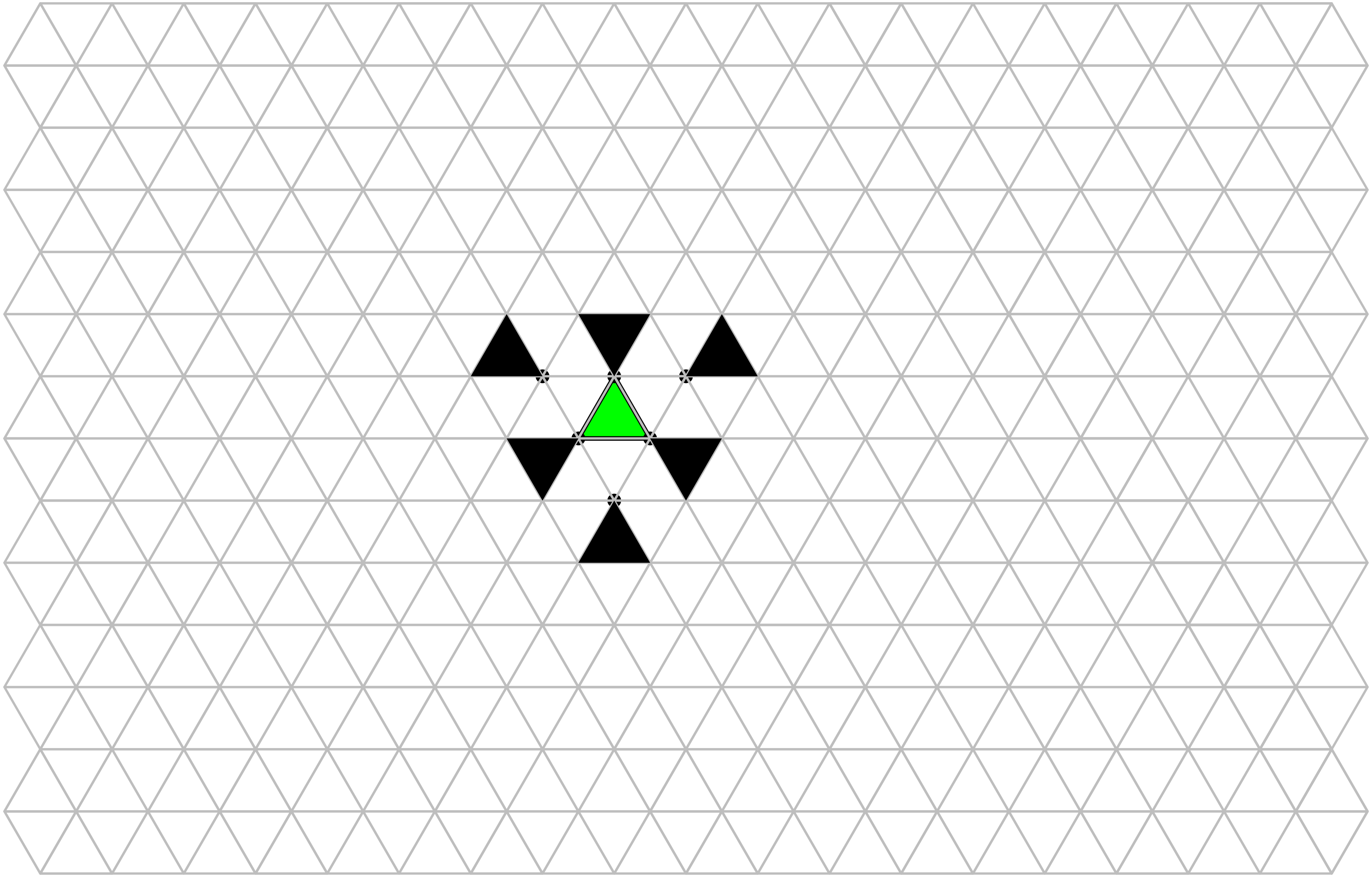

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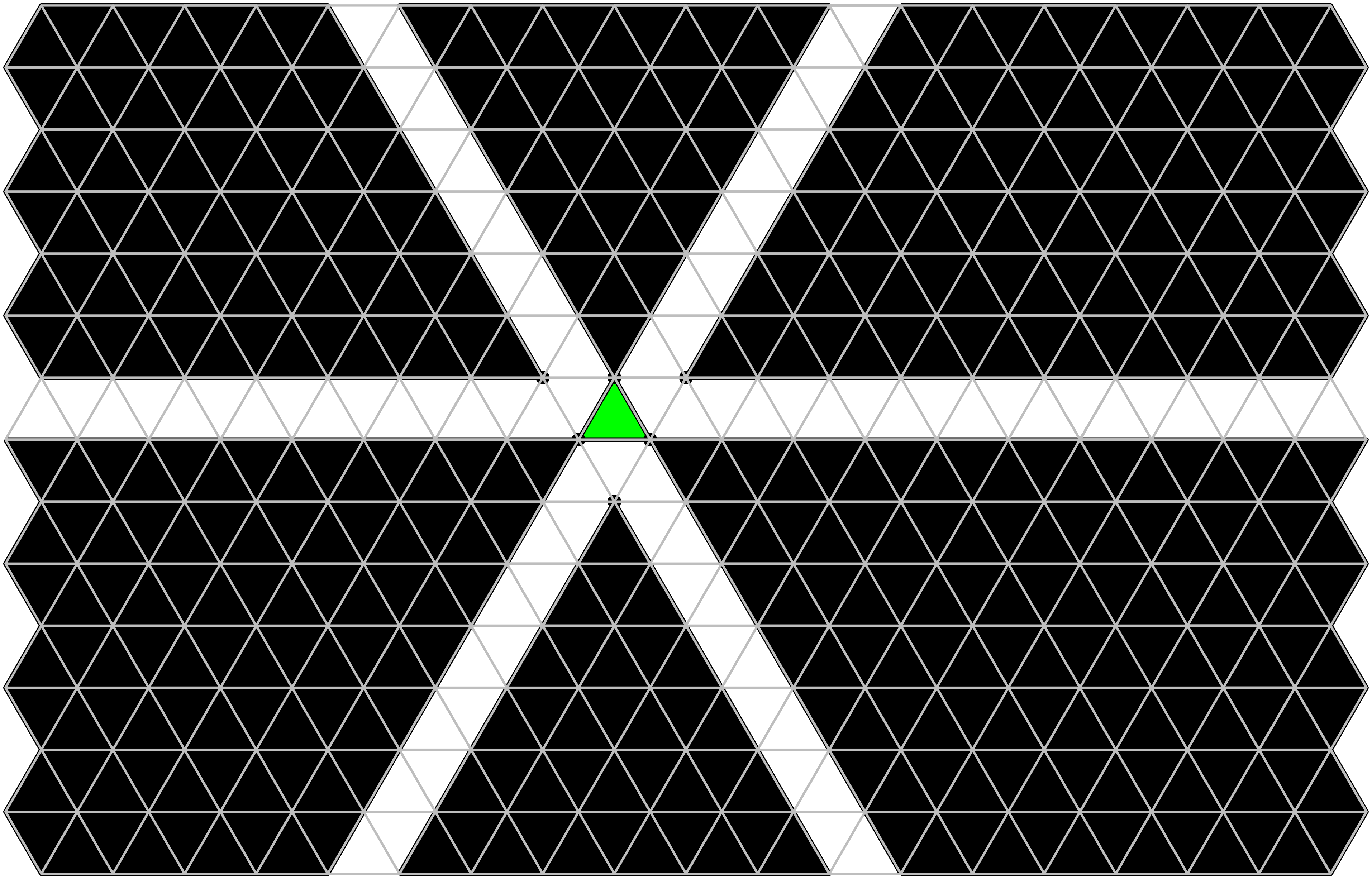

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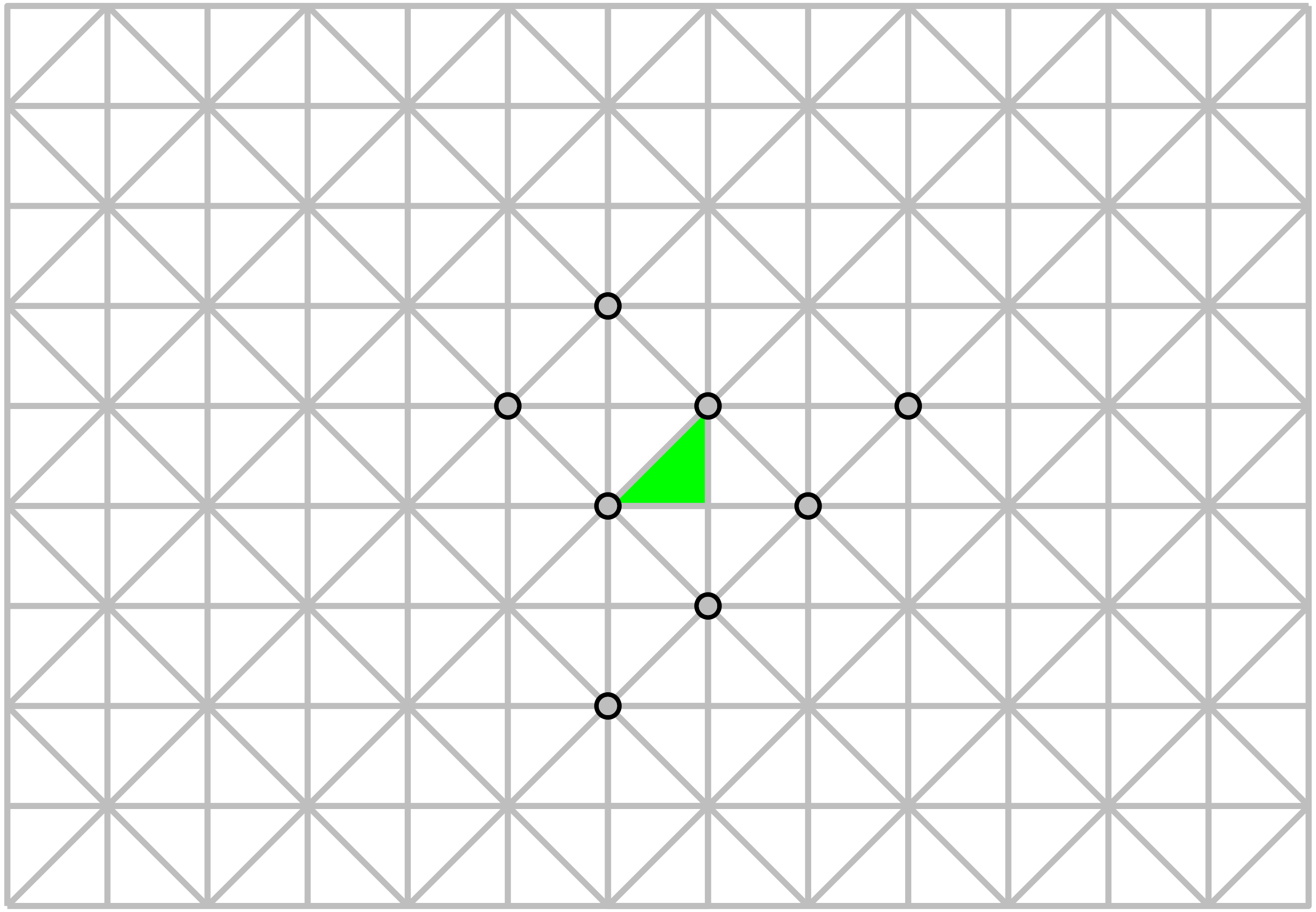

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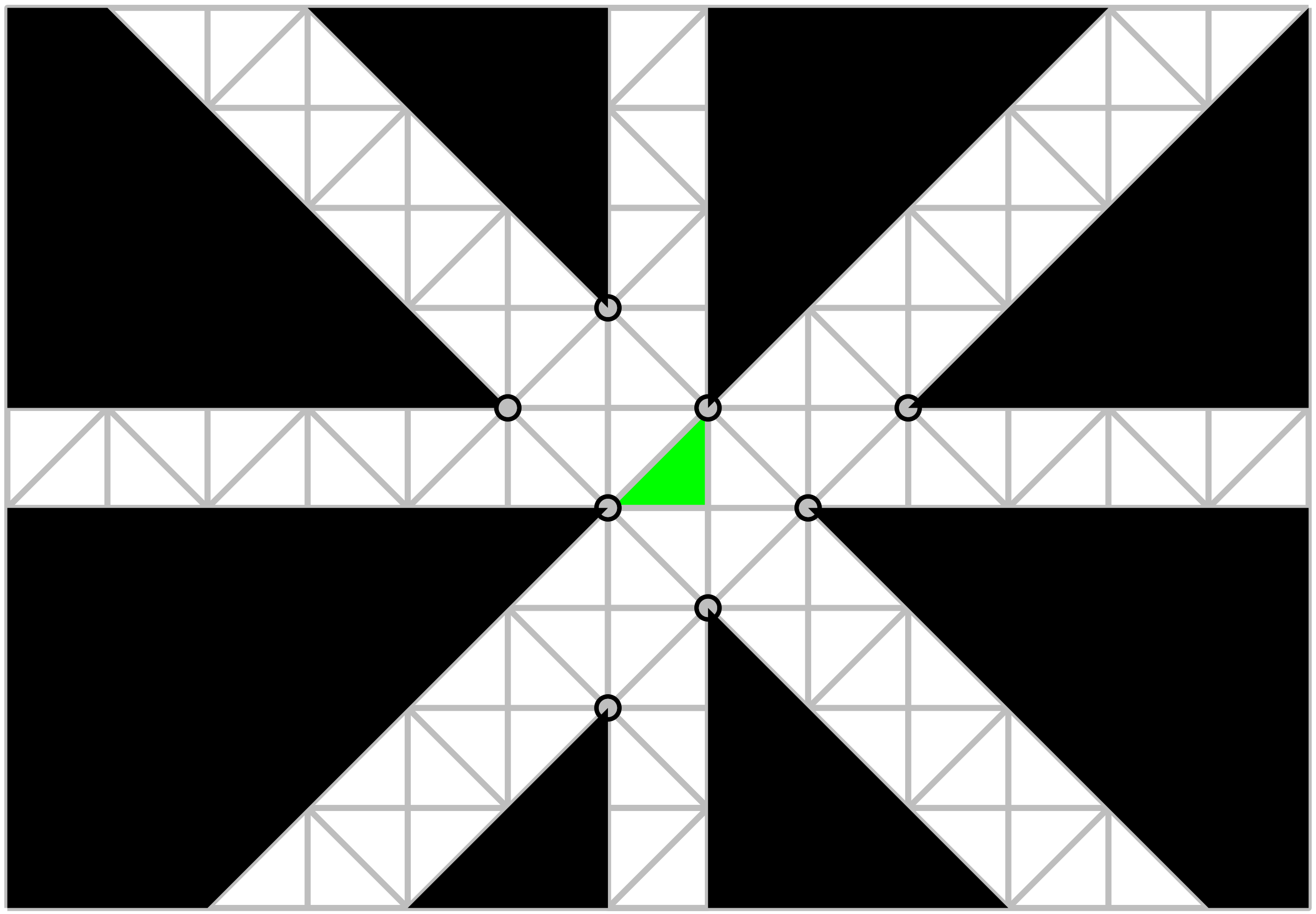
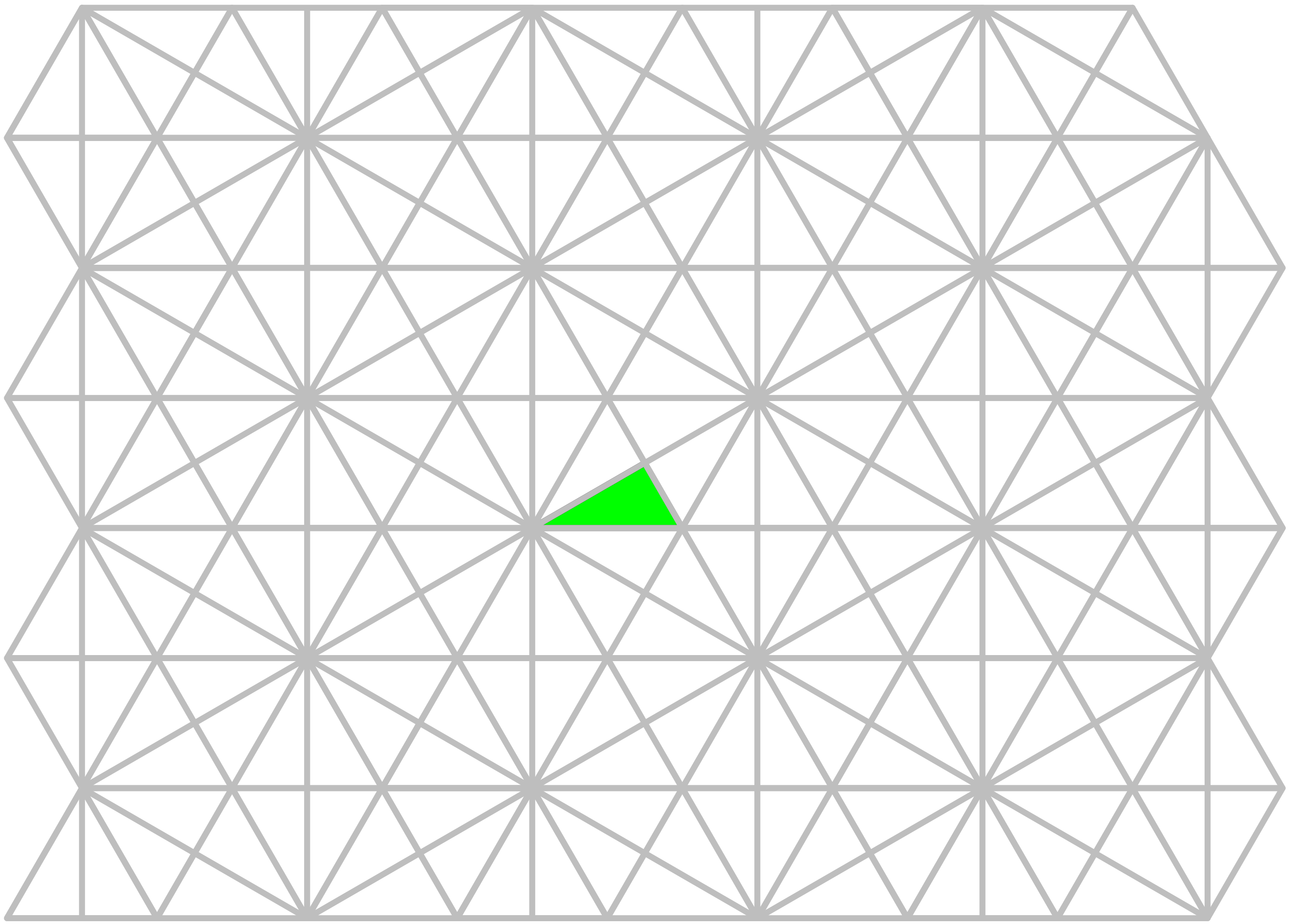


Type $\widetilde{G}_2$



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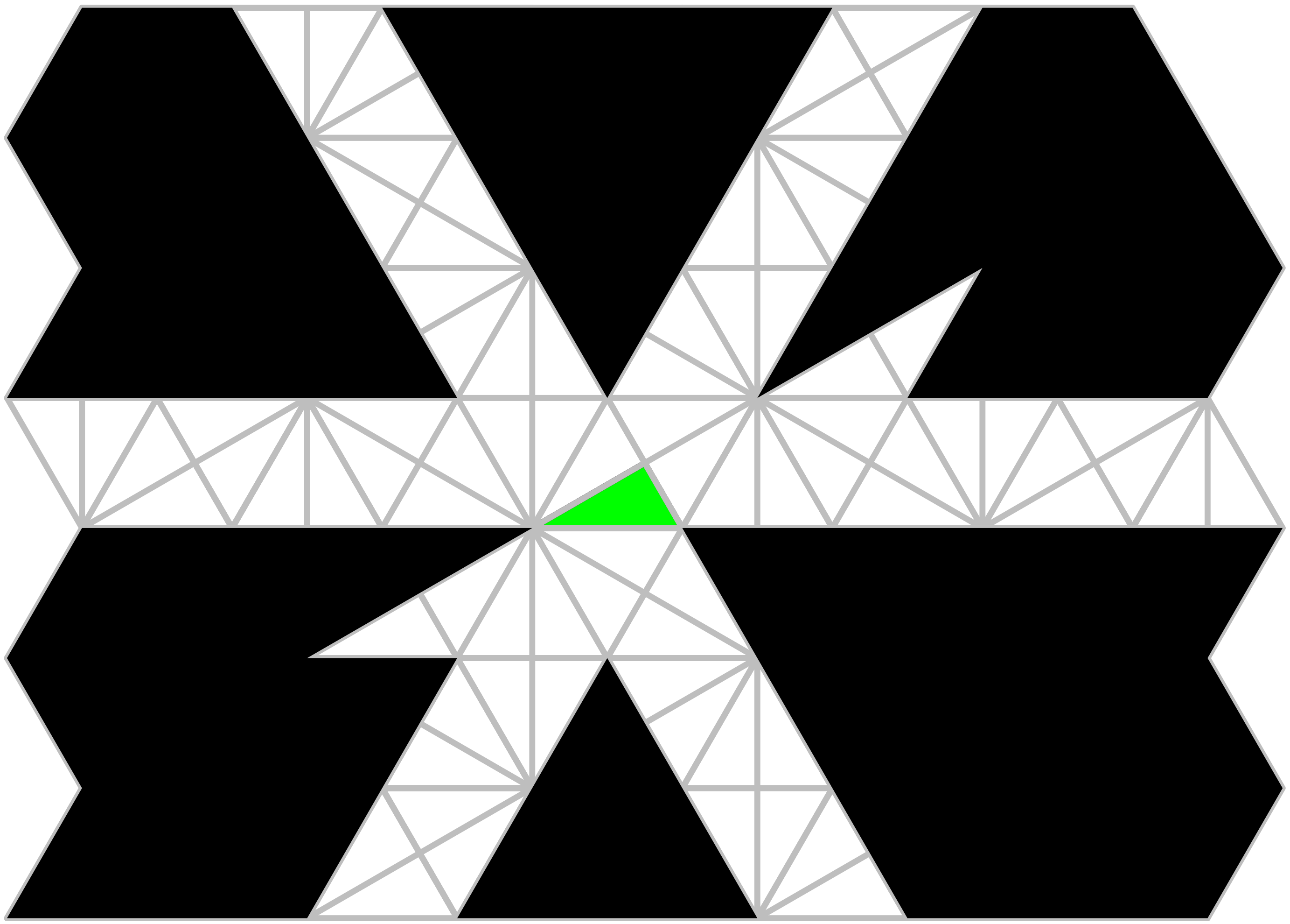

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