On some polynomials enumerating Fully Packed Loop configurations

Philippe Nadeau (Univ. of Vienna)
Joint work with Tiago Fonseca (LPTHE, Univ. Paris 6)

SLC 64, Lyon, March 29th 2010.
Introduction

Fully Packed Loop (FPL) configurations are combinatorial structures that arose in statistical mechanics, in simple bijection with known combinatorial objects.
Introduction

Fully Packed Loop (FPL) configurations are combinatorial structures that arose in statistical mechanics, in simple bijection with known combinatorial objects.

One can define certain polynomials $A_\pi(X)$ (indexed by noncrossing matchings $\pi$) which count FPLs when specialized to nonnegative integers.

We will here formulate conjectures for the polynomials $A_\pi(X)$, hinting at “combinatorial reciprocity”.

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]
Introduction

Start with the square grid $G_n$ with $n^2$ vertices and $4n$ external edges (here $n = 7$).

A FPL configuration of size $n$ is a subgraph of the grid $G_n$
(1) such that around each vertex of $G_n$, 2 edges out of 4 are selected; (“Fully Packed”)
(2) containing every other external edge. (“Boundary condition”)

$A_n =$ number of FPLs of size $n$
A link pattern $\pi$ of size $n$ is a set of $n$ noncrossing chords between $2n$ points on a disk.

To each FPL $F$ is associated a link pattern $\pi(F)$. 
We will write link patterns linearly, as noncrossing matchings:

Given a noncrossing matching $\pi$, we note $A_\pi$ the number of FPLs $F$ such that $\pi(F) = \pi$. 
Introduction

We will write link patterns linearly, as noncrossing matchings:

Given a noncrossing matching $\pi$, we note $A_\pi$ the number of FPLs $F$ such that $\pi(F) = \pi$.

Matchings with nested arches: for $p \geq 0$,

$$(\pi)_p :=$$
Introduction: the Razumov-Stroganov ex-conjecture

Definition: The operators $e_i, i \in [1, 2n]$, act on matchings by $\{i, j\}, \{i + 1, k\} \in \pi \rightarrow \{i, i + 1\}, \{j, k\} \in e_i(\pi)$. 

![Diagram showing the action of $e_i$ on a matching]
Introduction : the Razumov-Stroganov ex-conjecture

**Definition** : The operators $e_i, i \in [1, 2n]$, act on matchings by

$\{i, j\}, \{i + 1, k\} \in \pi \rightarrow \{i, i + 1\}, \{j, k\} \in e_i(\pi)$. 

**Markov chain** $\mathcal{M}$

- **States** = $LP_n$;
- **Transition probabilities** : $P(\pi \rightarrow \pi') = \frac{k}{2n}$ where $k$ is the number of $i \in \{1, \ldots, 2n\}$ such that $e_i(\pi) = \pi'$. 
Introduction: the Razumov-Stroganov ex-conjecture

**Definition**: The operators $e_i, i \in [1, 2n]$, act on matchings by $\{i, j\}, \{i + 1, k\} \in \pi \rightarrow \{i, i + 1\}, \{j, k\} \in e_i(\pi)$.

Markov chain $\mathcal{M}$
- **States** = $LP_n$
- **Transition probabilities** : $P(\pi \rightarrow \pi') = \frac{k}{2n}$ where $k$ is the number of $i \in \{1, \ldots, 2n\}$ such that $e_i(\pi) = \pi'$.

**Stationary distribution**: Let $P$ be the matrix defined by $P_{\pi\pi'} = P(\pi \rightarrow \pi')$ where $\pi, \pi' \in LP_n$. Then there is a unique probability distribution $(\psi)_\pi$ on $LP_n$ such that $P\psi = \psi$. 

6-3
RS conjecture ([Cantini and Sportiello '10]) :

\[ \forall \pi \in LP_n, \quad \psi_\pi = \frac{A_\pi}{A_n} \]

The proof consists in showing that the numbers \( A_\pi / A_n \) verify the stationary equations of \( M \), which can be written explicitly :

\[ \forall \pi, \quad 2nA_\pi = \sum_{(i, \pi'), e_i(\pi') = \pi} A_{\pi'} \]
Introduction: the Razumov-Stroganov ex-conjecture

RS conjecture ([Cantini and Sportiello ’10]):

\[ \forall \pi \in LP_n, \quad \psi_\pi = \frac{A_\pi}{A_n} \]

The proof consists in showing that the numbers \( A_\pi / A_n \) verify the stationary equations of \( \mathcal{M} \), which can be written explicitly:

\[ \forall \pi, \quad 2nA_\pi = \sum_{(i,\pi'), e_i(\pi') = \pi} A_{\pi'} \]

The numbers \( \psi_\pi \) were studied in detail by Di Francesco and Zinn-Justin \( \rightarrow \) integral expressions (up to a change of basis), multivariate versions, computations in special cases.
Introduction: the Razumov-Stroganov ex-conjecture

**RS conjecture ([Cantini and Sportiello '10]):**

\[ \forall \pi \in LP_n, \quad \psi_\pi = \frac{A_\pi}{A_n} \]

The proof consists in showing that the numbers \( A_\pi / A_n \) verify the stationary equations of \( \mathcal{M} \), which can be written explicitly:

\[ \forall \pi, \quad 2nA_\pi = \sum_{(i, \pi'), e_i(\pi') = \pi} A_{\pi'} \]

The numbers \( \psi_\pi \) were studied in detail by Di Francesco and Zinn-Justin → integral expressions (up to a change of basis), multivariate versions, computations in special cases.

**Consequence:** to prove results about the numbers \( A_\pi \) one can either use their combinatorial definitions or use the expressions from Di-Francesco and Zinn-Justin.
Theorem [Caselli, Krattenthaler, Lass, N. ’05]

For a fixed $\pi$, the quantity $A_{(\pi)}_p$ is polynomial in $p$; let $A_\pi(X)$ be the polynomial such that $A_\pi(p) = A_{(\pi)}_p$ for $p \in \mathbb{N}$. Then $A_\pi(X)$ is a polynomial of degree $d(\pi)$, with leading coefficient $\frac{1}{H_\pi}$, such that $d(\pi)! \cdot A_\pi(X)$ has integer coefficients.
Introduction

**Theorem** [Caselli, Krattenthaler, Lass, N. ’05]

For a fixed $\pi$, the quantity $A_{(\pi)_p}$ is polynomial in $p$; let $A_{\pi}(X)$ be the polynomial such that $A_{\pi}(p) = A_{(\pi)_p}$ for $p \in \mathbb{N}$. Then $A_{\pi}(X)$ is a polynomial of degree $d(\pi)$, with leading coefficient $\frac{1}{H_{\pi}}$, such that $d(\pi)! \cdot A_{\pi}(X)$ has integer coefficients.

$d(\pi)$ is the number of boxes in the Young diagram $Y(\pi)$:

![Diagram of Young diagrams]
Theorem [Caselli, Krattenthaler, Lass, N. ’05]
For a fixed $\pi$, the quantity $A_{(\pi)_p}$ is polynomial in $p$; let $A_{\pi}(X)$ be the polynomial such that $A_{\pi}(p) = A_{(\pi)_p}$ for $p \in \mathbb{N}$. Then $A_{\pi}(X)$ is a polynomial of degree $d(\pi)$, with leading coefficient $\frac{1}{H_\pi}$, such that $d(\pi)! \cdot A_{\pi}(X)$ has integer coefficients.

$d(\pi)$ is the number of boxes in the Young diagram $Y(\pi)$:

$H_\pi$ is the product of the hook lengths of $Y(\pi)$.  

8 7 5 4 1
6 5 3 2
5 4 2 1
2 1
Introduction

We have the following properties:

\[
A_\pi(X) = A_{\pi^*}(X).
\]
\[
A_{(\pi)}(X) = A_\pi(X + 1).
\]

Here \( \pi^* \) is the matching reflected vertically; this is proved by checking the evaluations \( X = p \in \mathbb{N} \).
Introduction

We have the following properties:

\[
A_\pi(X) = A_{\pi^*}(X).
\]
\[
A_{(\pi)}(X) = A_\pi(X + 1).
\]

Here $\pi^*$ is the matching reflected vertically; this is proved by checking the evaluations $X = p \in \mathbb{N}$.

We formulate several conjectures about the polynomials $A_\pi(X)$, and emphasize the combinatorics related to them.
Some simple combinatorics
Combinatorial constructions (1)

Let $\pi$ be a matching of size $|\pi| = n$, and consider an integer $i$ in $[1, n - 1]$.

Let $\hat{x} = 2n + 1 - x$, and $(I)$ be the interval $[i + 1, \hat{i} + 1]$, while $(O)$ is defined as $[1, 2n] - (I)$.

**Definition** $m_i(\pi) :=$ half the number of arcs in $\pi$ linking the regions $(O)$ and $(I)$. 
Let $\pi$ be a matching of size $|\pi| = n$, and consider an integer $i$ in $[1, n - 1]$.

Let $\hat{x} = 2n + 1 - x$, and $(I)$ be the interval $[i + 1, \hat{i} + 1]$, while $(O)$ is defined as $[1, 2n] - (I)$.

**Definition** $m_i(\pi) :=$ half the number of arcs in $\pi$ linking the regions $(O)$ and $(I)$.

$$m_4(\pi_0) = \frac{4}{2} = 2.$$
This is better visualized by folding $\pi$ on itself:

We obtain $m_i(\pi_0) = 0, 1, 2, 2, 2, 1, 1$ for $i = 1 \ldots 7$. 
This is better visualized by folding $\pi$ on itself:

![Diagram](image)

We obtain $m_i(\pi_0) = 0, 1, 2, 2, 2, 1, 1$ for $i = 1 \ldots 7$.

One has the easy properties:

- $m_i(\pi) = m_i(\pi^*)$, where $\pi^*$ is the reflected matching;
- $m_i(\pi) = m_{i+1}(\pi')$ where $\pi' = (\pi)$. 

Combinatorial constructions (1)
This is better visualized by folding $\pi$ on itself:

We obtain $m_i(\pi_0) = 0, 1, 2, 2, 2, 1, 1$ for $i = 1 \ldots 7$.

One has the easy properties:

- $m_i(\pi) = m_i(\pi^*)$, where $\pi^*$ is the reflected matching;
- $m_i(\pi) = m_{i+1}(\pi')$ where $\pi' = (\pi)$.

We now define other integers $m_i^{bis}(\pi)$, defined also for $i = 1, 2, \ldots, n - 1$. 
Combinatorial constructions (2)

We use here the Young diagram $Y(\pi)$ attached to $\pi$. 

![Young diagram](image-url)
Combinatorial constructions (2)

We use here the Young diagram $Y(\pi)$ attached to $\pi$.

1) Label the cells by putting $n - 1$ in the top left corner, and letting labels decrease by 1; decompose $Y(\pi)$ in rims.

<table>
<thead>
<tr>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Combinatorial constructions (2)

We use here the Young diagram $Y(\pi)$ attached to $\pi$.

1) Label the cells by putting $n - 1$ in the top left corner, and letting labels decrease by 1; decompose $Y(\pi)$ in rims.

2) For each rim $R$, construct the multiset union of $\{k\}$, $[k + 1, e_1]$ and $[k + 1, e_2]$, where $k$ is the minimum label in $R$, and $e_1, e_2$ are the labels at both extremities.
Combinatorial constructions (2)

We use here the Young diagram $Y(\pi)$ attached to $\pi$.

1) Label the cells by putting $n - 1$ in the top left corner, and letting labels decrease by 1; decompose $Y(\pi)$ in rims.

2) For each rim $R$, construct the multiset union of $\{k\}$, $[k + 1, e_1]$ and $[k + 1, e_2]$, where $k$ is the minimum label in $R$, and $e_1, e_2$ are the labels at both extremities.

3) Now do the union $U$ of all these multisets :

$$m_i^{bis}(\pi) := \text{multiplicity of } i \text{ in } U.$$

We have $U = \{2, 3^2, 4^2, 5^2, 6, 7\}$

$$\rightarrow m_i^{bis} = 0, 1, 2, 2, 2, 1, 1$$

for $i = 1 \ldots 7$. 

\[ \begin{array}{cccccc}
7 & 6 & 5 & 4 & 3 \\
6 & 5 & 4 & 3 \\
5 & 4 & 3 & 2 \\
4 & 3 \\
\end{array} \]

\{6, 7\}

\{4, 5, 5\}

\{2, 3, 4, 3\}
Combinatorial constructions (2)

In fact we have:

**Proposition**: For all $\pi, i$, we have $m_i(\pi) = m_i^{bis}(\pi)$

One can show this by induction on the number of rims in the rim decomposition of $Y(\pi)$. 
Combinatorial constructions (2)

In fact we have:

**Proposition**: For all $\pi, i$, we have $m_i(\pi) = m_i^{bis}(\pi)$

One can show this by induction on the number of rims in the rim decomposition of $Y(\pi)$.

**Proposition**: For any $\pi$, we have
• $\sum_i m_i(\pi) \leq d(\pi)$;
• $\sum_i m_i(\pi) \equiv d(\pi) \pmod{2}$.

**Proof**: Use the $m_i^{bis}(\pi)$ construction.

By convention, we set $m_i(\pi) = 0$ if $i \in [1, n - 1]$.
The conjectures
The conjectures

We will formulate conjectures about the polynomials $A_{\pi}(X)$, which concern:

• the real roots;
• the values at negative integers;
• the coefficients.
The conjectures

We will formulate conjectures about the polynomials $A_\pi(X)$, which concern:

- the real roots;
- the values at negative integers;
- the coefficients.

The $A_\pi(X)$ have been explicitly computed for $|\pi| \leq 8$, and the conjectures hold for these polynomials.

(There are $C_8 = 1430$ matchings $\pi$ with 8 arches. The maximal degree $d(\pi)$ of the corresponding polynomials $A_\pi(X)$ is 28.)
**The root conjecture**

**Root conjecture**: All real roots of $A_\pi (X)$ are negative integers. The multiplicity of $-i$ is exactly $m_i (\pi)$. Equivalently, we have the factorization:

$$A_\pi (X) = \left( \prod_i (X + i)^{m_i (\pi)} \right) \cdot Q_\pi (X)$$

where $Q_\pi (X)$ has no real roots.
The root conjecture

Root conjecture: All real roots of $A_\pi(X)$ are negative integers. The multiplicity of $-i$ is exactly $m_i(\pi)$.

Equivalently, we have the factorization:

$$A_\pi(X) = \left( \prod_i (X + i)^{m_i(\pi)} \right) \cdot Q_\pi(X)$$

where $Q_\pi(X)$ has no real roots.

(Remark: A consequence of the conjecture is that it gives the sign variations of the real function $x \mapsto A_\pi(x)$.)

Proposition: If $(1, 2n)$ is not an arc in $\pi$, then $A_\pi(-1) = 0$.

This is a very special case of the conjecture.
The root conjecture

We need to check that the root conjecture is compatible with what we know about $A_\pi(X)$ and $m_i(\pi)$.

1) $A_\pi(X)$ has degree $d(\pi)$, so $\sum_i m_i(\pi)$ cannot be larger than $d(\pi)$. Furthermore $Q_\pi(X)$ has even degree, so $d(\pi) - \sum_i m_i(\pi)$ is even.
\[ \Rightarrow \text{We already checked both facts.} \]

2) We have $A_\pi(X) = A_{\pi^*}(X)$.
\[ \Rightarrow \text{indeed } m_i(\pi) = m_i(\pi^*) \text{ for all } i. \]

3) $A_{(\pi)}(X) = A_\pi(X + 1)$
\[ \Rightarrow \text{we have } m_i((\pi)) = m_{i+1}(\pi) \text{ as expected.} \]
The ghost value conjecture

1) **Definition** For any $\pi$ we define $G_\pi := A_\pi (-|\pi|)$.

By the root conjecture, its sign is $(-1)^{d(\pi)}$.

2) **Composition of matchings**

\[ \alpha \circ \beta \]
The ghost value conjecture

1) **Definition** For any \( \pi \) we define \( G_\pi := A_\pi(-|\pi|) \).

By the root conjecture, its sign is \((-1)^{d(\pi)}\).

2) **Composition of matchings**

\[
\begin{align*}
\alpha & \quad \beta & \quad \alpha \circ \beta
\end{align*}
\]

**Ghost value conjecture**: Let \( i \in [1, n - 1] \) such that \( m_i(\pi) = 0 \), and write \( \pi = \alpha \circ \beta \) where \( |\alpha| = i, \ |\beta| = n - i \). Then

\[
A_\pi(-i) = G_\alpha A_\beta.
\]
The ghost value conjecture

The values $G_{\pi}$ play thus a special role, and we have conjectures for them:

**Conjecture:** For all $n \geq 1$, we have

$$\sum_{\pi : |\pi| = n} |G_{\pi}| = A_n$$

Here $A_n$ is the total number of FPLs of size $n$.

So, like the $A_{\pi}$, the values $G_{\pi}$ seem to be associated to a partition of FPLs indexed by non-crossing matchings (It is easily checked that $A_{\pi} \neq G_{\pi}$ in general).
The positivity conjecture

**Positivity conjecture:** The polynomial $A_\pi(X)$ has nonnegative coefficients.

This is already known for:
- the constant term $A_\pi(0) = A_\pi$, and
- the leading coefficient in degree $d(\pi)$ which is $1/H_\pi$. 
The positivity conjecture

**Positivity conjecture**: The polynomial $A_\pi(X)$ has nonnegative coefficients.

This is already known for:
- the constant term $A_\pi(0) = A_\pi$, and
- the leading coefficient in degree $d(\pi)$ which is $1/H_\pi$.

**Theorem** The positivity conjecture holds for the coefficient of $X^{d(\pi) - 1}$ in $A_\pi(X)$.

Two proofs can be given: either using the formula for $\psi_\pi$, or an expression for the polynomials $A_\pi(X)$ based on “FPL configurations in a triangle” ([N. ’10]).

**Remark**: As a byproduct of these proofs, we obtained certain summation formulas involving hook products $H_\pi$. 
A last word of support

We sum up the various sources of supporting evidence for the conjectures:

- Computation of the $A_{\pi}(X)$ for small $|\pi|$;
- Compatibility of the conjectures with known facts;
- Coherence of the conjectures among themselves.
- Proof of the conjectures for special values;
- Proof of the conjectures for certain families of matchings $\pi$, for which $A_{\pi}(X)$ is known explicitly [Di-Francesco and al '04, Caselli and Krattenthaler '05].
A last word of support

We sum up the various sources of supporting evidence for the conjectures:

- Computation of the $A_{\pi}(X)$ for small $|\pi|$;
- Compatibility of the conjectures with known facts;
- Coherence of the conjectures among themselves.
- Proof of the conjectures for special values;
- Proof of the conjectures for certain families of matchings $\pi$, for which $A_{\pi}(X)$ is known explicitly [Di-Francesco and al '04, Caselli and Krattenthaler '05].

I wish this were enough to turn them into theorems; it’s not… But anyway, they are true.
A last word of support

We sum up the various sources of supporting evidence for the conjectures:

- Computation of the $A_\pi(X)$ for small $|\pi|$;
- Compatibility of the conjectures with known facts;
- Coherence of the conjectures among themselves.
- Proof of the conjectures for special values;
- Proof of the conjectures for certain families of matchings $\pi$, for which $A_\pi(X)$ is known explicitly [Di-Francesco and al '04, Caselli and Krattenthaler '05].

I wish this were enough to turn them into theorems; it’s not...
But anyway, they are true.

I guess.
Conclusion

• The conjectures lead us to believe that there is *combinatorial reciprocity* result underlying the $A_{\pi}(X)$, à la Ehrhart polynomial:

  ⇒ there “should be” nice objects enumerated by the values $A_{\pi}(-i)$.

Especially interesting is to conjecture/prove:

What do the numbers $G_{\pi}$ count?
Conclusion

• The conjectures lead us to believe that there is combinatorial reciprocity result underlying the $A_\pi(X)$, à la Ehrhart polynomial:

$\Rightarrow$ there “should be” nice objects enumerated by the values $A_\pi(-i)$.

Especially interesting is to conjecture/prove:

What do the numbers $G_\pi$ count?

• The $\tau$ case: There exists a refinement of the probabilities $\psi_\pi$ to polynomials in $\tau$, with no known equivalent for the $A_\pi$, which specialize to our previous setting for $\tau = 1$. Our conjectures all have “$\tau$ versions” dealing with bivariate polynomials $\psi_\pi(X, \tau)$. 
\[ A_{\pi_0}(X) = \frac{(2 + X)(3 + X)^2(4 + X)^2(5 + X)^2(6 + X)(7 + X)}{145152000} \times (9X^6 + 284X^5 + 4355X^4 + 39660X^3 + 225436X^2 + 757456X + 123120) \]