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Ramanujan, Robin, highly composite numbers, and the Riemann Hypothesis

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Abstract. We provide an historical account of equivalent conditions for the Riemann Hypothesis arising from the work of Ramanujan and, later, Guy Robin on generalized highly composite numbers. The first part of the paper is on the mathematical background of our subject. The second part is on its history, which includes several surprises.

1. Mathematical Background

Definition. The *sum-of-divisors function* σ is defined by

$$\sigma(n) := \sum_{d|n} d = n \sum_{d|n} \frac{1}{d}.$$

In 1913, Grönwall found the maximal order of σ.

Grönwall's Theorem [8]. *The function*

$$G(n) := \frac{\sigma(n)}{n \log\log n} \qquad (n > 1)$$

satisfies

$$\limsup_{n\to\infty} G(n) = e^{\gamma} = 1.78107\ldots ,$$

where

$$\gamma := \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) = 0.57721\ldots$$

is the Euler-Mascheroni constant.

Grönwall's proof uses:

Mertens's Theorem [10]. *If p denotes a prime, then*

$$\lim_{x\to\infty} \frac{1}{\log x} \prod_{p\le x} \left(1 - \frac{1}{p}\right)^{-1} = e^{\gamma}.$$

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FIGURE 1. Thomas Hakon GRÖNWALL (1877–1932)

FIGURE 2. Franz MERTENS (1840–1927)

Now we come to:

Ramanujan's Theorem [2, 15, 16]. *If RH is true, then for n_0 large enough,* $n > n_0 \implies G(n) < e^\gamma$.

To prove his theorem, Ramanujan introduces a real non-negative parameter s, considers the multiplicative function $n \mapsto \sigma_{-s}(n) = \sum_{d|n} d^{-s}$, and calls an integer N a *generalized highly composite number* if

$$N' < N \implies \sigma_{-s}(N') < \sigma_{-s}(N).$$

When $s = 1$ these numbers have been called *superabundant* by Erdős and Alaoglu [1], while for $s \neq 1$ they have only been studied by Ramanujan. Further, Ramanujan

FIGURE 3. Srinivasa RAMANUJAN (1887–1920)

calls an integer N a *generalized superior highly composite number* of parameter $\varepsilon > 0$ if

$$N' < N \implies \frac{\sigma_{-s}(N)}{N^{\varepsilon}} \geq \frac{\sigma_{-s}(N')}{(N')^{\varepsilon}}$$

and

$$N' > N \implies \frac{\sigma_{-s}(N)}{N^{\varepsilon}} > \frac{\sigma_{-s}(N')}{(N')^{\varepsilon}}.$$

When $s = 1$ these numbers have been called *colossally abundant* by Erdős and Alaoglu. It is easily seen that all generalized superior highly composite numbers are generalized highly composite.

The prime factorization of a generalized superior highly composite number N can be obtained from the value of the parameter ε. For $r = 1, 2, 3 \ldots$, Ramanujan defines x_r by

$$x_r^{\varepsilon} = \frac{1 - x_r^{-s(r+1)}}{1 - x_r^{-sr}}$$

and then

$$N = \prod_{r=1}^{R} e^{\vartheta(x_r)}$$

where $\vartheta(x) = \sum_{p \leq x} \log p$ denotes Chebyshev's function and R is the largest integer such that $x_R \geq 2$. One has

$$\sigma_{-s}(N) = \prod_{r=1}^{R} \prod_{p \leq x_r} \frac{1 - p^{-s(r+1)}}{1 - p^{-sr}},$$

and, to estimate $\sigma_{-s}(N)$, one needs an estimate of

$$\sum_{p \leq x} \log\left(1 - \frac{1}{p^s}\right) = -\sum_{p \leq x} \int_s^{\infty} \frac{\log p}{p^t - 1} \mathrm{d}t$$

whence the idea of considering the sum $\sum_{p \leq x} \frac{\log p}{p^s - 1}$.

Here is an excerpt from Ramanujan's proof: Ramanujan [**16**, p. 133]: ... *assume that* ... $s > 0$ ... *if* p *is the largest prime not greater than* x*, then*

$$\frac{\log 2}{2^s - 1} + \frac{\log 3}{3^s - 1} + \frac{\log 5}{5^s - 1} + \cdots + \frac{\log p}{p^s - 1}$$
$$= C + \int^{\theta(x)} \frac{dx}{x^s - 1} - s\int \frac{x - \vartheta(x)}{x^{1-s}(x^s - 1)^2}\, dx + O\{x^{-s}(\log x)^4\}.$$

But it is known that

$$x - \theta(x) = \sqrt{x} + x^{\frac{1}{3}} + \sum \frac{x^\rho}{\rho} - \sum \frac{x^{\frac{1}{2}\rho}}{\rho} + O(x^{\frac{1}{5}})$$

where ρ is a complex root of $\zeta(s)$. ...

The last equation is a variant of the classical explicit formula in prime number theory. This shows "explicitly" how Ramanujan used RH in his proof.

From the estimate for $\sigma_{-s}(N)$, Ramanujan deduces that, for $1/2 < s < 1$ and all sufficiently large integers n, the upper bound

$$\begin{aligned}\sigma_{-s}(n) \quad \leq \quad & |\zeta(s)| \exp\left\{ \mathrm{Li}((\log n)^{1-s}) - \frac{2s(2^{\frac{1}{2s}} - 1)}{2s-1} \frac{(\log n)^{\frac{1}{2}-s}}{\log\log n} \right. \\ & \left. - \frac{s}{\log\log n} \sum_\rho \frac{(\log n)^{\rho - s}}{\rho(\rho - s)} + O\left\{ \frac{(\log n)^{\frac{1}{2}-s}}{(\log\log n)^2} \right\} \right\}\end{aligned}$$

holds. Finally, making s tend to 1, he gets

$$\limsup_{n\to\infty} (\sigma_{-1}(n) - e^\gamma \log\log n)\sqrt{\log n} \leq -e^\gamma(2\sqrt{2} - 4 - \gamma + \log 4\pi) = -1.393\ldots < 0.$$

Since $G(n) = \sigma_{-1}(n)/\log\log n$, this proves Ramanujan's Theorem.

Next we have:

Robin's Theorem [18, 19]. *RH is true if and only if* $n > 5040$ $(= 7!)$ $\implies$ $G(n) < e^\gamma$.

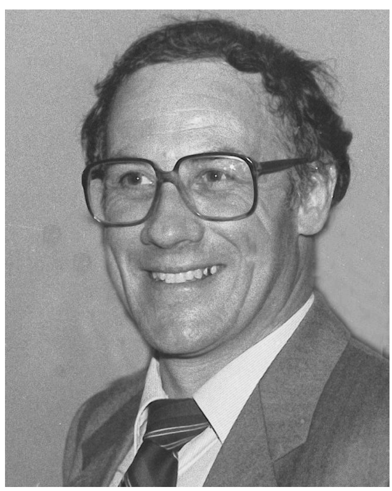


FIGURE 4. Guy ROBIN (photo courtesy of Guy Robin)

J. Math. pures et appl.,
63, 1984, p. 187 à 213.

GRANDES VALEURS DE LA FONCTION SOMME DES DIVISEURS ET HYPOTHÈSE DE RIEMANN

G. ROBIN

RÉSUMÉ. — Soit σ la fonction $\sigma(n)=\sum_{d\mid n} d$. Le but de cet article est de prouver que si l'hypothèse de Riemann est vraie, on a $\sigma(n)<e^\gamma n \log\log n$ pour $n\geqq 5041$. (γ est la constante d'Euler). Si l'hypothèse de Riemann est fausse, on montre aussi qu'il existe une infinité de n tels que $\sigma(n)>e^\gamma n \log\log n$ et $\forall n\geqq 3$, $\sigma(n)<e^\gamma n\log\log n+0{,}6483\, n/\log\log n$.

ABSTRACT. — Let be the function $\sigma(n)=\sum_{d\mid n} d$. The aim of this paper is to prove that under Riemann's hypothesis, $\sigma(n)<e^\gamma n\log\log n$ for $n\geqq 5041$. (γ is Euler's constant). If Riemann's hypothesis is false it is shown that for infinitely many n, $\sigma(n)>e^\gamma n\log\log n$ and $\forall n\geqq 3$, $\sigma(n)<e^\gamma n\log\log n+0{,}6483\, n/\log\log n$.

FIGURE 5. Robin's paper on σ and RH, Journal de Mathématiques Pures et Appliquées, 1984

To prove his theorem, Robin uses generalized superior highly composite numbers only with $s = 1$, i.e., colossally abundant numbers (CA for short). First, he shows that if $N' < N''$ are two consecutive CA numbers, then

$$N' < n < N'' \implies G(n) \le \max(G(N'), G(N'')).$$

Second, by numerical computation, he checks that $G(N) < e^\gamma$ for all integers N with $5041 < N < 55440$, as well as for all CA numbers $N \ge 55440$ whose largest prime factor $P^+(N)$ is < 20000.

Further, if a CA number N satisfies $P^+(N) > 20000$, then getting an upper bound for $\sigma_{-1}(N)$ requires a precise estimate of the Mertens product

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}.$$

The sum-of-divisors function σ and Euler's totient function ϕ, defined as

$$\phi(n) := \sum_{\substack{1\le k\le n\\(k,n)=1}} 1 = n\prod_{p\mid n}\left(1-\frac{1}{p}\right),$$

are related by the inequalities

$$\frac{6}{\pi^2} < \frac{\sigma(n)}{n}\cdot\frac{\phi(n)}{n} < 1,$$

which hold for all $n > 1$. Mertens's Theorem implies that the minimal order of ϕ is given by

$$\limsup_{n\to\infty} \frac{n/\log\log n}{\phi(n)} = e^{\gamma}.$$

To estimate the Mertens product, Robin used ideas from a result on the ϕ function proved by his thesis advisor Nicolas in 1983.

Nicolas's Theorem [11, 12]. *RH is true if and only if*

$$\text{prime } p > 2 \implies \frac{p\#/\log\log p\#}{\phi(p\#)} > e^{\gamma},$$

where $p\# := 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdots p$ denotes a primorial.

FIGURE 6. Jean-Louis NICOLAS (photo courtesy of Jean-Louis Nicolas)

Nicolas in turn used Landau's Oscillation Theorem [9], which Landau had applied in 1905 to prove Chebyshev's bias in the form $\pi(x;4,3) - \pi(x;4,1) = \Omega_+(\sqrt{x}/\log x)$.

FIGURE 7. Edmund Georg Hermann LANDAU (1877–1938)

Recently, Caveney and the authors gave two reformulations of Robin's Theorem.

Caveney, Nicolas, and Sondow's Theorems [5, 6].
Define an integer $N > 1$ to be a GA1 number *if N is composite and $G(N) \geq G(N/p)$ for all prime factors p. Call an integer N a* GA2 number *if $G(N) \geq G(aN)$ for all multiples aN. Then:*
1. *RH is true if and only if* 4 *is the only number that is both GA1 and GA2.*
2. *A GA2 number $N > 5040$ exists if and only if RH is false, in which case N is even and $> 10^{8576}$.*

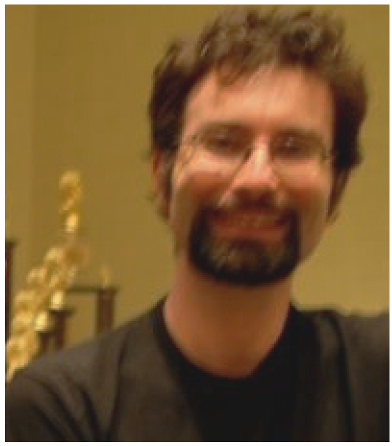


FIGURE 8. Geoffrey CAVENEY (photo courtesy of Geoffrey Caveney)

2. Historical Survey

Our story begins in 1915, when Ramanujan published the first part of his dissertation "Highly Composite Numbers" (HCN for short).

Papers Published in the Proceedings of the London Mathematical Society
S. Ramanujan
Highly Composite Numbers
Proc. London Math. Soc. (1915) s2_14(1): 347-409 doi:10.1112/plms/s2_14.1.347

FIGURE 9. Ramanujan's HCN Part 1, Proceedings of the London Mathematical Society, 1915

Ramanujan (in [**14**]): *I define a highly composite number as a number whose number of divisors exceeds that of all its predecessors.*

His thesis was written at Trinity College, University of Cambridge. According to The Mathematics Genealogy Project [**13**], his advisors were Hardy and Littlewood. However, Littlewood served in World War I and was not in Cambridge for almost all of the time that Ramanujan was there.

In 1944, Erdős published a paper "On highly composite and similar numbers" with Alaoglu [**1**].

Erdős (in "Ramanujan and I" [**7**]): *Ramanujan had a very long manuscript on highly composite numbers but some of it was not published due to a paper shortage during the First World War.*

Dyson (email to Sondow, 2012): *Hardy told me, "Even Ramanujan could not make highly composite numbers interesting." He said it to discourage me from working on H. C. numbers myself. I think he was right.*

FIGURE 10. Freeman John DYSON (photo courtesy of Freeman Dyson)

In 1982 Rankin published a paper on "Ramanujan's manuscripts and notebooks." He quoted Hardy's mention of "the suppressed part of HCN" in a 1930 letter to Watson.

Rankin (in [**17**]): *The most substantial manuscript consists of approximately 30 pages of HCN carrying on from where the published paper stops.*

By a curious coincidence, 1981–1982 is also the year of Séminaire Delange-Pisot-Poitou's exposition [**18**] of Robin's Theorem, in which he improved on Ramanujan's Theorem without ever having heard of it!

Berndt (email to Sondow, 2012): *It is doubtful that Rankin took notice of Robin's paper. I definitely did not.*

After I began to edit Ramanujan's notebooks, I wrote Trinity College in 1978 for a copy of the notes that Watson and Wilson made in their efforts to edit the notebooks. I also decided to write for copies of all the Ramanujan material that was in the Trinity College Library. Included in their shipment to me was the completion of Ramanujan's paper on highly composite numbers. I put all of this on display during the Ramanujan centenary meeting at Illinois in June, 1987.

FIGURE 11. Bruce Carl BERNDT holding Ramanujan's slate (permission by Bruce Berndt)

Nicolas (email to Sondow, 2012): *I keep a very strong souvenir of the conference organised in Urbana-Champaign in 1987 for the one hundred anniversary of Ramanujan. It is there that I discovered the hidden part of "Highly Composite Numbers"* [first published in [**15**], later in [**16**], and again in [**2**]].

What I have not written is that there was an error of calculus in Ramanujan's manuscript which prevented him from seeing Robin's Theorem. Soon after discovering the hidden part, I read it and saw the difference between Ramanujan's result and Robin's one. Of course, I would have bet that the error was in Robin's paper, but after recalculating it several times and asking Robin to check, it turned out that there was an error of sign in what Ramanujan had written.

Thus it happened that Robin avoided the fate of the many mathematicians who have found that (Berndt in [**3, 4**], quoting Gosper): *Ramanujan reaches his hand from his grave to snatch your theorems from you.*

Ramanujan's Theorem was not explicitly stated by him in HCN Parts 1 or 2. Nicolas and Robin formulated it for him in Note 71 of their annotated and corrected version of HCN Part 2.

Nicolas and Robin (in [**16**]): *It follows from (382)* [(the corrected version of Ramanujan's formula)] *that under the Riemann hypothesis, and for* n_0 *large enough,*

$$n > n_0 \implies \sigma(n)/n < e^\gamma \log\log n.$$

It has been shown in [**19**] *that the above relation with* $n_0 = 5040$ *is equivalent to the Riemann hypothesis.*

Here [**19**] is Robin's paper, which he published three years *before* learning of Ramanujan's Theorem. However, a reader of [**16**] who neglects to look up [**19**] in the References is left with the misimpression "that the above relation with $n_0 = 5040$ is equivalent to the Riemann hypothesis" was proven *after* whoever proved it learned of "the above relation"!

In 1993, HCN Part 2 was submitted to Proceedings of the London Mathematical Society, which had published Part 1 in 1915. The paper was accepted, but could not be published, because Trinity College did not own the rights to Ramanujan's papers and was not able to obtain permission from his widow, Janaki.

Janaki passed away in 1994, and HCN Part 2 was eventually published by Alladi in the first volume of his newly-founded Ramanujan Journal. Later, the paper was republished as Chapter 9 in the book by Andrews and Berndt [**2**], who expanded the commentary and brought it up-to-date.

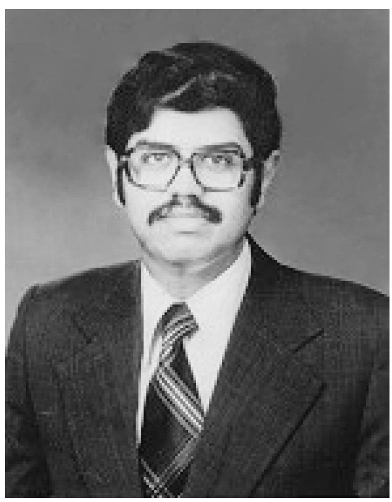
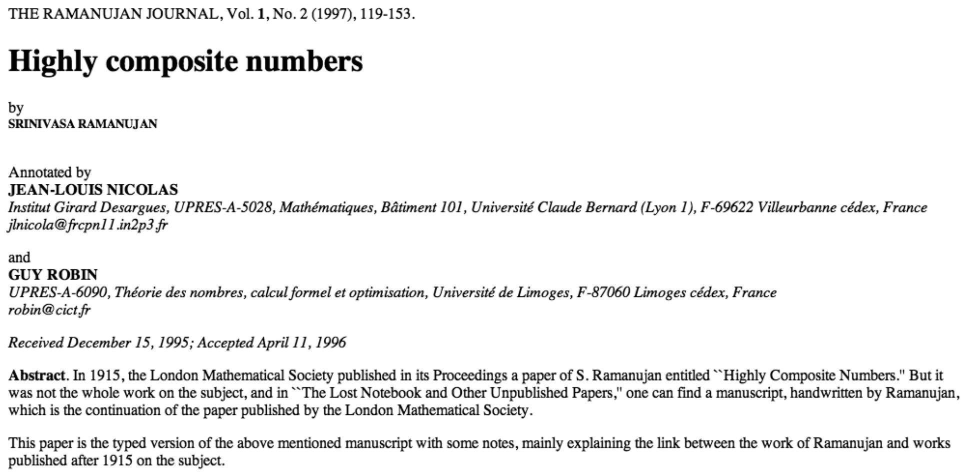


FIGURE 12. Krishnaswami ALLADI, founder of The Ramanujan Journal (photo courtesy of Krishnaswami Alladi)

Here our story ends. If it has offended anyone, we apologize.



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Highly composite numbers

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Abstract. In 1915, the London Mathematical Society published in its Proceedings a paper of S. Ramanujan entitled ``Highly Composite Numbers.'' But it was not the whole work on the subject, and in ``The Lost Notebook and Other Unpublished Papers,'' one can find a manuscript, handwritten by Ramanujan, which is the continuation of the paper published by the London Mathematical Society.

This paper is the typed version of the above mentioned manuscript with some notes, mainly explaining the link between the work of Ramanujan and works published after 1915 on the subject.

FIGURE 13. Ramanujan's HCN Part 2, annotated by Nicolas and Robin, The Ramanujan Journal, 1997

FIGURE 14. Jonathan SONDOW (left) and Ramjee RAGHAVAN, Ramanujan's grandnephew, by chance (!) seatmates on a flight to Charlotte, where S. took a connecting flight to RAMA125 in Gainesville and R. took one to a business meeting in Chicago (photo courtesy of Hisayo Foster)

Acknowledgments

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References

[1] L. Alaoglu and P. Erdös, *On highly composite and similar numbers*, Trans. Amer. Math. Soc. **56** (1944), 448–469. MR0011087 (6,117c)

[2] George E. Andrews and Bruce C. Berndt, *Ramanujan's lost notebook. Part III*, Springer, New York, 2012. MR2952081

[3] Bruce C. Berndt, *Ramanujan reaches his hand from his grave to snatch your theorems from you*, Asia Pac. Math. Newsl. **1** (2011), no. 2, 8–13. MR2817235

[4] Bruce C. Berndt, Heng Huat Chan, Song Heng Chan, and Wen-Chin Liaw, *Cranks—really, the final problem*, Ramanujan J. **23** (2010), no. 1-3, 3–15, DOI 10.1007/s11139-008-9143-z. MR2739201 (2011j:11193)

[5] Geoffrey Caveney, Jean-Louis Nicolas, and Jonathan Sondow, *Robin's theorem, primes, and a new elementary reformulation and the Riemann hypothesis*, Integers **11** (2011), A33, 10, DOI 10.1515/INTEG.2011.057. MR2798609 (2012c:11199)

[6] Geoffrey Caveney, Jean-Louis Nicolas, and Jonathan Sondow, *On SA, CA, and GA numbers*, Ramanujan J. **29** (2012), no. 1-3, 359–384, DOI 10.1007/s11139-012-9371-0. MR2994107

[7] P. Erdős. Ramanujan and I, *Number Theory, Proc. Int. Ramanujan Cent. Conf., Madras 1987*, Lect. Notes Math. 1395, Springer Verlag, 1989, pp. 1–17.

[8] T. H. Gronwall, *Some asymptotic expressions in the theory of numbers*, Trans. Amer. Math. Soc. **14** (1913), no. 1, 113–122, DOI 10.2307/1988773. MR1500940

[9] Edmund Landau, *Über einen Satz von Tschebyschef* (German), Math. Ann. **61** (1906), no. 4, 527–550, DOI 10.1007/BF01449495. MR1511360

[10] F. Mertens. Über einige asymptotische Gesetze der Zahlentheorie. *J. Reine Angew. Math.* **77** (1874), 289–338.

[11] J.-L. Nicolas, *Petites valeurs de la fonction d'Euler et hypothèse de Riemann* (French), Seminar on number theory, Paris 1981–82 (Paris, 1981/1982), Progr. Math., vol. 38, Birkhäuser Boston, Boston, MA, 1983, pp. 207–218. MR729170 (86b:11060)

[12] Jean-Louis Nicolas, *Petites valeurs de la fonction d'Euler* (French, with English summary), J. Number Theory **17** (1983), no. 3, 375–388, DOI 10.1016/0022-314X(83)90055-0. MR724536 (85h:11053)

[13] North Dakota State University, The Mathematics Genealogy Project. `http://genealogy.math.ndsu.nodak.edu/id.php?id=91561`. Accessed 21 October 2013.

[14] S. Ramanujan, *Highly composite numbers [Proc. London Math. Soc. (2)* **14** *(1915), 347–409]*, Collected papers of Srinivasa Ramanujan, AMS Chelsea Publ., Providence, RI, 2000, pp. 78–128. MR2280858

[15] Srinivasa Ramanujan, *The lost notebook and other unpublished papers*, Springer-Verlag, Berlin; Narosa Publishing House, New Delhi, 1988. With an introduction by George E. Andrews. MR947735 (89j:01078)

[16] Srinivasa Ramanujan, *Highly composite numbers*, Ramanujan J. **1** (1997), no. 2, 119–153, DOI 10.1023/A:1009764017495. Annotated and with a foreword by Jean-Louis Nicolas and Guy Robin. MR1606180 (99b:11112)

[17] R. A. Rankin, *Ramanujan's manuscripts and notebooks*, Bull. London Math. Soc. **14** (1982), no. 2, 81–97, DOI 10.1112/blms/14.2.81. MR647187 (83h:01059)

[18] G. Robin, *Sur l'ordre maximum de la fonction somme des diviseurs*, Seminar on number theory, Paris 1981–82 (Paris, 1981/1982), Progr. Math., vol. 38, Birkhäuser Boston, Boston, MA, 1983, pp. 233–244. MR729173 (85f:11070)

[19] G. Robin, *Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann* (French, with English summary), J. Math. Pures Appl. (9) **63** (1984), no. 2, 187–213. MR774171 (86f:11069)

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