On some polynomials enumerating Fully Packed Loop configurations

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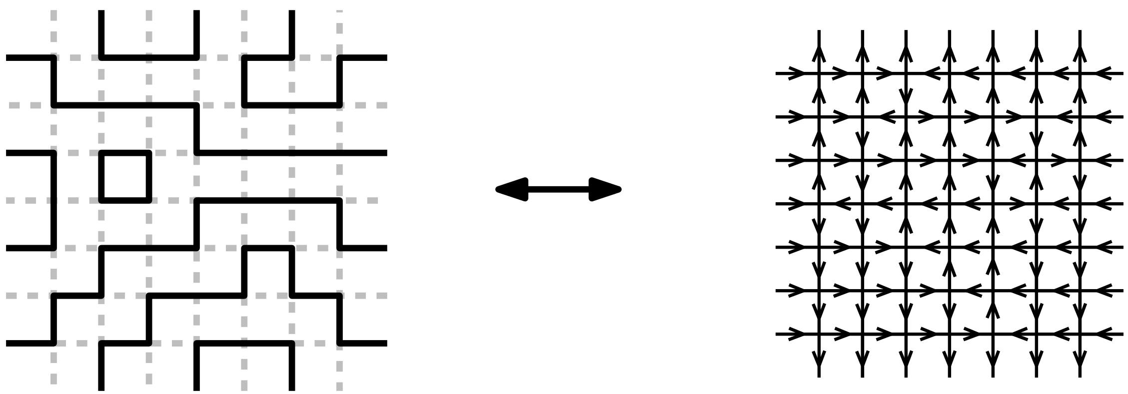
Introduction

Fully Packed Loop (FPL) configurations are combinatorial structures that arose in statistical mechanics, in simple bijection with known combinatorial objects.



$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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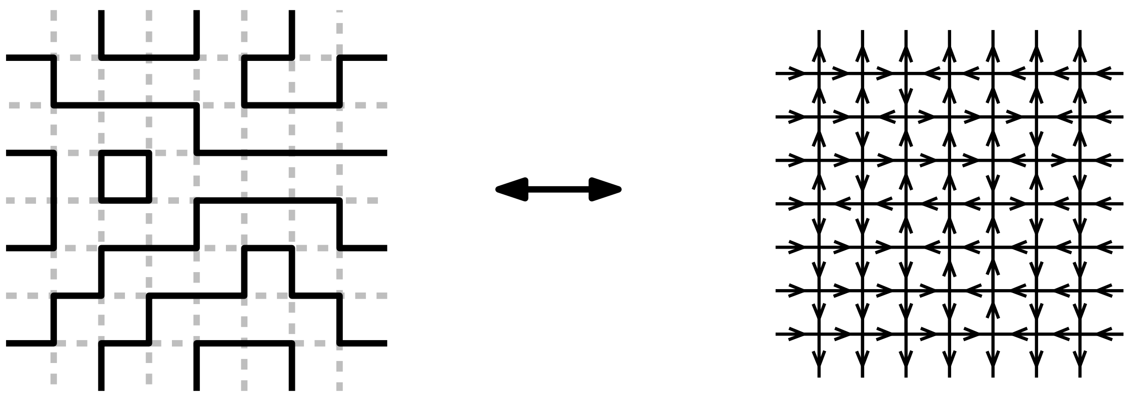

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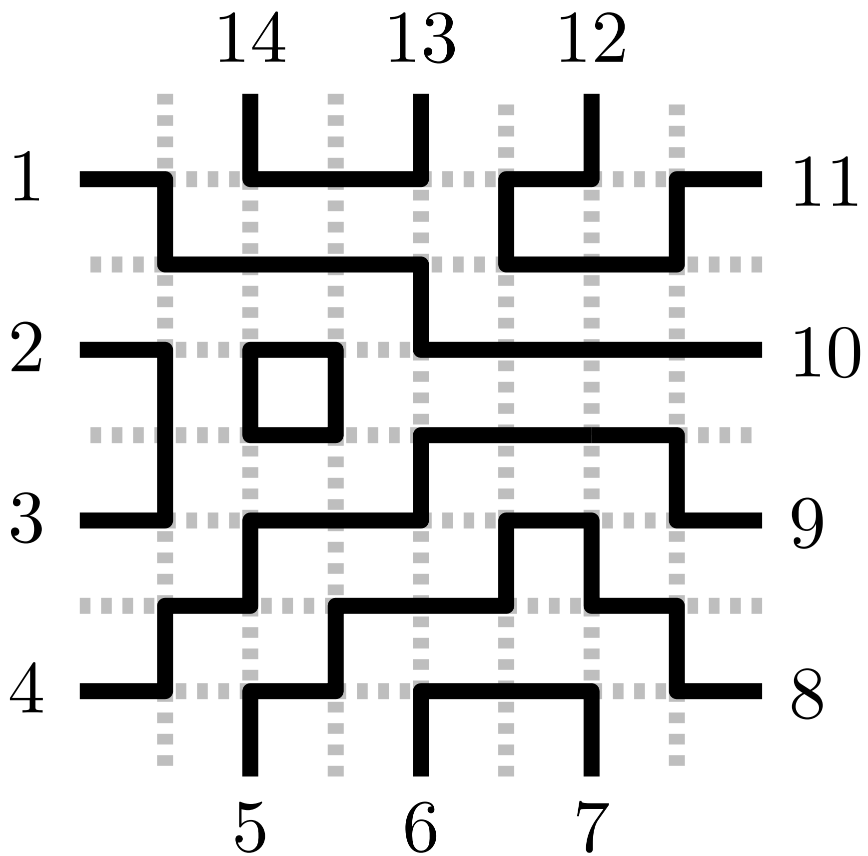
One can define certain polynomials $A_\pi(X)$ (indexed by noncrossing matchings π) which count FPLs when specialized to nonnegative integers.

We will here formulate conjectures for the polynomials $A_\pi(X)$, hinting at "combinatorial reciprocity".

Introduction

Start with the square grid G_n with n^2 vertices and $4n$ external edges (here $n = 7$).



A FPL configuration of size n is a subgraph of the grid G_n

(1) such that around each vertex of G_n, 2 edges out of 4 are selected ; ("Fully Packed")

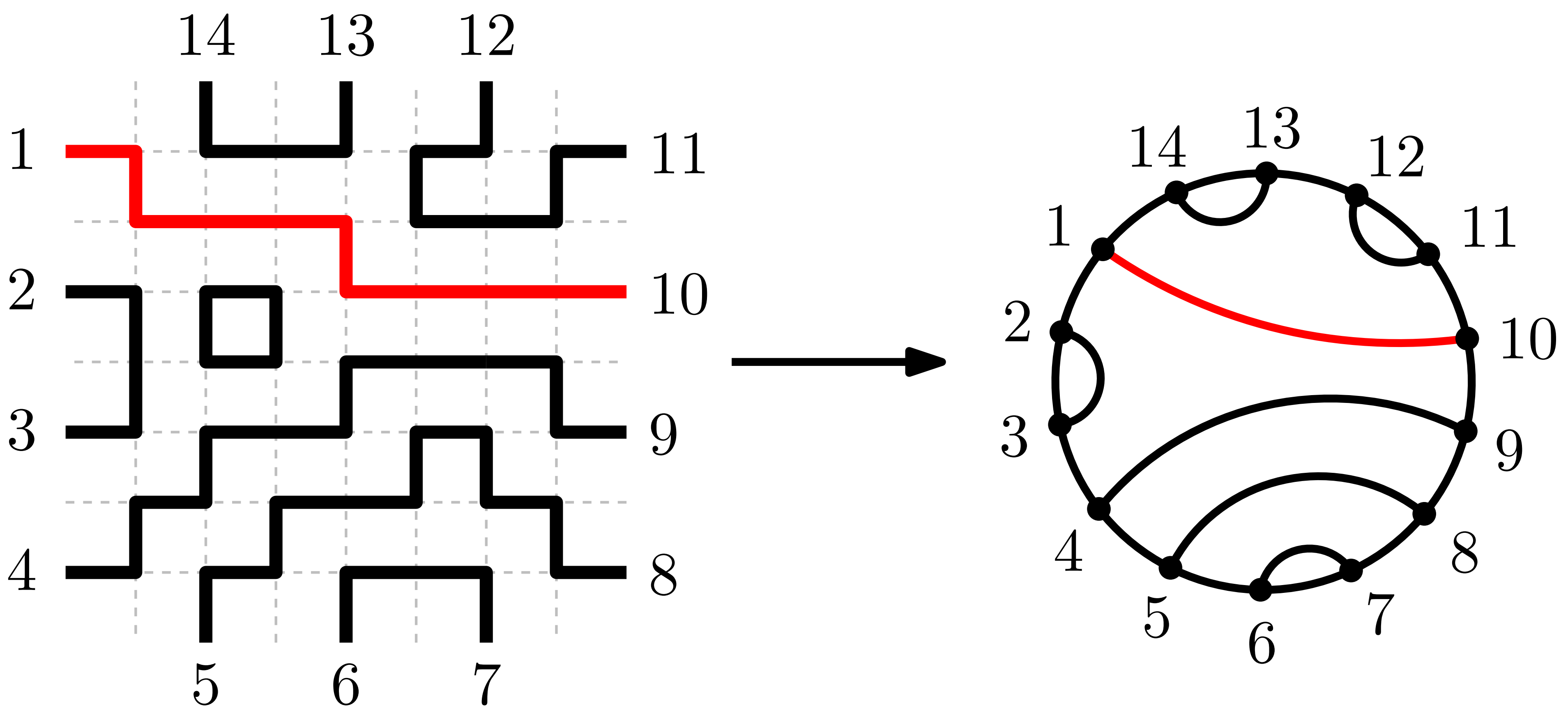
(2) containing every other external edge. ("Boundary condition")

A_n =number of FPLs of size n

Introduction

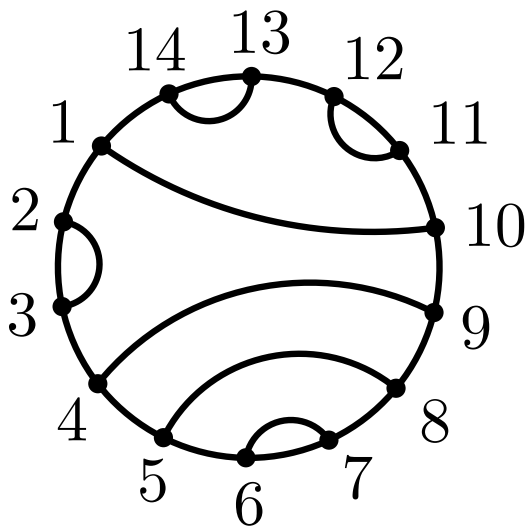
A link pattern π of size n is a set of n noncrossing chords between $2n$ points on a disk.

To each FPL F is associated a a link pattern $\pi(F)$.

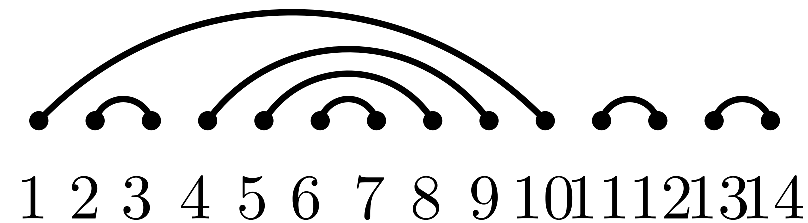


Introduction

We will write link patterns linearly, as noncrossing matchings :



$\Leftrightarrow$



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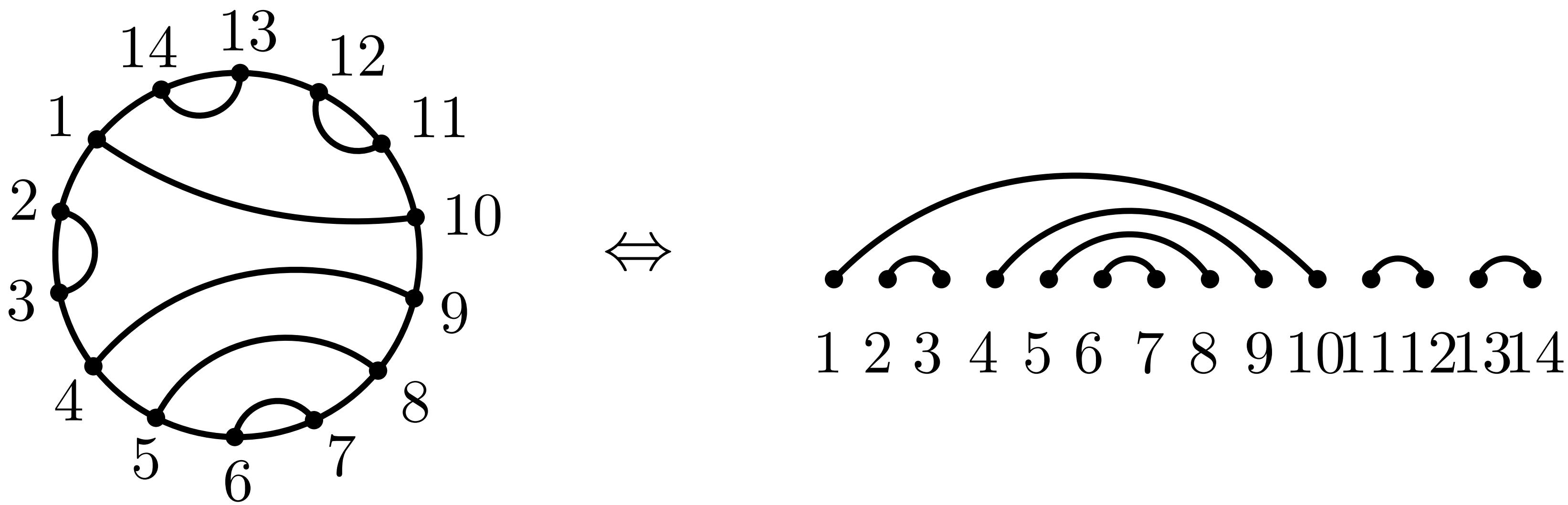

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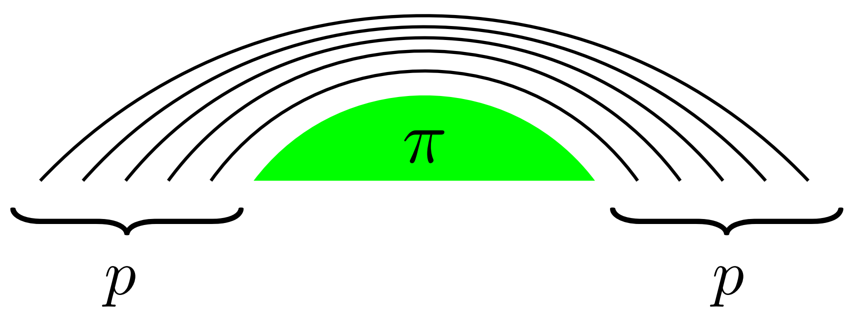
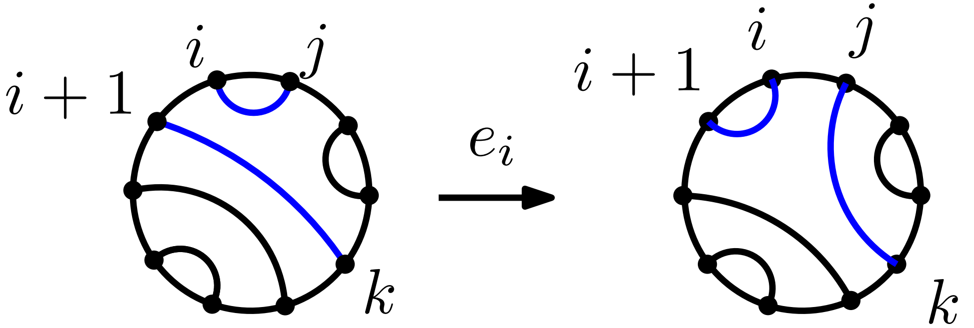
Matchings with nested arches : for $p \geq 0$,

$(\pi)_p :=$



Introduction : the Razumov-Stroganov ex-conjecture

Definition : The operators e_i,$i \in [\![1, 2n]\!]$, act on matchings by $\{i, j\}, \{i+1, k\} \in \pi \to \{i, i+1\}, \{j, k\} \in e_i(\pi)$.



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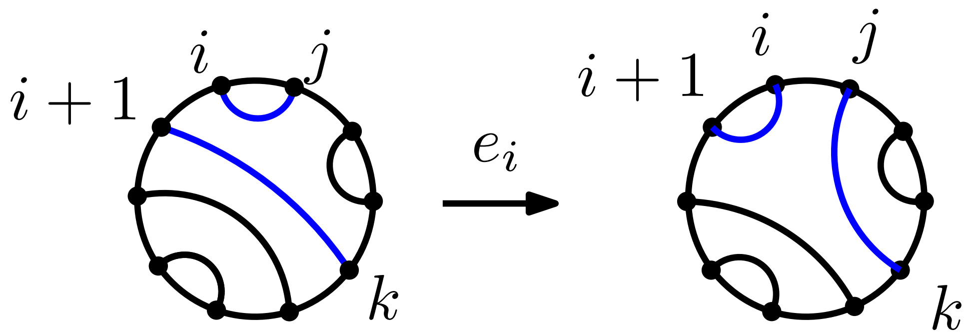

Markov chain $\mathcal{M}$

- States $= LP_n$;
- Transition probabilities : $P(\pi \to \pi') = \frac{k}{2n}$ where k is the number of $i \in \{1, \dots, 2n\}$ such that $e_i(\pi) = \pi'$.

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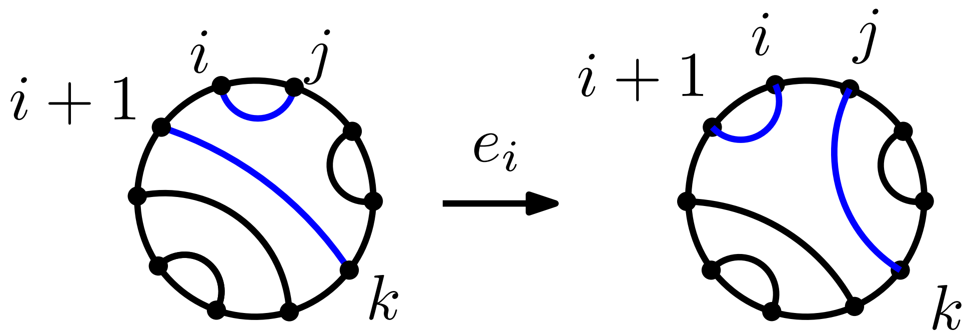

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Stationary distribution : Let P be the matrix defined by $P_{\pi\pi'} = P(\pi \to \pi')$ where $\pi, \pi' \in LP_n$.
Then there is a unique probability distribution $(\psi)_\pi$ on LP_n such that $P\psi = \psi$.

Introduction : the Razumov-Stroganov ex-conjecture

RS conjecture ([Cantini and Sportiello '10]) :

$$\forall \pi \in LP_n, \quad \psi_\pi = \frac{A_\pi}{A_n}$$

The proof consists in showing that the numbers A_π / A_n verify the stationary equations of $\mathcal{M}$, which can be written explicitly :

$$\forall \pi, \quad 2nA_\pi = \sum_{(i,\pi'), e_i(\pi')=\pi} A_{\pi'}$$

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The numbers ψ_π were studied in detail by Di Francesco and Zinn-Justin $\rightarrow$ integral expressions (up to a change of basis), multivariate versions, computations in special cases.

Consequence : to prove results about the numbers A_π one can either use their combinatorial definitions or use the expressions from Di-Francesco and Zinn-Justin.

Introduction

Theorem[Caselli,Krattenthaler,Lass,N. '05]

For a fixed π, the quantity $A_{(\pi)_p}$ is polynomial in p; let $A_\pi(X)$ be the polynomial such that $A_\pi(p) = A_{(\pi)_p}$ for $p \in \mathbb{N}$. Then $A_\pi(X)$ is a polynomial of degree $d(\pi)$, with leading coeffient $\frac{1}{H_\pi}$, such that $d(\pi)! \cdot A_\pi(X)$ has integer coefficients.

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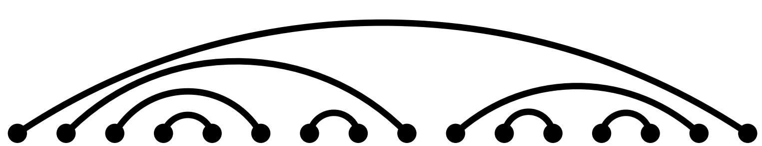

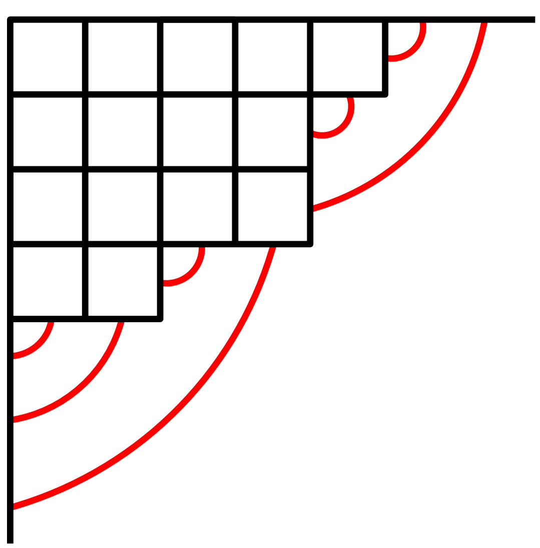

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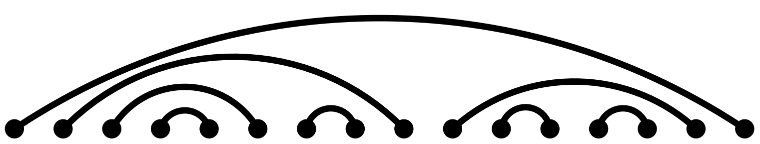

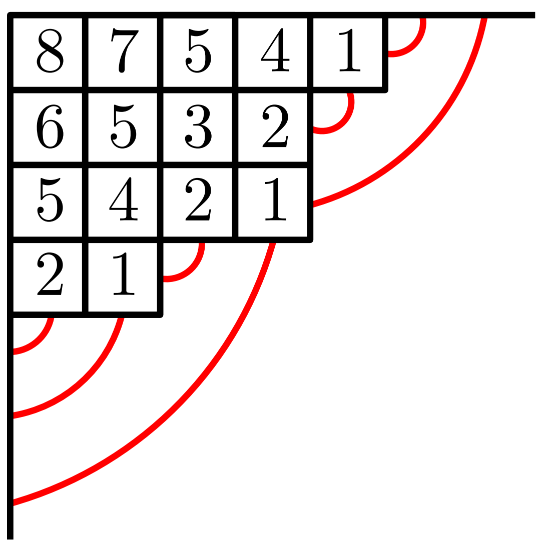

H_π is the product of the hook lengths of $Y(\pi)$.

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We have the following properties :

$$A_\pi(X) = A_{\pi^*}(X).$$
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Here π^* is the matching reflected vertically ; this is proved by checking the evaluations $X = p \in \mathbb{N}$.

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We formulate several conjectures about the polynomials $A_\pi(X)$, and emphasize the combinatorics related to them.

Some simple combinatorics

Combinatorial constructions (1)

Let π be a matching of size $|\pi| = n$, and consider an integer i in $[\![1, n-1]\!]$.

Let $\widehat{x} = 2n + 1 - x$., and (I) be the interval $[\![i+1, \widehat{i+1}]\!]$, while (O) is defined as $[\![1, 2n]\!] - (I)$.

Definition $m_i(\pi) :=$ half the number of arcs in π linking the regions (O) and (I).

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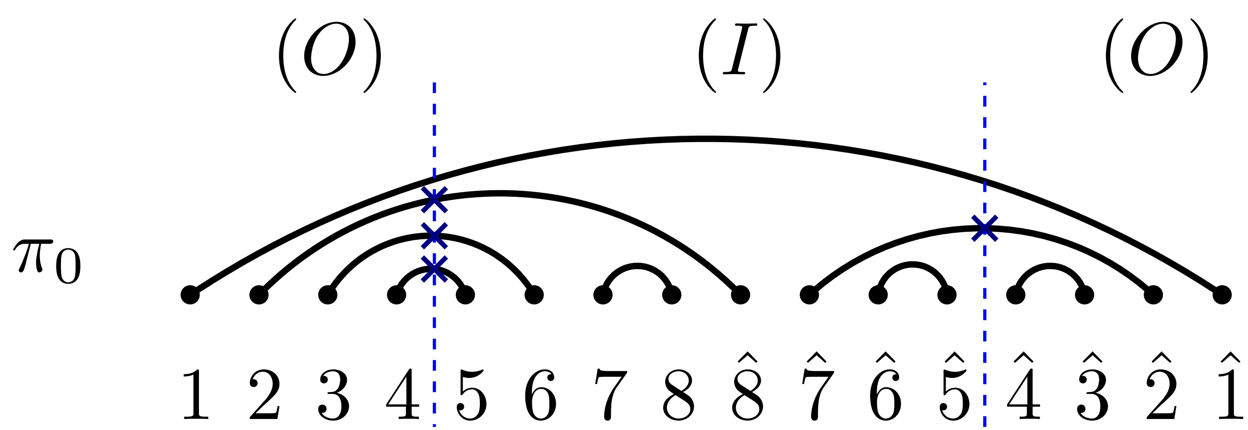
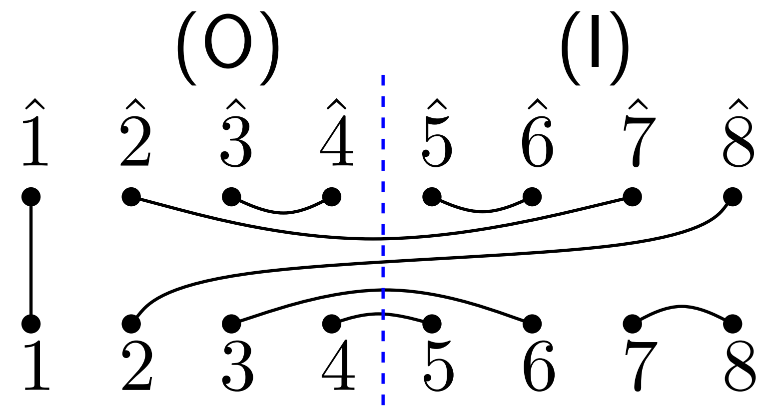


$m_4(\pi_0) = \frac{4}{2} = 2.$

Combinatorial constructions (1)

This is better visualized by folding π on itself :



We obtain $m_i(\pi_0) = 0, 1, 2, 2, 2, 1, 1$ for $i = 1 \ldots 7$.

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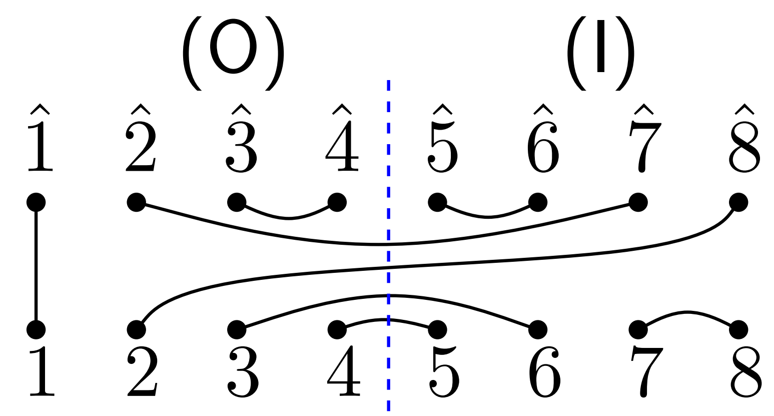

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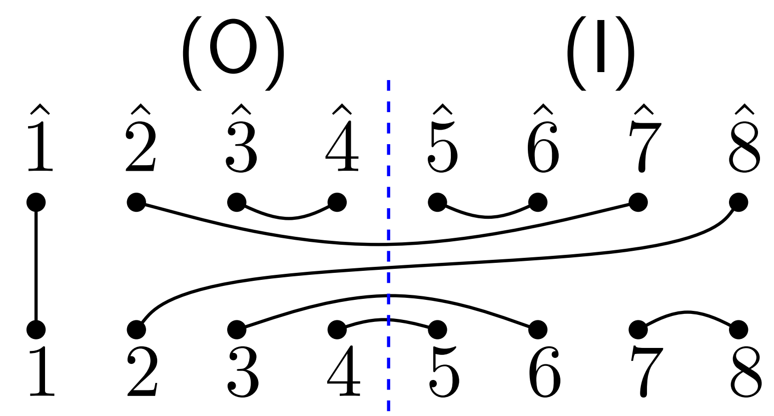

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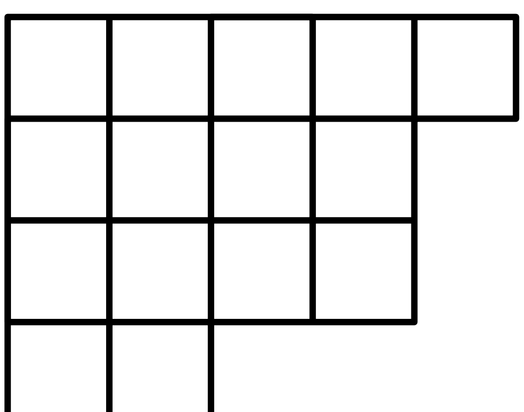
We now define other integers $m_i^{bis}(\pi)$, defined also for $i = 1, 2, \dots, n-1$.

Combinatorial constructions (2)

We use here the Young diagram $Y(\pi)$ attached to π.



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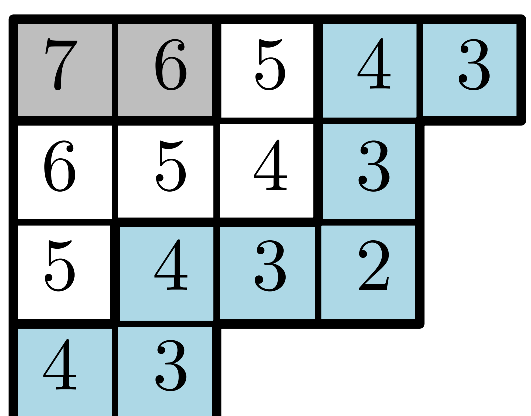

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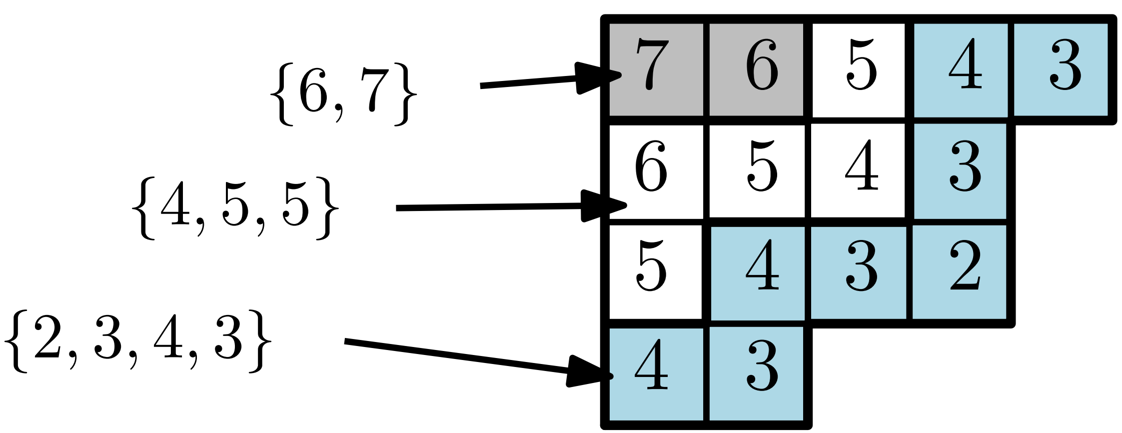
2) For each rim R, construct the multiset union of $\{k\}$, $[\![k+1, e_1]\!]$ and $[\![k+1, e_2]\!]$, where k is the minimum label in R, and e_1, e_2 are the labels at both extremities.



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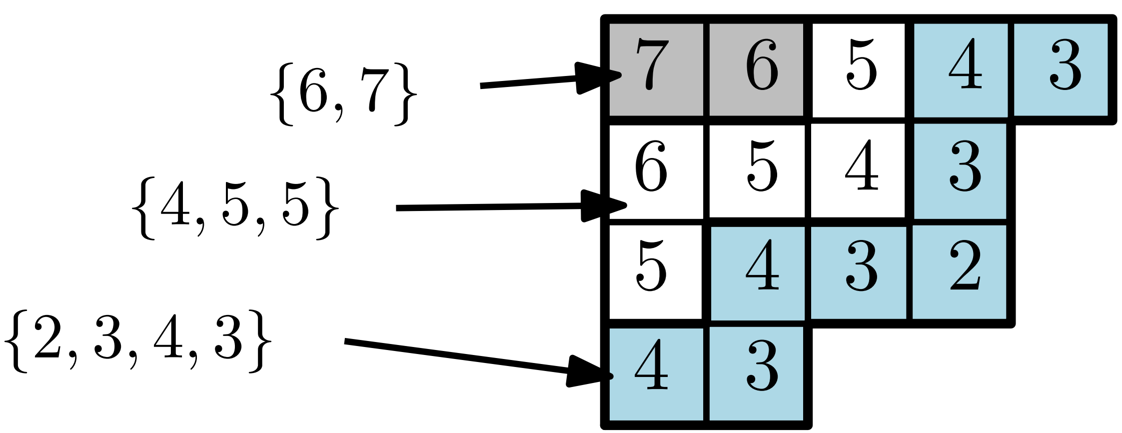
3) Now do the union U of all these multisets :

$$m_i^{bis}(\pi) := \text{multiplicity of } i \text{ in } U.$$



$U = \{2, 3^2, 4^2, 5^2, 6, 7\}$
$\to m_i^{bis} = 0, 1, 2, 2, 2, 1, 1$
for $i = 1 \ldots 7$.

Combinatorial constructions (2)

In fact we have :

Proposition : For all π, i, we have $m_i(\pi) = m_i^{bis}(\pi)$

One can show this by induction on the number of rims in the rim decomposition of $Y(\pi)$.

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Proposition : For any π, we have
- $\sum_i m_i(\pi) \leq d(\pi)$;
- $\sum_i m_i(\pi) \equiv d(\pi)$ (mod 2).

Proof : Use the $m_i^{bis}(\pi)$ construction.

By convention, we set $m_i(\pi) = 0$ if $i \in [\![1, n-1]\!]$.

The conjectures

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- the real roots ;
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The $A_\pi(X)$ have been explicitly computed for $|\pi| \leq 8$, and the conjectures hold for these polynomials.

(There are $C_8 = 1430$ matchings π with 8 arches. The maximal degree $d(\pi)$ of the corresponding polynomials $A_\pi(X)$ is 28.)

The root conjecture

Root conjecture : All real roots of $A_\pi(X)$ are negative integers. The multiplicity of $-i$ is exactly $m_i(\pi)$.

Equivalently, we have the factorization :

$$A_\pi(X) = \left(\prod_i (X+i)^{m_i(\pi)} \right) \cdot Q_\pi(X)$$

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(**Remark :** A consequence of the conjecture is that it gives the *sign variations* of the real function $x \mapsto A_\pi(x)$.)

Proposition : If $(1, 2n)$ is not an arc in π, then $A_\pi(-1) = 0$.

This is a very special case of the conjecture.

The root conjecture

We need to check that the root conjecture is compatible with what we know about $A_\pi(X)$ and $m_i(\pi)$.

1) $A_\pi(X)$ has degree $d(\pi)$, so $\sum_i m_i(\pi)$ cannot be larger than $d(\pi)$. Furthermore $Q_\pi(X)$ has even degree, so $d(\pi) - \sum_i m_i(\pi)$ is even.
$\Rightarrow$ We already checked both facts.

2) We have $A_\pi(X) = A_{\pi^*}(X)$.
$\Rightarrow$ indeed $m_i(\pi) = m_i(\pi^*)$ for all i.

3) $A_{(\pi)}(X) = A_\pi(X+1)$
$\Rightarrow$ we have $m_i\left((\pi)\right) = m_{i+1}(\pi)$ as expected.

The ghost value conjecture

1) **Definition** For any π we define $G_\pi := A_\pi(-|\pi|)$.

By the root conjecture, its sign is $(-1)^{d(\pi)}$.

2) **Composition of matchings**

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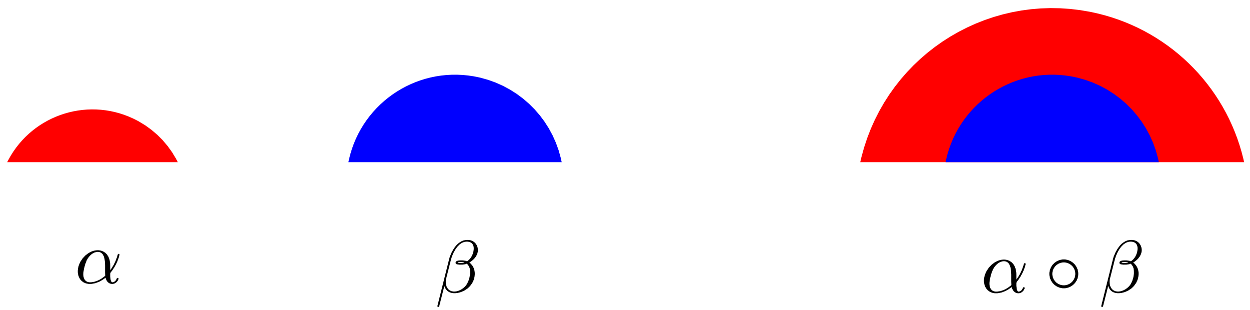

Ghost value conjecture : Let $i \in [\![1, n-1]\!]$ such that $m_i(\pi) = 0$, and write $\pi = \alpha \circ \beta$ where $|\alpha| = i$, $|\beta| = n - i$. Then

$$A_\pi(-i) = G_\alpha A_\beta.$$

The ghost value conjecture

The values G_π play thus a special role, and we have conjectures for them :

Conjecture : For all $n \geq 1$, we have

$$\sum_{\pi : |\pi| = n} |G_\pi| = A_n$$

Here A_n is the total number of FPLs of size n.

So, like the A_π, the values G_π seem to be associated to a partition of FPLs indexed by non-crossing matchings (It is easily checked that $A_\pi \neq G_\pi$ in general).

The positivity conjecture

Positivity conjecture : The polynomial $A_\pi(X)$ has nonnegative coefficients.

This is already known for :
– the constant term $A_\pi(0) = A_\pi$, and
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Theorem The positivity conjecture holds for the coefficient of $X^{d(\pi)-1}$ in $A_\pi(X)$.

Two proofs can be given : either using the formula for ψ_π, or an expression for the polynomials $A_\pi(X)$ based on "FPL configurations in a triangle" ([N. '10]).

Remark : As a byproduct of these proofs, we obtained certain summation formulas involving hook products H_π.

A last word of support

We sum up the various sources of supporting evidence for the conjectures :

• Computation of the $A_\pi(X)$ for small $|\pi|$;

• Compatibility of the conjectures with known facts ;

• Coherence of the conjectures among themselves.

• Proof of the conjectures for special values ;

• Proof of the conjectures for certain families of matchings π, for which $A_\pi(X)$ is known explicitly [Di-Francesco and al '04, Caselli and Krattenthaler '05].

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• The conjectures lead us to believe that there is *combinatorial reciprocity* result underlying the $A_\pi(X)$, à la Ehrhart polynomial :
$\Rightarrow$ there "should be" nice objects enumerated by the values $A_\pi(-i)$.

Especially interesting is to conjecture/prove :

What do the numbers G_π count ?

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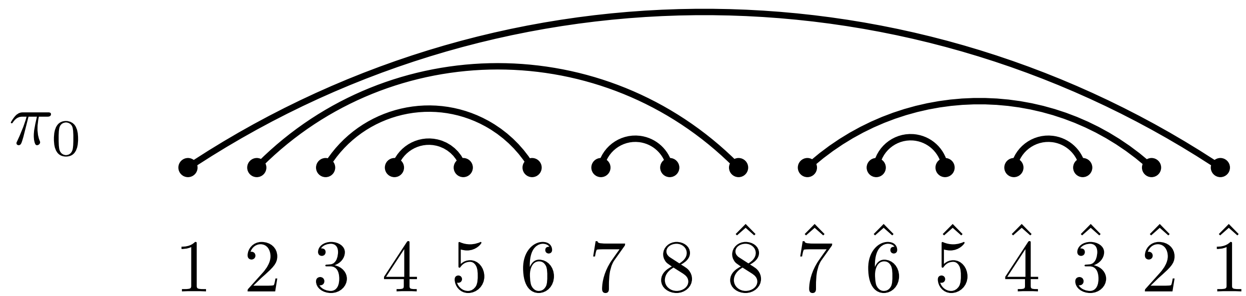
• **The τ case :** There exists a refinement of the probabilities ψ_π to polynomials in τ, with no known equivalent for the A_π, which specialize to our previous setting for $\tau = 1$.
Our conjectures all have "τ versions" dealing with bivariate polynomials $\psi_\pi(X, \tau)$.



$$A_{\pi_0}(X) = \frac{(2+X)(3+X)^2(4+X)^2(5+X)^2(6+X)(7+X)}{145152000} \times (9X^6 + 284X^5 + 4355X^4 + 39660X^3 + 225436X^2 + 757456X + 123120)$$