Reprinted from

Analytic Number Theory

Proceedings of a Conference in Honor of Paul T. Bateman

Edited by
Bruce C. Berndt Harold G. Diamond
Heini Halberstam Adolf Hildebrand

1990 Birkhäuser
Boston · Basel · Berlin

On the Number of Partitions of n Without a Given Subsum, II

P. ERDÖS, J. L. NICOLAS, AND A. SÁRKÖZY

Dedicated to Professor Paul T. Bateman for his seventieth birthday

Abstract

Let $R(n,a)$ denote the number of unrestricted partitions of n whose subsums are all different of a, and $Q(n,a)$ the number of unequal partitions (i.e. each part is allowed to occur at most once) with the same property. In a preceding paper, we considered $R(n,a)$ and $Q(n,a)$ for $a \le \lambda_1\sqrt{n}$, where λ_1 is a small constant. Here we study the case $a \ge \lambda_2\sqrt{n}$. The behaviour of these quantities depends on the size of a, but also on the size of $s(a)$, the smallest positive integer which does not divide a.

1. Introduction

Let us denote by $p(n)$ the number of unrestricted partitions of n, by $r(n,m)$ the number of partitions of n whose parts are at least m, and by $R(n,a)$ the number of those partitions

$$n = n_1 + \cdots + n_t \quad (n_1 \le \cdots \le n_t)$$

of n which do not represent a, i.e. whose subsums $n_{i_1} + \cdots + n_{i_j}$ are all different from a.

We shall consider also partitions of n into distinct parts. In that case the above notations will be change to $q(n)$, $\rho(n,m)$ and $Q(n,a)$.

In [5] J. Dixmier considered $R(n,a)$ when a is fixed, and in [7], we studied $R(n,a)$ and $Q(n,a)$ when $a < \lambda_1\sqrt{n}$, where λ_1 is a small constant. Here we shall consider the case $\lambda_2\sqrt{n} \le a \le n/2$, where λ_2 is a large constant. (Since $R(n,a) = R(n,n-a)$ and $Q(n,a) = Q(n,n-a)$, we may suppose that $a \le n/2$.) We shall prove:

Research partially supported by Hungarian National Foundation for Scientific Research, grant no. 1811, and by C.N.R.S., Greco Calcul Formel and PRC Math.-Info.

Theorem 1. *For $n > n_0$ and*

$$10^{18}\sqrt{n} \le a \le n^{5/7}, \tag{1.1}$$

we have

$$Q(n,a) \le q([n/2]) \exp(5 \cdot 10^3 a^{-1/3} n^{2/3} \log(a^{1/3} n^{-1/6})) \tag{1.2}$$

and

$$R(n,a) \le p([n/2]) \exp(5 \cdot 10^3 a^{-1/3} n^{2/3} \log(a^{1/3} n^{-1/6})) \tag{1.3}$$

where $[x]$ denotes the integral part of x.

Theorem 2. *For $n > n_0$ and*

$$n^{5/7} < a \le n/2 \tag{1.4}$$

we have

$$Q(n,a) \le q([n/2]) \exp(n^{1/2-1/30}) \tag{1.5}$$

and

$$R(n,a) \le p([n/2]) \exp(n^{1/2-1/30}). \tag{1.6}$$

It follows from Theorems 1 and 2 that

Corollary. *If $a = a(n)$ is such that $a/\sqrt{n} \to \infty$ and $a \le n/2$, we have $Q(n,a) = (q([n/2]))^{1+o(1)}$ and $R(n,a) = (p([n/2]))^{1+o(1)}$.*

Theorem 3. *Let $s(a)$ denote the smallest positive integer which does not divide a. For $n \ge (2500)^2$, $s(a) \ge 40000$ and*

$$\frac{7}{100} n^{1/2} (s(a))^{3/2} \le a \le \frac{1}{40} n (s(a))^{-1} \tag{1.7}$$

we have

$$Q(n,a) < \exp(201 n^{1/2} (s(a))^{-1/2} \log(s(a))) \tag{1.8}$$

and

$$R(n,a) < \exp(301 n^{1/2} s(a)^{-1/2} \log(s(a))). \tag{1.9}$$

Remark: By Lemma 1 below, (1.7) holds for all a's such that

$$(7/10) n^{1/2} (\log n)^{3/2} \le a \le (1/200) n (\log n)^{-1}.$$

To give lower bounds for $R(n,a)$ and $Q(n,a)$, first we note that if a is odd and n is even, then multiplying the parts of a partition of $n/2$ by 2 we get a partition of n whose subsums are all even and thus different from a. Hence

$$R(n,a) \ge p([n/2]) \tag{1.10}$$

and

$$Q(n,a) \ge q([n/2]) \tag{1.11}$$

which show the exponent $1+o(1)$ in the above corollary to be best possible.

This argument can be extended, and yields:

Theorem 4. *Let $h = h(n,a,m)$ be given by $h = 0$ if $m|n$ and*

$$h \equiv n \pmod m, \quad \text{if} \quad a < h \le a + m.$$

Then

$$Q(n,a) \ge q\left(\frac{n - h(n,a,s(a))}{s(a)}\right) \tag{1.12}$$

and

$$R(n,a) \ge p\left(\frac{n - h(n,a,s(a))}{s(a)}\right). \tag{1.13}$$

Proof: When $s(a)$ divides n, we consider all the partitions of $n/s(a)$, and we multiply their parts by $s(a)$. In this way we obtain a partition of n whose subsums are all divisible by $s(a)$ and they cannot be equal to a.

When $s(a)$ does not divide n, let us set $h = h(n,a,s(a))$. We consider all the partitions of $(n-h)/s(a)$, we multiply their parts by $s(a)$, and we complete them with a part equal to h to obtain a partition of n. As $h > a$, the subsums of such a partition are all different from a.

Taking into account the results of Hardy and Ramanujan (cf. [9]):

$$\begin{aligned} p(n) &\sim \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right), \\ q(n) &\sim \frac{1}{4(3n^3)^{1/4}} \exp\left(\pi\sqrt{\frac{n}{3}}\right), \end{aligned} \tag{1.14}$$

we observe that for $a = o(n)$, the upper bounds given in Theorem 3 for $\log Q(n,a)$ and $\log R(n,a)$ are of the same order of magnitude as the lower bounds given in Theorem 4 (apart from a factor $\log s(a)$). This shows that the behavior of $Q(n,a)$ and $R(n,a)$ depends on the arithmetical structure of a if a is large.

In [7], we gave bounds for $R(n,a)$, and a lower bound for $Q(n,a)$ when a is $\le \lambda_1\sqrt{n}$. Here we will prove the following upper bound:

Theorem 5. *For $a \le \frac{3}{5}\sqrt{n}$ and n large enough, we have*

$$Q(n,a) \le q(n) \exp\left(-a \log\frac{2}{\sqrt{3}} + \frac{\pi a^2}{8\sqrt{3}\sqrt{n}}\right). \tag{1.15}$$

The proof follows the same principle as in [7] for unrestricted partitions: if a partition π does not represent a, then i and $a-i$ cannot belong simultaneously to π. So, for every i, $1 \le i < a/2$, there are three possibilities: $i \in \pi$ and $a - i \notin \pi$, $i \notin \pi$ and $a - i \in \pi$, $i \notin \pi$ and $a - i \notin \pi$. When a is even, and $i = a/2$, there are only two possibilities. Therefore, the number of possible sets $\mathcal{A}$ of parts $< a$ is at most $3^{a/2}$. For such a set $\mathcal{A}$, there are $\rho(n - \sum_{x \subset \mathcal{A}} x,\ a+1)$ possibilities of completing $\mathcal{A}$ to a partition of n. As already observed in [8], $\rho(n,m)$ is nondecreasing in n for $n \geq m$, is 0 for $1 \leq m < n$, and 1 for $n = 0$. Thus we have

$$Q(n,a) \leq 3^{a/2} \rho(n, a+1).$$

From Theorem 1 of [8], we have

$$\rho(n, a+1) \leq \rho(n,a) \leq \frac{1}{2^{a-2}} q\left(n + \frac{a^2}{4}\right),$$

and from Lemma 3 of [8] (which is an easy consequence of (1.14)),

$$q\left(n + \frac{a^2}{4}\right) \sim q(n) \exp \frac{\pi a^2}{8\sqrt{3}\sqrt{n}}$$

and Theorem 5 is proved. For $a \geq 0.64\sqrt{n}$, the quantity in the exponent in (1.15) is positive, and thus the trivial bound $Q(n,a) \leq q(n)$ is better.

Now consider the case when a is of the order of magnitude $\sqrt{n}$. In [4], Theorem 2.18 claims that if a is odd and $a \sim \sqrt{n}$, then we have for n large enough

$$\log R(n,a) \geq 2.0138\sqrt{n}. \tag{1.16}$$

This result can be extended to the case when a is odd, $a \sim \lambda\sqrt{n}$, and we obtain

$$\log R(n,a) \geq \varphi(\lambda)\sqrt{n} \tag{1.17}$$

for some function φ.

Our guess is that, when a is odd, such a result is best possible. But when a is even, we have no precise conjecture. J. Dixmier has proved (cf. [6]) that for $\epsilon > 0$ there exists $\delta < 1$ such that, for n large enough,

$$\epsilon\sqrt{n} \leq a \leq n - \epsilon\sqrt{n} \Rightarrow R(n,a) \leq (p(n))^{\delta} \tag{1.18}$$

for all n. The proof is short, and starts with the results of [7].

In the same way, it can be deduced from Theorem 5 above that for $\epsilon > 0$, there exists $\delta < 1$ such that, for n large enough,

$$\epsilon\sqrt{n} \leq a \leq n - \epsilon\sqrt{n} \Rightarrow Q(n,a) \leq (q(n))^{\delta}. \tag{1.19}$$

The aim of [6] is to give a fairly good estimation of $R(2n,n)$, and to study $R(n,a)$ for $\lambda_2 n \leq a \leq n/2$, where λ_2 is a fixed constant.

The proofs of Theorems 1, 2 and 3 are based on results from additive number theory (cf. [11] and [12]). In §2, we shall give some estimates involving partitions. In §3, we shall prove some lemmas on additive properties of dense sequences, from which, in §4, the proof of our three theorems will follow.

All these proofs are effective, but the constants are rather large and we did not attempt to optimize them.

A table of $R(n,a)$ for $n \leq 40$ was given in [7]. Here in the appendix, we give a table of $Q(n,a)$ for $n \leq 40$. It has been computed by M. DeLéglise, and we are very pleased to thank him. For a fixed a, first he determines, by a backtracking programming method, all the subsets $\mathcal{A}$ of $\{1,2,\ldots,a-1\}$ having no subsum equal to a. Then for all $\mathcal{A}$ such that $S(\mathcal{A}) = \sum_{x\in\mathcal{A}} x$ is smaller than n,

$$Q(n,a) = \sum_{\mathcal{A}} \rho(n - S(\mathcal{A}), a+1).$$

As can be seen in [8], $\rho(n,m)$ is easy to calculate.

We thank J. Dixmier for many helpful remarks, and for an improvement of Lemma 11 below.

Notations: $p(n,m)$ will denote the number of unrestricted partitions of n into parts $\leq m$ (or into atmost m parts); here m is not necessarily an integer.

$\mathbb{N}$ is the set of positive integers $\{1,2,\ldots\}$.

$$\mathbb{N}_M = \{1,2,\ldots,M\}.$$

If $\mathcal{A}$ is a finite set of not necessarily distinct integers, then $|\mathcal{A}|$ denotes cardinality of $\mathcal{A}$, $\mathcal{A}'$ the set of distinct elements of $\mathcal{A}$, $S(\mathcal{A}) = \sum_{a\in\mathcal{A}} a$,

$$P(\mathcal{A}) = \left\{ \sum_{a\in\mathcal{A}} \epsilon_a a; \quad \epsilon_a = 0 \quad \text{or} \quad 1, \sum_{a\in\mathcal{A}} \epsilon_a \neq 0 \right\}$$

the set of the nonzero subsums of $\mathcal{A}$,

$$\mathcal{L}(\mathcal{A},d) = \{i;\ 1 \leq i \leq d, \text{ there exist at least 2 elements of } \mathcal{A} \text{ which are } \equiv i \pmod d\}$$

and

$$L(\mathcal{A},d) = |\mathcal{L}(\mathcal{A},d)|.$$

2. Partition Lemmas

Lemma 1. *Let $s(m)$ be the smallest integer which does not divide m. Then for all $m \geq 2$, we have*

$$s(m) < \frac{3}{\log 2} \log m < 4.5 \log m. \tag{2.1}$$

Proof: First, if m is odd, $s(m) = 2$ and (2.1) holds. So we may suppose that m is even, and $s(m) \geq 3$. Let $\psi(x)$ denote the Chebychef function:

$$\psi(x) = \sum_{p^k \leq x} \log p.$$

It follows from Chebychef's results that for all integers $n \geq 2$,

$$\psi(n)/n \geq (\log 2)/2.$$

Then

$$\log m \geq \psi(s(m) - 1) \geq \frac{\log 2}{2}(s(m) - 1)$$

which implies

$$s(m) \leq 1 + \frac{2 \log m}{\log 2} < \frac{3}{\log 2} \log m.$$

It can be shown similarly that

$$m \geq m_0(\epsilon) \Longrightarrow s(m) < (1 + \epsilon) \log m. \tag{2.2}$$

Lemma 2. *Let α be a real number satisfying $0 < \alpha \leq 1.05$. For $m \leq \alpha\sqrt{n}$ we have*

$$p(n, m) < \exp\left(\left(2\alpha \log \frac{3.6}{\alpha}\right) \sqrt{n}\right). \tag{2.3}$$

This inequality can be used to obtain upper bounds for $\rho(n, m)$ and $r(n, m)$ since

$$\rho(n, m) \leq r(n, m) \leq p(n, [n/m]). \tag{2.4}$$

Proof: From the classical inequality

$$m!\, p(n, m) \leq \binom{n + \frac{m(m+1)}{2} - 1}{m - 1},$$

it has been proved in [3] that for all $\alpha > 0$, and $m \leq \alpha\sqrt{n}$,

$$p(n, m) \leq \exp\left((\alpha^3/2 + 2\alpha(1 - \log \alpha)) \sqrt{n}\right).$$

Observing that $\alpha \leq 1.05$ implies $\alpha^2/4 + 1 < \log (3.6)$, (2.3) follows easily.

For $\alpha \geq 1.06$, the obvious inequality $p(n, m) \leq p(n)$ and (1.14) give a better upper bound.

As $p(n, m)$ is also the number of partitions of n with at most m parts, (2.4) can be proved easily.

Lemma 3. *Let $Y(n, t, m)$ denote the number of partitions of n into unequal parts such that at most t parts not exceeding m may occur. We have*

$$Y(n, t, m) \leq p(n, t + n/m). \tag{2.5}$$

Proof: A partition counted in $Y(n, t, m)$ has at most t parts $\leq m$, and n/m parts $> m$. The right–hand side of (2.5) is certainly greater than the number of partitions of n with at most $t + n/m$ unequal parts.

Lemma 4. *Let $Z(n, t, m)$ denote the number of unrestricted partitions of n such that at most t distinct parts not exceeding m may occur. For $1 \leq t \leq m \leq n$ we have*

$$Z(n, t, m) \leq 6tn^2 \binom{m}{min(t, [m/2])} p(n, t) p(n, n/m). \tag{2.6}$$

proof: For $\mathcal{A} \subset \{1, 2, \ldots, m\}$, $\mathcal{A} = \{a_1 < a_2 < \cdots < a_s\}$, let $P(n, \mathcal{A}, m)$ denote the number of partitions of n with all the parts not exceeding m in $\mathcal{A}$, i.e., $P(n, \mathcal{A}, m)$ denotes the number of solutions of

$$a_1 x_1 + \cdots + a_s x_s + \sum_{i=1}^{n-m} (m + i) x_{s+i} = n, \quad (x_i \geq 0, 1 \leq i \leq n). \tag{2.7}$$

Then we have

$$P(n, \mathcal{A}, m) \leq \sum_{k=0}^{n} P(k, \{1, 2, \ldots, s\}, m)$$

since replacing a_j by j in (2.7), we have

$$x_1 + 2x_2 + \cdots + s x_s + \sum_{i=1}^{n-m} (m + i) x_{s+i} = k \tag{2.8}$$

for some $k \leq n$. It follows that

$$Z(n, t, m) \leq \sum_{s=0}^{t} \binom{m}{s} \sum_{k=0}^{n} P(k, \{1, \ldots, s\}, m) \tag{2.9}$$

since $\mathcal{A}$ with $|\mathcal{A}| = s$ can be selected in $\binom{m}{s}$ ways from $\{1, \ldots, m\}$. Hence

$$Z(n, t, m) \leq (t + 1) \binom{m}{min(t, [m/2])} \sum_{k=0}^{n} P(k, \{1, \ldots, t\}, m). \tag{2.10}$$

Now, counting the partitions according to the sum j of the parts not exceeding t, we obtain

$$P(k,\{1,\dots,t\},m)=\sum_{j=0}^{k}p(j,t)r(k-j,m+1)$$
$$\leq (k+1)p(k,t)r(k,m+1)$$

since it is easy to see that $p(n,m)$ and $r(n,m)$ are not decreasing in n. Then (2.10) yields (2.6) observing that $t+1\leq 2t$,

$$\sum_{k=0}^{n}(k+1)\leq 3n^2,$$

and using (2.4).

Lemma 5. *Given integers $M\geq 2$, $D\geq 2$, let $V(n,M,D)$ denote the number of partitions of $n\geq M$ into distinct parts:*

$$n=n_1+\dots+n_t \quad (n_1<\dots<n_t)$$

with the set $\mathcal{N}=\{n_1,\dots,n_t\}$ of parts of n having the following property: there exists an integer d,

$$2\leq d\leq D, \tag{2.11}$$

and integers $i_1,\dots,i_{[d/2]}$ satisfying

$$1\leq i_1<\dots<i_{[d/2]}\leq d, \tag{2.12}$$

such that, if $\mathcal{N}_1=\{n_\ell : n_\ell\in\mathcal{N},\ d\leq n_\ell\leq M,\ n_\ell\equiv i_j \mod d \text{ for some } j\}$, the cardinality of the set $\mathcal{N}_2=\{n_\ell : n_\ell\in\mathcal{N}\backslash\mathcal{N}_1,\ n_\ell\leq M\}$ satisfies

$$|\mathcal{N}_2|\leq 2D, \tag{2.13}$$

then

$$V(n,M,D)\leq n^{5D}q([n/2])p(n,n/M). \tag{2.14}$$

Proof: Let $\mathcal{N}_3=\mathcal{N}\backslash(\mathcal{N}_1\cup\mathcal{N}_2)=\{n_\ell : n_\ell\in\mathcal{N}, n_\ell>M\}$.

Let us fix $d,i_1,i_2,\dots,i_{[d/2]}$ in (2.11), (2.12) and (2.13). By the defintion of $\mathcal{N}_1$, every element m of $\mathcal{N}_1$ can be written in the form

$$m=i_{j(m)}+\ell(m)d$$

where

$$1\leq j(m)\leq[d/2] \quad\text{and}\quad 1\leq\ell(m). \tag{2.15}$$

To every $m\in\mathcal{N}_1$ we assign the integer

$$m^*=j(m)+(\ell(m)-1)[d/2],$$

and write $\mathcal{N}_1^*=\{m^* : m\in\mathcal{N}_1\}$. Clearly, (2.15) implies that to distinct elements of $\mathcal{N}_1$, distinct elements of $\mathcal{N}_1^*$ are assigned. Furthermore, we have

$$m^*=j(m)+(\ell(m)-1)[d/2]\leq[d/2]+(\ell(m)-1)[d/2]$$
$$=\ell(m)[d/2]\leq\ell(m)d/2<m/2$$

whence

$$S(\mathcal{N}_1^*)=\sum_{m^*\in\mathcal{N}_1^*}m^*<\frac{1}{2}\sum_{m\in\mathcal{N}_1}m=\frac{1}{2}S(\mathcal{N}_1).$$

Thus writing $S(\mathcal{N}_1)=u$, the elements of $\mathcal{N}_1^*$ form partition of an integer $v<u/2$ into distinct parts, so that for fixed $d,i_1,\dots,i_{[d/2]}$ and u, $\mathcal{N}_1$ can be selected in at most

$$\sum_{v=0}^{[u/2]}q(v)\leq([n/2]+1)q([n/2])\leq nq([n/2])$$

ways.

Furthermore, the elements of $\mathcal{N}_2$ are selected from $\{1,2,\dots,M\}$ and, by (2.13), their number is at most $2D$, so that $\mathcal{N}_2$ can be chosen in at most $M^{2D}\leq n^{2D}$ ways.

Finally, if $d,i_1,\dots,i_{[d/2]}$ and u are fixed, then

$$S(\mathcal{N}_3)=(S(\mathcal{N}_1)+S(\mathcal{N}_2)+S(\mathcal{N}_3))-S(\mathcal{N}_1)-S(\mathcal{N}_2)$$
$$=n-u-S(\mathcal{N}_2)\leq n-u$$

so that

$$\sum_{n_i\in\mathcal{N}_3}n_i=z$$

is a partition of $z(\leq n-u)$ into parts $\geq M+1$. Thus $\mathcal{N}_3$ can be chosen in at most

$$\sum_{z=0}^{n-u}\rho(z,M+1)\leq(n+1)\rho(n,M)\leq n^2\rho(n,M)$$

ways (note that $\rho(n,m)$ is non-decreasing function of n for $n\geq m$).

Collecting the results above, we obtain that for fixed $d,i_1,\dots,i_{[d/2]}$, the partition $\mathcal{N}$ can be chosen in at most

$$n^{2D+3}q([n/2])\rho(n,M)$$

ways. Furthermore, for fixed d, the numbers $i_1,i_2,\dots,i_{[d/2]}$ in (2.12) can be chosen in at most 2^d ways, and summation over the d's in (2.11) gives

$$\sum_{d\leq D}2^d<2^{D+1}\leq n^{D+1}$$

whence, by (2.4), the result follows.

Lemma 6. *With the notation of Lemma 5 but considering unrestricted partitions $n = n_1 + \cdots + n_t$ ($n_1 \le \cdots \le n_t$) of $n \ge M$, so that now parts in $\mathcal{N}_1$ and $\mathcal{N}_2$ have to be counted according to multiplicity, suppose here that*

$$|\mathcal{N}_2'| \le 2D. \tag{2.16}$$

Then the number $W(n, M, D)$ of unrestricted partitions of n that corresponds to $V(n, M, D)$ in Lemma 5 satisfies

$$W(n, M, D) \le n^{7D} p([n/2]) p(n, n/M).$$

Proof: Again first fix $d, i_1, i_2, \dots, i_{[d/2]}$, and define $\mathcal{N}_3, m^*, \mathcal{N}_1^*$ in the same way as in the proof of Lemma 5. The same argument shows that, writing $S(\mathcal{N}_1) = u$, $\mathcal{N}_1$ can be chosen in at most

$$\sum_{v=0}^{[u/2]} p(v) \le np([n/2])$$

ways.

Furthermore, the elements of $\mathcal{N}_2$ are selected from $\{1, 2, \dots, M\}$ and, by (2.16), the number of distinct elements of $\mathcal{N}_2$ is at most $2D$, so that they can be chosen in at most $M^{2D} \le n^{2D}$ ways; and if we have selected the distinct elements of $\mathcal{N}_2$, then the multiplicity of each of them can be chosen in at most n ways, so that the multiplicities of the distinct elements of $\mathcal{N}_2$ can be chosen altogether in at most n^{2D} ways. Thus $\mathcal{N}_2$ can be chosen in at most $n^{2D} \cdot n^{2D} = n^{4D}$ ways.

Finally, the same argument as in Section 2 shows that $\mathcal{N}_3$ can be chosen in at most

$$\sum_{z=0}^{n-u} r(z, M+1) \le n^2 r(n, M)$$

ways.

Collecting the results above and using also (2.4), we obtain that for fixed $d, i_1, \dots, i_{[d/2]}$, the partition $\mathcal{N}$ can be chosen in at most

$$n^{4D+3} p([n/2]) p(n, n/M)$$

ways. Finally, as in Lemma 5, $d, i_1, \dots, i_{[d/2]}$ can be chosen in at most n^{D+1} ways whence the result follows.

3. Additive Lemmas

First we need the following well known fact (see, e.g. [**12**], Lemma 3).

Lemma 7. *If $d \in \mathbb{N}$ and $n_1, n_2, \dots, n_d$ are integers, then there is a sum of the form $n_{i_1} + \cdots + n_{i_t}$ ($1 \le i_1 < \cdots < i_t \le d$) such that $d | (n_{i_1} + \cdots + n_{i_t})$.*

The next lemma is variant of Lemma 4 in [**12**].

Lemma 8. *If $N \in \mathbb{N}$, $d \in \mathbb{N}$, and $\mathcal{B}$ is a finite set of not necessarily distinct positive integers such that*

$$\textit{the elements of } \mathcal{B} \textit{ do not exceed } N, \tag{3.1}$$

then for every integer n such that $0 \le n \le \frac{1}{d}S(\mathcal{B}) - N$ there is a number x_n in the set $\{n+1, n+2, \dots, n+N\}$ such that $dx_n \in \mathcal{P}(\mathcal{B})$.

Proof: It suffices to show the existence of integers $y_1, y_2, \dots, y_t$ such that $0 < y_1 \le N$, $0 < y_i - y_{i-1} \le N$ (for $i = 2, 3, \dots, t$), $\frac{1}{d}S(\mathcal{B}) - N < y_t$ and $dy_i \in \mathcal{P}(\mathcal{B})$ for $i = 1, 2, \dots, t$. Afterwards we shall define x_n by

$$x_n = \begin{cases} y_1 & \text{for } 0 \le n < y_1, \\ y_i & \text{for } y_{i-1} \le n < y_i \ (2 \le i \le t-1), \\ y_t & \text{for } y_{t-1} \le n \le \frac{1}{d}S(\mathcal{B}) - N. \end{cases}$$

We are going to define these integers y_i by recursion.

We may suppose that $S(\mathcal{B}) \ge dN$ which, by (3.1), implies $|\mathcal{B}| \ge d$. Let $\mathcal{B}_1 \subset \mathcal{B}$, $|\mathcal{B}_1| = d$. Then by Lemma 7, there is a (non-empty) subset $\mathcal{B}_1^*$ of $\mathcal{B}_1$ such that $d | S(\mathcal{B}_1^*)$; write $S(\mathcal{B}_1^*)/d = y_1$. Then $dy_i \in \mathcal{P}(\mathcal{B}_1^*) \subset \mathcal{P}(\mathcal{B}_1)$, $0 < y_1$, and

$$dy_1 = S(\mathcal{B}_1^*) = \sum_{b \in \mathcal{B}_1^*} b \le \sum_{b \in \mathcal{B}_1^*} N = N|\mathcal{B}_1^*| \le N|\mathcal{B}_1| = Nd$$

so that $y_1 \le N$.

Assume now that $y_1, y_2, \dots, y_{i-1}$ have been defined and

$$y_{i-1} \le \frac{1}{d} S(\mathcal{B}) - N. \tag{3.2}$$

By the definition of y_{i-1}, there is a subset $\mathcal{B}_{i-1}^* \subset \mathcal{B}$ such that $S(\mathcal{B}_{i-1}^*) = dy_{i-1}$. Then by (3.2) we have

$$\begin{aligned} S(\mathcal{B} \setminus \mathcal{B}_{i-1}^*) &= S(\mathcal{B}) - S(\mathcal{B}_{i-1}^*) = S(\mathcal{B}) - dy_{i-1} \\ &\ge S(\mathcal{B}) - (S(\mathcal{B}) - Nd) = Nd. \end{aligned} \tag{3.3}$$

(3.1) implies that

$$S(\mathcal{B}\backslash\mathcal{B}^*_{i-1}) = \sum_{b\in(\mathcal{B}\backslash\mathcal{B}^*_{i-1})} b \le \sum_{b\in(\mathcal{B}\backslash\mathcal{B}^*_{i-1})} N = N|\mathcal{B}\backslash\mathcal{B}^*_{i-1}|.$$

It follows from (3.3) and (3.4) that

$$|\mathcal{B}\backslash\mathcal{B}^*_{i-1}| > d.$$

Thus there is a subset $\mathcal{B}_i$ of $\mathcal{B} - \mathcal{B}^*_{i-1}$ with $|\mathcal{B}_i| = d$. By Lemma 7, there is a (non-empty) subset $\mathcal{B}'_i$ of $\mathcal{B}_i$ such that $d|S(\mathcal{B}'_i)$; let $y_i = y_{i-1} + d^{-1}S(\mathcal{B}'_i)$. Then we have

$$\begin{aligned} y_{i-1} < y_i &= y_{i-1} + d^{-1}S(\mathcal{B}'_i) = y_{i-1} + d^{-1}\sum_{b\in\mathcal{B}'_i} b \\ &\le y_{i-1} + d^{-1}\sum_{b\in\mathcal{B}'_i} N = y_{i-1} + Nd^{-1}|\mathcal{B}'_i| \le y_{i-1} + Nd^{-1}|\mathcal{B}_i| \\ &= y_{i-1} + N \end{aligned}$$

and

$$\begin{aligned} dy_i &= dy_{i-1} + S(\mathcal{B}'_i) = S(\mathcal{B}^*_{i-1}) + S(\mathcal{B}'_i) \\ &= \sum_{b\in\mathcal{B}^*_{i-1}} b + \sum_{b\in\mathcal{B}'_i} b \in \mathcal{P}(\mathcal{B}) \end{aligned}$$

which completes the proof of the lemma.

Lemma 9. *Let $N \in \mathbb{N}$,*

$$N > 2500, \tag{3.5}$$

$A \subset \mathbb{N}_N$ and

$$|A| > 100(N\log N)^{1/2}. \tag{3.6}$$

Then there exist integers d, y, z such that

$$1 \le d < 10^4\frac{N}{|A|}, \tag{3.7}$$

$$z > \frac{|A|^2}{7\cdot 10^4}, \tag{3.8}$$

$$1 \le y < 7\cdot 10^4\frac{N}{|A|^2}z \tag{3.9}$$

and

$$\{yd, (y+1), ..., zd\} \subset P(A). \tag{3.10}$$

Proof: This is Theorem 4 in [12].

Lemma 10. *Let $N \in \mathbb{N}$, $N \le 2500$, $m \in \mathbb{N}$,*

$$7Ns(m) \le m \le 10^3\frac{N^2}{s(m)^s}, \tag{3.11}$$

$$\mathcal{A} \subset \mathbb{N}_N \tag{3.12}$$

and

$$|\mathcal{A}| \ge 10^4\frac{N}{s(m)}. \tag{3.13}$$

Then we have

$$m \in \mathcal{P}(\mathcal{A}). \tag{3.14}$$

Remarks: This lemma is non-trivial only when $s(m) > 10^4$, which implies that m must be a multiple of all integers up to 10^4, and N must be much greater than 2500.

It is easy to see that, from Lemma 1, (3.11) holds for every m and N such that $N > 2500$ and

$$63\,N\log N < m < 12\,N^2(\log N)^{-2}.$$

A slightly weaker version of this lemma follows from the results of Alon, Freiman, Lipkin [1], [2], [10].

Proof: It follows from (3.13) and Lemma 1 that

$$|\mathcal{A}| \ge 10^4\frac{N}{s(m)} \ge \frac{10^4 N}{4\cdot 5\log m}. \tag{3.15}$$

Now by (3.11), $\log m \le \log(10^3N^2) \le 3\log N$, and thus (3.15) yields

$$|\mathcal{A}| \ge \frac{10^4}{13.5}\frac{N}{\log N} \ge \frac{8600}{13.5}\sqrt{N\log N},$$

because $\frac{N}{\log N} \ge 0.86\sqrt{N\log N}$ for all $N > 1$. So (3.6) holds and thus we may apply Lemma 9. We obtain that there exist integers d, y, z, satisfying (3.7), (3.8), (3.9) and (3.10). It follows from (3.7) and (3.13) that

$$d < 10^4\frac{N}{|\mathcal{A}|} \le 10^4\frac{N}{10^4N/s(m)} = s(m)$$

so that

$$d|m. \tag{3.16}$$

Furthermore, by (3.10) we have $zd \in \mathcal{P}(\mathcal{A})$ whence

$$zd \le S(\mathcal{A}) = \sum_{a\in\mathcal{A}} a \le \sum_{a\in\mathcal{A}} N = N|\mathcal{A}|. \tag{3.17}$$

It follows from (3.9), (3.17), (3.13) and (3.11) that

$$yd < 7.10^4 \frac{N}{|\mathcal{A}|^2} zd \le 7.10^4 \frac{N^2}{|\mathcal{A}|} \le 7Ns(m) \le m. \tag{3.18}$$

Now, (3.8), (3.13) and (3.11) imply

$$zd \ge z > \frac{|\mathcal{A}|^2}{7.10^4} \ge 10^3 \frac{N^2}{s(m)^2} \ge m, \tag{3.19}$$

and (3.14) follows from (3.10), (3.16), (3.18) and (3.19). This completes the proof of the lemma.

Lemma 11. *Assume that* $N \in \mathbb{N}$,

$$N > 10^{10}, \tag{3.20}$$

δ *is a real number with*

$$0 < \delta \le \frac{1}{2}, \tag{3.21}$$

$\mathcal{A}$ *is a finite set of (not necessarily distinct) integers not exceeding* N,

$$|\mathcal{A}'| > 10^3(\delta^{-1}N)^{3/4} \tag{3.22}$$

and there is an integer m *with*

$$2\cdot 10^7\delta^{-2}N^2|\mathcal{A}'|^{-1} < m < (1-\delta)S(\mathcal{A}) \quad \text{and} \quad m \notin \mathcal{P}(\mathcal{A}). \tag{3.23}$$

Then there is an integer d *with*

$$d < 11\cdot 10^4\delta^{-1}N|\mathcal{A}'|^{-1} \tag{3.24}$$

and

$$L(\mathcal{A}, d) \le d/2. \tag{3.25}$$

Proof: In a first step we shall prove Lemma 11 when $\delta = 1/2$. In this step (3.21) should be read $\delta = 1/2$.

Let $D = 11\cdot 10^4\delta^{-1}N|\mathcal{A}'|^{-1}$. To every $d \le D$, $i \in \mathcal{L}(\mathcal{A}, d)$, we assign two numbers $a(d,i) \in \mathcal{A}$, $a'(d,i) \in \mathcal{A}$, $a(d,i) \equiv a'(d,i) \equiv i \pmod d$ so that either $a(d,i) \ne a'(d,i)$, or $a(d,i) = a'(d,i)$ and $a(d,i)$ occurs with multiplicity at least 2 in $\mathcal{A}$. Let

$$\mathcal{A}_0 = \cup_{d\le D} \cup_{i\in\mathcal{L}(\mathcal{A},d)} \{a(d,i), a'(d,i)\}.$$

Then clearly we have

$$|\mathcal{A}_0| \le 2\sum_{d\le D} d \le 2D^2 < 3\cdot 10^{10}\delta^{-2}N^2|\mathcal{A}'|^{-2} \tag{3.26}$$

(where in $|\mathcal{A}_0|$ we count the elements of $\mathcal{A}_0$ with multiplicity). Furthermore we have

$$S(\mathcal{A}) \ge S(\mathcal{A}') = \sum_{a\in\mathcal{A}} a \ge \sum_{a\le|\mathcal{A}'|} a > \frac{1}{2}|\mathcal{A}'|^2. \tag{3.27}$$

It follows from (3.22), (3.26) and (3.27) that

$$\begin{aligned} S(\mathcal{A}_0) &= \sum_{a\in\mathcal{A}_0} a \le \sum_{a\in\mathcal{A}_0} N \le |\mathcal{A}_0|N \\ &< 3\cdot 10^{10}\delta^{-2}N^3|\mathcal{A}'|^{-2} < 3\cdot 10^{10}\delta^{-2}N^3|\mathcal{A}'|^{-2}\cdot 2S(\mathcal{A})|\mathcal{A}'|^{-2} \\ &= 6\cdot 10^{-2}\delta(10^{12}(\delta^{-1}N)^3|\mathcal{A}'|^{-4})S(\mathcal{A}) < 10^{-1}\delta S(\mathcal{A}). \end{aligned} \tag{3.28}$$

Let us write $(\mathcal{A}\backslash\mathcal{A}_0)' = \{a_1, a_2, \dots, a_t\}$ where $a_1 < a_2 < \dots < a_t$. (Here and in what follows, $\mathcal{A}\backslash\mathcal{A}_0$ is defined so that the multiplicity of a in $\mathcal{A}\backslash\mathcal{A}_0$ is the difference of the multiplicities of a in $\mathcal{A}$ and $\mathcal{A}_0$, respectively.) By (3.20), (3.21), (3.22), (3.26) and (3.28) we have

$$\begin{aligned} t &= |(\mathcal{A}\backslash\mathcal{A}_0)'| \ge |\mathcal{A}'| - |\mathcal{A}_0'| \\ &> |\mathcal{A}'| - 3\cdot 10^{10}\delta^{-2}N^2|\mathcal{A}'|^{-2} = |\mathcal{A}'|(1 - 3\cdot 10^{10}\delta^{-2}N^2|\mathcal{A}'|^{-3}) \\ &> |\mathcal{A}'|(1 - 3\cdot 10^{10}\delta^{-2}N^2 10^{-9}(\delta^{-1}N)^{-9/4}) \\ &= |\mathcal{A}'|(1 - 3(\delta N^{-1})^{1/4}) > \frac{10}{11}|\mathcal{A}'| \end{aligned} \tag{3.29}$$

and

$$S(\mathcal{A}\backslash\mathcal{A}_0) = S(\mathcal{A}) - S(\mathcal{A}_0) > S(\mathcal{A}) - 10^{-1}\delta S(\mathcal{A}) = (1 - 10^{-1}\delta)S(\mathcal{A}). \tag{3.30}$$

Let $u = [\frac{\delta}{8}t]$. It follows easily from (3.20), (3.21), (3.22) and (3.29) that

$$\frac{\delta}{10}t < u \le \frac{\delta}{8}t < \frac{t}{2}. \tag{3.31}$$

Write $\mathcal{A}_1 = \{a_1, a_2, \dots, a_n\}$, $\mathcal{A}_2 = (\mathcal{A}\backslash\mathcal{A}_0)\backslash\mathcal{A}_1$ so that

$$\mathcal{A}\backslash\mathcal{A}_0 = \mathcal{A}_1 \cup \mathcal{A}_2 \tag{3.32}$$

(in the sense that the multiplicity of a in $\mathcal{A}\backslash\mathcal{A}_0$ is the sum of the multiplicities of a in $\mathcal{A}_1$ and $\mathcal{A}_2$). Clearly,

$$\begin{aligned} S(\mathcal{A}) &= \sum_{a\in\mathcal{A}} a \geq \sum_{a\in(\mathcal{A}\backslash\mathcal{A}_0)'} a = \sum_{i=1}^{t} a_i \\ &\geq \sum_{j=0}^{[t/u]-1}\left(\sum_{i=1}^{u} a_{ju+i}\right) \geq \sum_{j=0}^{[t/u]-1}\left(\sum_{i=1}^{u} a_i\right) = \left[\frac{t}{u}\right] S(\mathcal{A}_1). \end{aligned} \tag{3.33}$$

It follows from (3.29), (3.30), (3.31) and (3.33) that

$$|\mathcal{A}_1| = u > \frac{\delta}{10}t > \frac{\delta}{11}|\mathcal{A}'| \tag{3.34}$$

and

$$S(\mathcal{A}_1) \leq \frac{S(\mathcal{A})}{[t/u]} < \frac{2uS(\mathcal{A})}{t} < \frac{\delta}{4}S(\mathcal{A}). \tag{3.35}$$

By (3.21), (3.22) and (3.34) we have

$$|\mathcal{A}_1| > \frac{\delta}{11}|\mathcal{A}'| > \frac{\delta}{11}\cdot 10^3(\delta^{-1}N)^{3/4} > 90\delta^{3/4}N^{3/4} > 75N^{3/4}$$

so that (3.6) in Lemma 9 holds with $\mathcal{A}_1$ in place of $\mathcal{A}$. (Note that also (3.5) holds by (3.20).) Thus by Lemma 9, there exist integers d, y, z satisfying (3.7), (3.8), (3.9) and (3.10) (with $\mathcal{A}_1$ in place of $\mathcal{A}$) so that, in view of (3.34), we have

$$1 \leq d < 10^4\frac{N}{|\mathcal{A}_1|} < 11\cdot 10^4\delta^{-1}N|\mathcal{A}'|^{-1}, \tag{3.36}$$

$$z > \frac{|\mathcal{A}_1|^2}{7\cdot 10^4} > 10^{-7}\delta^2|\mathcal{A}'|^2, \tag{3.37}$$

$$y < 7\cdot 10^4\frac{N}{|\mathcal{A}_1|^2} < 10^7\delta^{-2}N|\mathcal{A}'|^{-2}z \tag{3.38}$$

and

$$\{yd, (y+1)d, \dots, zd\} \subset \mathcal{P}(\mathcal{A}_1). \tag{3.39}$$

This integer d satisfies (3.24) by (3.36). It remains to show that if there is an m satisfying (3.23), then this implies (3.25). To show this, we start out from the indirect assumption

$$L(\mathcal{A}, d) > d/2. \tag{3.40}$$

First we shall show that

$$\nu \in \mathbb{N}, y \leq \nu < (1-\delta)S(\mathcal{A})/d \quad \text{imply} \quad \nu d \in \mathcal{P}(\mathcal{A}_1 \cup \mathcal{A}_2). \tag{3.41}$$

It follows from (3.20), (3.21), (3.22), (3.37) and (3.38) that

$$\begin{aligned} z - y &> z(1 - 10^7\delta^{-2}N|\mathcal{A}'|^{-2}) \\ &> 10^{-7}(\delta|\mathcal{A}'|)^2(1 - 10^7\delta^{-2}N(10^3(\delta^{-1}N)^{3/4})^{-2}) \\ &> 10^{-7}(10^3\delta^{1/4}N^{3/4})^2(1 - 10(\delta N)^{-1/2}) \\ &> 7\cdot 10^{-2}N^{3/2}(1 - 10(2N)^{-1/2}) > 3\cdot 10^{-2}N^{3/2} > N \end{aligned}$$

so that (3.39) implies

$$\{yd, (y+1)d, \dots, (y+N-1)d\} \subset \mathcal{P}(\mathcal{A}_1). \tag{3.42}$$

Thus (3.41) holds for $y \leq \nu < y + N$. Assume now that

$$y + N \leq \nu < (1-\delta)S(\mathcal{A})/d. \tag{3.43}$$

Let us write $n = \nu - y - N$. Then by (3.43) we have

$$0 \leq n. \tag{3.44}$$

Furthermore, it follows from (3.30), (3.32) and (3.35) that

$$\begin{aligned} S(\mathcal{A}_2) &= S(\mathcal{A}\backslash\mathcal{A}_0) - S(\mathcal{A}_1) > (1 - 10^{-1}\delta)S(\mathcal{A}) - \delta S(\mathcal{A})/4 \\ &= \left(1 - \frac{7}{20}\delta\right) S(\mathcal{A}) > (1-\delta)S(\mathcal{A}). \end{aligned} \tag{3.45}$$

(3.43) and (3.45) imply

$$n = \nu - y - N < \nu - N < (1-\delta)S(\mathcal{A})/d - N < S(\mathcal{A}_2)/d - N. \tag{3.46}$$

By (3.44) and (3.46), Lemma 8 can be applied with $\mathcal{A}_2$ in place of $\mathcal{B}$. We obtain that there is a number $x_n \in \mathbb{N}$ such that

$$n + 1 \leq x_n \leq n + N \tag{3.47}$$

and

$$dx_n \in \mathcal{P}(\mathcal{A}_2). \tag{3.48}$$

(3.47) can be rewritten in the equivalent form

$$0 \leq n + N - x_n \leq N - 1. \tag{3.49}$$

Furthermore, we have

$$\nu - x_n = (n + y + N) - x_n = y + (n + N - x_n). \tag{3.50}$$

By (3.49) and (3.50), $(\nu - x_n)d$ belongs to the arithmetic progression $\{yd, (y+1)d, \ldots, (y+N-1)d\}$ and thus by (3.42) we have

$$(\nu - x_n)d \in \mathcal{P}(\mathcal{A}_1). \tag{3.51}$$

It follows from (3.48) and (3.51) that

$$\nu d = dx_n + (\nu - x_n)d \in \mathcal{P}(\mathcal{A}_1 \cup \mathcal{A}_2)$$

which proves (3.41).

Assume now that m satisfies (3.23). Let

$$\mathcal{L}_m(\mathcal{A}, d) = \{m - i : i \in \mathcal{L}(\mathcal{A}, d)\}.$$

The the elements of both $\mathcal{L}(\mathcal{A}, d)$ and $\mathcal{L}_m(\mathcal{A}, d)$ are pairwise incongruent modulo d and, in view of the indirect assumption (3.40), the total number of them is

$$|\mathcal{L}(\mathcal{A}, d)| + |\mathcal{L}_m(\mathcal{A}, d)| = 2L((\mathcal{A}, d) > d.$$

Thus by the box principle, there is a number i_1 in $\mathcal{L}(\mathcal{A}, d)$ which is congruent to a number $m - i_2$ in $\mathcal{L}_m(\mathcal{A}, d)$ modulo d:

$$i_1 \equiv m - i_2 \pmod d$$

whence

$$m - i_1 - i_2 \equiv 0 \pmod d. \tag{3.52}$$

Let us write $a_m = a(d, i_1)$ and

$$a'_m = \begin{cases} a(d, i_2) & \text{for } i_1 \neq i_2 \\ a'(d, i_1) = a'(d, i_2) & \text{for } i_1 = i_2. \end{cases}$$

Then if follows from (3.52) that

$$d | m - a_m - a'_m \tag{3.53}$$

and from the definition of $\mathcal{A}_0$ that $a_m \in \mathcal{A}_0$, $a'_m \in \mathcal{A}_0$, and thus

$$a_m + a'_m \in \mathcal{P}(\mathcal{A}_0). \tag{3.54}$$

Furthermore, in view of (3.21) and (3.23) we have

$$m - a_m - a'_m < m < (1 - \delta)S(\mathcal{A}) \tag{3.55}$$

and

$$\begin{aligned} m - a_m - a'_m &\geq 2 \cdot 10^7 \delta^{-2} N^2 |\mathcal{A}'|^{-1} - N - N \\ &= 2N(10^7 \delta^{-2} N |\mathcal{A}'|^{-1} - 1) \\ &\geq 2N(10^7 \delta^{-2} N |\mathcal{A}'|^{-1} - \delta^{-2}(N|\mathcal{A}'|^{-1})) \\ &> 2N \cdot 5 \cdot 10^6 \delta^{-2} N |\mathcal{A}'|^{-1} = 10^7 \delta^{-2} N^2 |\mathcal{A}'|^{-1}. \end{aligned} \tag{3.56}$$

It follows from (3.38), (3.39) and (3.56) that

$$\begin{aligned} yd &< 10^7 \delta^{-2} N |\mathcal{A}'|^{-2} zd = 10^7 \delta^{-2} N |\mathcal{A}'|^{-2} (zd) \\ &< 10^7 \delta^{-2} N |\mathcal{A}'|^{-2} (|\mathcal{A}_1| N) < 10^7 \delta^{-2} N |\mathcal{A}'|^{-2} |\mathcal{A}'| N \\ &= 10^7 \delta^{-2} N^2 |\mathcal{A}'|^{-1} < m - a_m - a'_m. \end{aligned} \tag{3.57}$$

By (3.53), $(m - a_m - a'_m)/d = \nu$ is an integer. It follows from (3.41), (3.55) and (3.57) that

$$\nu d = m - a_m - a'_m \in \mathcal{P}(\mathcal{A}_1 \cup \mathcal{A}_2).$$

Thus

$$m = \nu d + (a_m + a'_m)$$

where $\nu d \in \mathcal{P}(\mathcal{A}_1 \cup \mathcal{A}_2)$ and, in view of (3.54), $a_m + a'_m \in \mathcal{P}(\mathcal{A}_0)$. This implies

$$m \in \mathcal{P}(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_0) = P(\mathcal{A})$$

which contradicts (3.23) and this completes the proof of the lemma, when in (3.21) it is assumed $\delta = 1/2$.

In a second step, we have to prove Lemma 11, with $0 < \delta \leq 1/2$. Let us refer to the particular case of Lemma 11 with $\delta = \delta_0 \stackrel{\text{def}}{=} 1/2$ as Lemma 11_o. So, Lemma 11_o has been proved. We assert that Lemma 11 is an easy consequence of Lemma 11_o.

Since $\delta \leq \delta_0$, one has:

$$|\mathcal{A}'| > 10^3 (\delta_0^{-1} N)^{3/4}.$$

If $m \leq \frac{1}{2} S(\mathcal{A})$, one has:

$$2 \cdot 10^7 \delta_0^{-2} N^2 |\mathcal{A}'|^{-1} < m \leq (1 - \delta_0) S(\mathcal{A}).$$

If $m > \frac{1}{2}S(\mathcal{A})$ replace m by $m' = S(\mathcal{A}) - m$ (use the symmetry of $\mathcal{P}(\mathcal{A})$). Then $m' \le (1-\delta_0)S(\mathcal{A})$. Since $m \le (1-\delta)S(\mathcal{A})$, one has (see proof of (3.27))

$$m' \ge \delta S(\mathcal{A}) > \frac{1}{2}\delta|\mathcal{A}'|^2 > \frac{1}{2}\delta 10^6\delta^{-3/2}N^{3/2} = \frac{10^6}{2}\delta^{-1/2}N^{3/2}.$$

Now, by (3.20), (3.21), and (3.22),

$$\frac{(1/2)10^6\delta^{-1/2}N^{3/2}}{2\cdot 10^7\delta_0^{-2}N^2|\mathcal{A}'|^{-1}} = \frac{1}{10}\delta^{-1/2}N^{-1/2}|\mathcal{A}'| > 10^2\delta^{-5/4}N^{1/4}$$
$$\ge 10^2\delta_0^{-5/4}N^{1/4} \ge 75000 > 1$$

whence Lemma 11_o can be applied with either m or m'. We get d with

$$d < 11\cdot 10^4\delta_0^{-1/2}N|\mathcal{A}'|^{-1} < 11\cdot 10^4\delta^{-1}N|\mathcal{A}'|^{-1}$$

and $L(\mathcal{A}, d) \le d/2$, and Lemma 11 is completely proved.

4. Proofs of Theorems 1, 2, and 3

To every (unequal or unrestricted) partition

$$n = n_1 + n_2 + \cdots + n_t$$

of n which does not represent a, we assign the set $\mathcal{N} = \{n_1, n_2, \ldots, n_t\}$ (so that in case of unrestricted partitions the parts are taken with multiplicity). For an M which will be defined later, let $\mathcal{N}_0$ denote the set of the parts not exceeding M, and let $\mathcal{N}_0'$ denote the set of the distinct parts not exceeding M (so that, in case of unequal partitions, we have $\mathcal{N}_0 = \mathcal{N}_0' = \mathcal{N} \cap \{1, 2, \ldots, M\}$).

To prove Theorem 1, we shall choose

$$M = [10^{-3}(an)^{1/3}]. \tag{4.1}$$

We have to distinguish two cases.

CASE 1: Assume that

$$|\mathcal{N}_0'| > 8\cdot 10^7 M^2 a^{-1} \stackrel{\text{def}}{=} B. \tag{4.2}$$

We are going to show that, if n is large enough, then in this case, Lemma 11 can be applied with M, N_0, N_0', $1/2$ and a in place of N, $\mathcal{A}$, $\mathcal{A}'$, δ, and m, respectively.

In fact, (3.20) follows from (1.1) and (4.1) for n large enough, and (3.21) holds trivially. Furthermore, it follows from (1.1), (4.1) and (4.2) that for large n we have

$$|\mathcal{N}_0'| > 8\cdot 10^7 M^2 a^{-1} = (4\cdot 10^4 M^{5/4}a^{-1})(2\cdot 10^3 M^{3/4})$$
$$> 3\cdot 10^4(10^{-3}(an)^{1/3})^{5/4}a^{-1}10^3(\delta^{-1}M)^{3/4}$$
$$> 3a^{-7/12}n^{5/12}10^3(\delta^{-1}M)^{3/4}$$
$$\ge 3\cdot 10^3(\delta^{-1}M)^{3/4}$$

and thus (3.22) is verfied. The left-hand side inequality of (3.23) follows immediately from (4.2), and by (1.1), (4.1) and (4.2), we have for large n

$$(1-\delta)S(\mathcal{N}_0) = \frac{1}{2}S(\mathcal{N}_0) \ge \frac{1}{2}S(\mathcal{N}_0') \ge \frac{1}{2}\sum_{i=1}^{|\mathcal{N}_0'|} i > \frac{|\mathcal{N}_0'|^2}{4}$$
$$> 16\cdot 10^{14}M^4a^{-2} > 10\cdot 10^{14}(10^{-3}(an)^{1/3})^4a^{-2}$$
$$= 10^3a^{-5/3}n^{4/3}a$$
$$> 10^3(n^{5/7})^{-5/3}n^{4/3}a = 10^3n^{1/7}a > a.$$

Thus all the assumptions in Lemma 11 hold so that the lemma can be applied. We deduce that there is an integer d with

$$d < 11\cdot 10^4\cdot 2\cdot M\cdot|\mathcal{N}_0'|^{-1} \tag{4.3}$$

and

$$L(\mathcal{N}_0, d) \le d/2.$$

It follows from (4.1), (4.2) and (4.3) that, if we set

$$D = 4a^{2/3}n^{-1/3}, \tag{4.4}$$

then

$$d < 22\cdot 10^4M(8\cdot 10^7M^2a^{-1})^{-1} < 3\cdot 10^{-3}M^{-1}a < D. \tag{4.5}$$

Now let $\{i_1, \ldots, i_{[d/2]}\}$ be any set containing $\mathcal{L}(\mathcal{N}_0, d)$ and such that $1 \le i_1 < \cdots < i_{[d/2]} \le d$. As in Lemma 5 or 6, we can define $\mathcal{N}_1$ and $\mathcal{N}_2$, and it follows from the definition of $\mathcal{L}(\mathcal{N}_0, d)$ that

$$|\mathcal{N}_2| \le (d-1) + d - [d/2] \le 2d \le 2D.$$

Therefore the number of partitions of n which do not represent a and satisfy (4.2) is smaller than $V(n, M, D)$ or $W(n, M, D)$.

CASE 2: Assume that

$$|\mathcal{N}_0'| \le 8\cdot 10^7 M^2 a^{-1} \stackrel{\text{def}}{=} B. \tag{4.6}$$

With the notation of Lemma 3 or 4, the total number of partitions of n is certainly smaller than $Y(n,B,M)$ or $Z(n,B,M)$.

So, we have proved that

$$Q(n,a) \le V(n,M,D) + Y(n,B,M) \tag{4.7}$$

and

$$R(n,a) \le W(n,M,D) + Z(n,B,M). \tag{4.8}$$

By Lemma 3 and Lemma 5, (4.7) yields

$$Q(n,a) \le p(n, B+n/M) + n^{5D} q([n/2]) p(n,n/M). \tag{4.9}$$

Now, by (4.1) we have

$$n/M \le (3/2)10^3 a^{-1/3} n^{1/6}\sqrt{n},$$
$$20a^{-1/3}n^{2/3} \le B \le 80a^{-1/3}n^{2/3}, \tag{4.10}$$

and if we set $\alpha = 2\cdot 10^3 a^{-1/3}n^{1/6}$, we have

$$n/M + B \le \alpha\sqrt{n}.$$

By (1.1), we have $\alpha \le 1$, so that we may apply Lemma 2. We obtain

$$\begin{aligned} Q(n,a) &\le 2n^{5D} q([n/2]) p(n, \alpha\sqrt{n}) \\ &\le q([n/2]) exp\left(5D\log n + \log 2 + \left(2\alpha \log\frac{3.6}{\alpha}\right)\sqrt{n}\right). \end{aligned}$$

But, from (4.4) and (1.1) we have $D = O(n^{1/7})$, and

$$\alpha \ge 2\cdot 10^3 n^{-1/14},$$

and thus, for n large enough, (1.2) is proved.

It remains to deduce (1.3) from (4.8). We are going to apply Lemma 4. First observe that by (4.1) and (1.1) we have

$$B = 8\cdot 10^7 M a^{-1} M \le 8\cdot 10^4 a^{-2/3} n^{1/3} M \le 8\cdot 10^{-2} M \le M/2,$$

and thus

$$Z(n,B,M) \le 6B\binom{M}{B} n^2 p(n,B) p(n,n/M). \tag{4.11}$$

By (1.1) and (4.10), one has

$$B \le 8\cdot 10^{-5} n^{1/2}, \tag{4.12}$$

so that Lemma 2 gives

$$p(n,B) \le \exp(2\cdot 10^{-3}\sqrt{n}). \tag{4.13}$$

Now, using Stirling's formula, (4.1) and (4.10), we have

$$\begin{aligned} \binom{M}{B} &\le \frac{M^B}{B!} \le \left(\frac{Me}{B}\right)^B \le \left(\frac{10^{-5}e(an)^{1/3}}{20a^{-1/3}n^{2/3}}\right)^B \le (a^{2/3}n^{-1/3})^B \\ &\le \exp(80n^{2/3}a^{-1/3}\log(a^2/3n^{-1/2})). \end{aligned}$$

But, the above quantity is a decreasing function of a for $a \ge e^3\sqrt{n}$, so that by (1.1),

$$\binom{M}{B} \le \exp(8\cdot 10^{-5}\sqrt{n}\log(10^{12})) \le \exp(3\cdot 10^{-3}\sqrt{n}). \tag{4.14}$$

Therefore, for n large enough, (1.14), (4.11), (4.12), (4.13) and (4.14) give

$$Z(n,B,M) \le p([n/2]) p(n,n/M),$$

and by Lemma 6, (4.8) gives

$$R(n,a) \le 2\,n^{7D} p([n/2]) p(n,n/M).$$

The end of the proof of (1.3) is similar to the end of the proof of (1.2).

To prove Theorem 2, we shall choose

$$M = [n^{15/28}] \tag{4.15}$$

and

$$\delta = 10^4 \cdot n^{-1/28}. \tag{4.16}$$

Again we start to prove (1.5) and (1.6) simultaneously. Define $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{N}_0'$ in the same way as in the proof of Theorem 1, and also write $\mathcal{N}^* = \mathcal{N} - \mathcal{N}_0$ (so that $\mathcal{N}^*$ is the set of the parts greater than M). We have to distinguish three cases.

CASE 1: Assume that

$$(1-\delta)S(\mathcal{N}_0) \le n/2 \tag{4.17}$$

whence, for large n,

$$S(\mathcal{N}_0) \le \frac{n}{2}\frac{1}{1-\delta} \le (\frac{1}{2}+\delta)n.$$

If we fix $S(\mathcal{N}_0) = k$, then in the case of unequal partitions, $\mathcal{N}_0$ can be chosen in at most $q(k)$ ways, while $\mathcal{N}^* = \mathcal{N} - \mathcal{N}_0$ can be chosen in at most $\rho(n-k, M+1)$ ways (since $S(\mathcal{N}^*) = S(\mathcal{N}) - S(\mathcal{N}_0) = n-k$, and the elements of $\mathcal{N}^*$ are greater than M). Therefore the total number of unequal partitions with property (4.17) is at most

$$T \stackrel{\text{def}}{=} \sum_{k<(\frac{1}{2}+\delta)n} q(k)\rho(n-k, M+1). \tag{4.18}$$

Similarly the total number of unrestricted partitions with property (4.17) is at most

$$U \stackrel{\text{def}}{=} \sum_{k<(\frac{1}{2}+\delta)n} p(k)r(n-k, M+1). \tag{4.19}$$

We have

$$\rho(n, M) \le r(n, M) \le p(n, n/M), \tag{4.20}$$

and by Lemma 2,

$$p(n, n/M) < \exp((3n^{-1/28}\log 3.6n^{1/28})\sqrt{n}) \tag{4.21}$$

Now, from (1.14) and for n large enough, we have:

$$\begin{aligned} q([(\frac{1}{2}+\delta)n]) &< \exp\left(\frac{\pi}{\sqrt{3}}\sqrt{\left(\frac{1}{2}+\delta\right)n}\right) < \exp\left(\frac{\pi}{\sqrt{3}}\sqrt{\frac{n}{2}}(1+\delta)\right) \\ &< nq([n/2])\exp\left(\frac{\pi}{\sqrt{6}}\delta\sqrt{n}\right). \end{aligned} \tag{4.22}$$

From (4.18), (4.20), (4.21) and (4.22), we have

$$T < nq([(\frac{1}{2}+\delta)n])\rho(n, M) < q([n/2])\exp(n^{1/2-1/29}) \tag{4.23}$$

for n large enough.

Similarly, from (4.19) we obtain

$$U < p([n/2])\exp(n^{1/2-1/23}). \tag{4.24}$$

CASE 2: Assume that

$$\frac{n}{2} < (1-\delta)S(\mathcal{N}_0) \tag{4.25}$$

and

$$|\mathcal{N}_0'| > B \stackrel{\text{def}}{=} 10^3(\delta^{-1}M)^{3/4} > \frac{1}{2}n^{3/7}. \tag{4.26}$$

We are going to show that, if n is large enough, then Lemma 11 can be applied with M, $\mathcal{N}_0$, $\mathcal{N}_0'$ and a in place of N, $\mathcal{A}$, $\mathcal{A}'$, and m, respectively, and with $\delta = 10^4 n^{-1/28}$.

In fact, (3.20) and (3.21) hold trivially, while (3.22) holds by (4.26). Furthermore by (1.4), (4.15), (4.16) and (4.25), we have for large n

$$2\cdot 10^7\delta^{-2}M^2|\mathcal{N}_0'|^{-1} < 2\cdot 10^7\cdot 10^{-8}n^{2/28+30/28-3/7} < n^{5/7} < a. \tag{4.27}$$

The assumption (3.23) follows from (4.25) and (4.27), and thus Lemma 11 can be applied. We obtain that there is an integer d with

$$d < 11\cdot 10^4\delta^{-1}M|\mathcal{N}_0'|^{-1} \tag{4.28}$$

and

$$L(\mathcal{N}_0, d) \le d/2.$$

It follows from (4.15), (4.16), (4.26) and (4.28) that

$$d < 11\cdot 10^4\delta^{-1}M\cdot 2n^{-3/7} < 22n^{1/7} \stackrel{\text{def}}{=} D. \tag{4.29}$$

Therefore, as in the proof of Theorem 1, the number of partitions of n which do not represent a and satisfy (4.25) and (4.26), is smaller than $V(n, M, D)$ or $W(n, M, D)$.

CASE 3: Assume that

$$|\mathcal{N}_0'| \le B = n^{3/7}. \tag{4.30}$$

As in Case 2 in the proof of Theorem 1, the number of partitions of n is bounded above by $Y(n, B, M)$ or $Z(n, B, M)$.

So we have proved that, under the assumptions of Theorem 2, we have

$$Q(n, a) \le T + V(n, M, D) + Y(n, B, M) \tag{4.31}$$

and

$$R(n, a) \le U + W(n, M, D) + Z(n, B, M). \tag{4.32}$$

We have

$$B + n/M < B + 2n^{13/28} < 3n^{13/28},$$

and by Lemmas 2 and 3,

$$Y(n,B,M) \le p(n,B+n/M) < \exp(n^{1/2-1/29}) \tag{4.33}$$

for n large enough.

Moreover, by Lemma 5, (4.21) and (4.29),

$$V(n,M,D) < q([n/2])\exp(n^{1/2-1/29}), \tag{4.34}$$

and by (4.23), (4.31), (4.33) and (4.34), (1.5) holds.

Similarly, from Lemma 4, using

$$\binom{M}{B} \le n^B$$

and since $p(n,B) \le p(n,n/M)$ by $B < n/M$, we have

$$Z(n,B,M) < \exp(n^{1/2-1/29}). \tag{4.35}$$

From Lemma 6, we have

$$W(n,M,D) < p([n/2])\exp(n^{1/2-1/29}), \tag{4.36}$$

(4.24), (4.32), (4.35) and (4.36) yield (1.6).

To prove Theorem 3, we choose

$$M = [10^{-2}\sqrt{n\,s(a)}]. \tag{4.37}$$

To a partition of n which does not represent a, we associate $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{N}_0'$ in the same way as in the proofs of Theorems 1 and 2. We apply Lemma 10 with a, M, $\mathcal{N}_0'$ in place of m, N, $\mathcal{A}$, respectively. By $n \ge (2500)^2$ and $s(a) \ge 40000$, (4.37) yields $M \ge 2500$. It is easily seen that (1.7) implies (3.11), and we conclude that

$$|\mathcal{N}_0'| < 10^4 M/s(a) \le 10^2\sqrt{n/s(a)} \stackrel{\text{def}}{=} t. \tag{4.38}$$

So, with the notation of Lemmas 3 and 4, we have

$$Q(n,a) \le Y(n,t,M) \tag{4.39}$$

and

$$R(n,a) \le Z(n,t,M) \tag{4.40}$$

As $M \ge 2500$, from (4.37) we deduce that $n/M < 101\sqrt{n/s(a)}$. By Lemma 3, (4.39) gives

$$Q(n,a) \le p(n,t+n/M) \le p(n,201\sqrt{n/s(a)})),$$

and Lemma 2 yields (1.8).

Now, by Lemma 4 and (4.40), we have

$$R(n,a) \le 6tn^2\binom{M}{t}p(n,t)p(n,n/M) \tag{4.41}$$

since

$$t/M < 2\cdot 10^4/s(a) \le 1/2. \tag{4.42}$$

From (4.38) and Lemma 1, we have

$$\begin{aligned} n &= 10^{-4}t^2 s(a) < (4.5)\cdot 10^{-4}t^2\log a < (4.5)10^{-4}t^2\log n \\ &< t^2\log(n^{1/3}) < t^2 n^{1/3}, \end{aligned}$$

whence $n < t^3$ and

$$6tn^2 < 6t^7 = \exp(\log 6 + 7\log t) < \exp(\log 6 + 7(t-1)) < e^{7t}.$$

By Stirling's formula, (4.37) and (4.38), we have

$$6tn^2\binom{M}{t} < e^{7t}\left(\frac{Me}{t}\right)^t = (e^8 M t^{-1})^t \le (e^8 10^{-4}s(a))^t < s(a)^t$$

and (4.41) and Lemma 2 give (1.9).

TABLE OF Q(n,a)

n	$q(n)$	$a=1$	2	3	4	5	6	7	8	9	10
1	1	0									
2	1	1									
3	2	1									
4	2	1	2								
5	3	2	2								
6	4	2	2	3							
7	5	3	3	3							
8	6	3	4	3	5						
9	8	5	5	4	5						
10	10	5	6	5	6	7					
11	12	7	7	7	7	7					
12	15	8	9	8	8	8	11				
13	18	10	11	10	10	10	10				
14	22	12	13	11	13	11	12	15			
15	27	15	16	14	15	13	15	15			
16	32	17	19	16	19	16	17	16	23		
17	38	21	22	20	21	20	20	20	20		
18	46	25	27	23	26	23	23	23	25	30	
19	54	29	32	28	29	28	28	27	28	28	
20	64	35	37	32	35	32	34	31	34	31	43
21	76	41	44	38	41	38	38	35	38	37	38

n	$q(n)$	$a=1$ 11	2 12	3 13	4 14	5 15	6 16	7 17	8 18	9 19	10 20
22	89	48 57	52	44	48	43	46	42	45	42	45
23	104	56 51	60	52	56	50	52	51	50	49	50
24	122	66 57	70 79	60	66	58	62	57	57	55	59
25	142	76 67	82 67	70	75	68	70	67	69	65	67
26	165	89 74	95 78	81 102	88	77	81	76	81	73	77
27	192	103 88	110 88	94 90	101	91	93	89	91	81	88
28	222	119 96	127 99	108 97	116 138	104	107	101	106	97	99
29	256	137 110	146 113	124 114	134 114	119	123	116	119	114	114
30	296	159 126	169 133	143 127	154 133	137 174	140	131	139	127	126
31	340	181 144	194 145	164 145	176 147	157 149	161	150	156	147	150
32	390	209 160	221 166	188 161	202 166	177 162	184 232	170	180	164	173
33	448	239 177	254 187	214 185	231 188	204 188	209 191	196	201	189	194
34	512	273 210	291 212	245 203	262 213	232 205	239 215	219 192	229	211	221
35	585	312 241	331 239	279 233	300 240	262 232	271 239	251 242	258	241	247
36	668	356 267	377 260	318 260	340 267	299 259	307 271	284 265	294 375	272	281
37	760	404 305	429 308	360 293	386 299	340 293	348 298	321 302	333 303	306	314
38	864	460 337	487 350	409 327	438 334	383 322	394 336	363 329	373 341	341 471	357
39	982	522 385	553 387	463 356	496 376	434 364	445 375	412 375	424 376	387 386	399
40	1113	591 427	626 439	525 417	560 420	491 402	501 423	460 410	476 420	433 415	451 602

REFERENCES

[1] N. Alon, Subset sums, J. Number Theory **27** (1987), 196-205.
[2] N. Alon and G. Freiman, On sums of subsets of a set of integers, Combinatorica (to appear).
[3] J. Dixmier and J. L. Nicolas, Partitions without small parts, Number Theory, Coll. Math. Soc. J. Bolyai (to appear).
[4] J. Dixmier and J. L. Nicolas, Partitions sans petits sommants II. To be published.
[5] J. Dixmier, Sur les sous sommes d'une partition, Bull. Soc. Math. Fance. Mémoire no. 35, 1988.
[6] J. Dixmier, Sur l'évaluation de $R(n,a)$. Submitted to Bull. Soc. Math. Belgique.
[7] P. Erdös, J. L. Nicolas and A. Sárközy, On the number of partitions of n without a given subsum (I), Discrete Math **74** (1989), 155-166.
[8] P. Erdös, J. L. Nicolas and M. Szalay, Partitions into parts which are unequal and large. Proceedings of "Journeés Arithmétiques" of Ulm, Springer Verlag Lecture Notes, to appear.
[9] G. H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, Proc. London Math. Soc. **(2) 17** (1918), 75-115. (Also in Collected Papers of S. Ramanujan pp. 276-309. Cambridge University Press 1927, reprinted by Chelsea, New-York 1962).
[10] E. Lipkin, On representation of r- powers by subset sums, Acta Arithmetica (to appear).
[11] A. Sárközy, Finite addition theorems I, J. Number Theory **32** (1989), 114-130.
[12] ———, Finite addition theorems II, J. Number Theory (to appear).

J. L. Nicolas
Département de Mathématiques
Université Claude Bernard (Lyon 1)
F-69622 Villeurbanne Cedex
France

P. Erdös and A. Sárközy
Mathematical Institute of
the Hungarian Academy of Sciences
Reáltanoda u. 13-15, Pf. 127
H-1364 Budapest
Hungary