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On the Asymptotic Behaviour of General Partition Functions

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Abstract. For $A = \{a_1, a_2, \ldots\} \subset \mathbf{N}$, let $p_A(n)$ denote the number of partitions of n into a's and let $q_A(n)$ denote the number of partitions of n into *distinct* a's. The asymptotic behaviour of the quotient $\frac{\log p_A(n)}{\log q_A(n)}$ is studied.

Key words: partitions, generating functions, asymptotic estimate

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1. Introduction

N denotes the set of the positive integers. If $A = \{a_1, a_2, \ldots\}$ (with $a_1 < a_2 < \cdots$) is a set of positive integers, then $p_A(n)$ denotes the number of partitions of n into a's, i.e., the number of solutions of the equation

$$x_1 a_1 + x_2 a_2 + \cdots = n$$

in non-negative integers $x_1, x_2, \ldots,$ while $q_A(n, m)$ denotes the number of partitions such that each a occurs at most m times, i.e., the number of solutions with $x_i \le m$ for all i. In particular, we write $q_A(n, 1) = q_A(n)$, so that $q_A(n)$ denotes the number of partitions of n into *distinct* a's, i.e., the number of solutions of the equation

$$a_{i_1} + a_{i_2} + \cdots = n \quad (i_1 < i_2 < \cdots).$$

In [1], Bateman and Erdős gave a necessary and sufficient condition on A for $p_A(n)$ being increasing from a certain point on. They were probably the first authors to deal with a property of $p_A(n)$ other than the estimate of its magnitude. Some other properties of p_A, depending on A, are studied in [2, 3, 7, 8].

In this paper our goal is to study the connection between the partition functions $p_A(n)$ and $q_A(n)$ for general infinite sets A. (If A is finite, $q_A(n) = 0$ for n large enough.) First we will show

Theorem 1. *For every infinite set $A \subset \mathbf{N}$ we have*

$$\limsup_{n\to+\infty} \frac{\log(\max(2,\ p_A(n)))}{\log(\max(2,q_A(n)))} \geq \sqrt{2}. \tag{1.1}$$

Note that since it is well-known [4, 5] that

$$\log p(n) = (1+o(1))\pi(2/3)^{1/2}n^{1/2} \tag{1.2}$$

and

$$\log q(n) = (1+o(1))\pi(1/3)^{1/2}n^{1/2}$$

(where $p(n) = p_{\mathbf{N}}(n)$ and $q(n) = q_{\mathbf{N}}(n)$ are the classical partition functions), we have

$$\lim_{n\to+\infty} \frac{\log p(n)}{\log q(n)} = \sqrt{2},$$

so that (1.1) cannot be improved without additional assumption on A. However, we will prove that if A is "*thin*" then the limit in (1.1) is infinite:

Theorem 2. *If $A \subset \mathbf{N}$ is an infinite set with*

$$\liminf_{n\to+\infty} \frac{\log A(n)}{\log n} = 0, \tag{1.3}$$

then we have

$$\limsup_{n\to+\infty} \frac{\log(\max(2,\ p_A(n)))}{\log(\max(2,q_A(n)))} = \infty. \tag{1.4}$$

We will show that Theorem 2 is best possible in the sense that (1.3) cannot be replaced by a weaker assumption. Indeed, for all $\varepsilon > 0$ there is a set $A \subset \mathbf{N}$ such that the limit on the left hand side of (1.3) is $< \varepsilon$, and we even have

$$\limsup_{n\to+\infty} \frac{\log A(n)}{\log n} < \varepsilon,$$

but the limit in (1.4) is finite:

Theorem 3. *Let $r, m \in \mathbf{N}$ and $A = A_r = \{1^r, 2^r, 3^r, \dots\}$ be the set of the rth powers of the integers. Then*

$$\lim_{n\to\infty} \frac{\log p_{A_r}(n)}{\log q_{A_r}(n,m)} = \frac{1}{\left(1 - \frac{1}{(m+1)^{1/r}}\right)^{r/(r+1)}} \ . \tag{1.5}$$

Due to the following asymptotic expansion as $r \to \infty$:

$$\left(1 - \frac{1}{m^{1/r}}\right)^{-r/(r+1)} = \frac{r}{\log m} + \left(\frac{1}{2} + \frac{\log\log m}{\log m} - \frac{\log r}{\log m}\right) + O\left(\frac{\log^2 r}{r}\right), \qquad (1.6)$$

the right hand side in (1.5) can be as large as we wish for m fixed, and r large enough.

We remark that the lim sup in (1.1) cannot be replaced by lim inf; to show this we shall have to consider sets A that are very irregularly distributed, similar to the counterexample given in [3]. We hope to return to this problem in a subsequent paper.

Finally, we remark that the results above can all be extended and generalized to the function $q_A(n, m)$ in place of $q_A(n)$. In particular, we can prove the following extension of Theorem 1:

Theorem 4. *For any $m \in \mathbf{N}$ and for every infinite set $A \subset \mathbf{N}$ satisfying the condition*

$$\textit{for all } a \in A, \textit{ the gcd of the elements of } A\backslash\{a\} \textit{ is } 1, \qquad (1.7)$$

we have

$$\limsup_{n\to+\infty} \frac{\log(\max(2, p_A(n)))}{\log(\max(2, q_A(n, m)))} \geq \sqrt{\frac{m+1}{m}}. \qquad (1.8)$$

(Again, as in the special case $m = 1$, the case $A = \mathbf{N}$ shows that (1.8) is the best possible, cf. [2]. By the Bateman-Erdős theorem [1] the condition (1.7) implies that $p_A(n)$ is increasing from a certain point on.)

However, since the proofs of Theorems 1 and 4 are similar but the proof of the latter result is much more technical, we will give here a detailed proof of Theorem 1 and only sketch the proof of Theorem 4.

Let $n = n_1 + n_2 + \cdots + n_r$ $(n_1 \geq n_2 \geq \cdots \geq n_r)$ be a partition Π of n. This partition is said to represent an integer a, if a can be written as a subsum $a = n_{i_1} + n_{i_2} + \cdots + n_{i_j}$ $(1 \leq i_1 < i_2 < \cdots < i_j \leq r)$ of the partition Π. We define the set $\mathcal{S}(\Pi)$ as the set of all integers a represented by Π. In [8] and [2], the number of distinct sets $\mathcal{S}(\Pi)$ generated by the $p_A(n)$ partitions of n (with parts belonging to A) is denoted by $\hat{p}_A(n)$. Erdős asked the following question: is it true that for all $A \subset \mathbf{N}$, there exists a number $\beta < 1$ such that

$$\hat{p}_A(n) \leq (p_A(n))^{\beta}$$

holds for n large enough? In [2] it is proved (the proof is easy) that if A is m-stable (i.e., $a \in A \Rightarrow ma \in A$) with $m \geq 2$ then

$$\hat{p}_A(n) \leq q_A(n, 2m - 2)$$

so that, by Theorem 4, the answer to Erdős's question is yes for all sets A satisfying (1.7) and which are m-stable for some $m \geq 2$.

2. Proof of Theorem 1.

If the greatest common divisor, say d, of the elements of A is greater than 1, then dividing every element of A by d we may reduce the problem to the case when the elements of A are coprime. Writing $A = \{a_1, a_2, \dots\}$ (with $a_1 < a_2 < \cdots$), we may therefore assume that

$$(a_1, a_2, \dots) = 1. \tag{2.1}$$

It follows that there is a $k \in \mathbf{N}$ with

$$(a_1, a_2, \dots, a_k) = 1.$$

Then it is well-known that there is an $n_0 \in \mathbf{N}$ such that if $n \geq n_0$, $n \in \mathbf{N}$, then there are non-negative integers $x_1, \dots, x_k$ with

$$a_1 x_1 + \cdots + a_k x_k = n \quad (\text{for } n \geq n_0). \tag{2.2}$$

Write $n_1 = n_0 + a_{k+1}$. If $n \geq n_1$, $n \in \mathbf{N}$, then n has at least two different partitions into a's: one partition is obtained by applying (2.2) to n, and a second partition is obtained by applying (2.2) to the number $n - a_{k+1} \geq n_0$ and adding the part a_{k+1}. Thus we have

$$p_A(n) \geq 2 \quad \text{for } n \geq n_1. \tag{2.3}$$

By extending this method, or by using generating functions (cf. [1, Lemma 1], it can be shown that assuming (2.1), one has $\lim_{n\to\infty} p_A(n) = +\infty$.

We will prove (1.1) by contradiction: assume that

$$\limsup_{n\to+\infty} \frac{\log p_A(n)}{\log(\max(2, q_A(n)))} < \sqrt{2}. \tag{2.4}$$

Then there are numbers $\varepsilon > 0$, $n_2 \in \mathbf{N}$ such that for $n \geq n_2$ we have

$$\log q_A(n) > \left(\frac{1}{\sqrt{2}} + \varepsilon\right) \log p_A(n) \quad (\text{for } n \geq n_2). \tag{2.5}$$

Denote the generating functions of the functions $p_A(n)$ and $q_A(n)$ by $F_A(x)$ and $G_A(x)$, respectively. Thus

$$F_A(x) = \sum_{n=0}^{+\infty} p_A(n) x^n = \prod_{a\in A} \frac{1}{1 - x^a} \quad (|x| < 1) \tag{2.6}$$

and

$$G_A(x) = \sum_{n=0}^{+\infty} q_A(n) x^n = \prod_{a\in A} (1 + x^a) \quad (|x| < 1). \tag{2.7}$$

Then clearly we have

$$F_A(x^2) = \prod_{a\in A} \frac{1}{1-x^{2a}} = \prod_{a\in A} \frac{1}{1-x^a}\left(\prod_{a\in A}(1+x^a)\right)^{-1} = F_A(x)(G_A(x))^{-1}$$

whence

$$F_A(x^2)G_A(x) = F_A(x)$$

which, by (2.6) and (2.7), can be rewritten as

$$\left(\sum_{r=0}^{+\infty} p_A(r)x^{2r}\right)\left(\sum_{s=0}^{+\infty} q_A(s)x^s\right) = \sum_{t=0}^{+\infty} p_A(t)x^t.$$

It follows that

$$\sum_{\substack{2r+s=t\\ r,s\geq 0}} p_A(r)q_A(s) = p_A(t). \tag{2.8}$$

Substituting $t = 4n$ and keeping only the (roughly maximal) term with $r = n$, $s = 2n$ on the left hand side, we obtain that

$$p_A(n)q_A(2n) \leq p_A(4n) \quad (n \in \mathbf{N}).$$

By (2.3) and (2.5), it follows that for all $n \geq n_3 \overset{\text{def}}{=} \max\{n_1, n_2\}$ we have

$$\log p_A(4n) \geq \log p_A(n) + \log q_A(2n) > \log p_A(n) + \left(\frac{1}{\sqrt{2}} + \varepsilon\right) \log p_A(2n) \quad (n \geq n_3). \tag{2.9}$$

Now write

$$b = \min\left\{ \log p_A(n_3), \left(\sqrt{2} + \frac{\varepsilon}{2}\right)^{-1} \log p_A(2n_3)\right\} \tag{2.10}$$

so that

$$b > 0 \tag{2.11}$$

by (2.3) and since $n_3 > n_1$. We will prove by induction on k that

$$\log p_A(n_3 2^k) \geq b\left(\sqrt{2} + \frac{\varepsilon}{2}\right)^k \tag{2.12}$$

for $k = 0, 1, 2, \dots,$ Indeed, by (2.10), (2.12) holds for $k = 0$ and $k = 1$. Assume now that $k \geq 1, k \in \mathbf{N}$, and that (2.12) holds with $0, 1, \dots, k$ in place of k. Then by (2.9) it follows that

$$\begin{aligned}
\log p_A(n_3 2^{k+1}) &\geq \log p_A(n_3 2^{k-1}) + \left(\frac{1}{\sqrt{2}} + \varepsilon\right) \log p_A(n_3 2^k) \\
&\geq b\left(\sqrt{2} + \frac{\varepsilon}{2}\right)^{k-1} + \left(\frac{1}{\sqrt{2}} + \varepsilon\right) b\left(\sqrt{2} + \frac{\varepsilon}{2}\right)^{k} \\
&= b\left(\sqrt{2} + \frac{\varepsilon}{2}\right)^{k-1}\left(1 + \left(\frac{1}{\sqrt{2}} + \varepsilon\right)\left(\sqrt{2} + \frac{\varepsilon}{2}\right)\right) \\
&= b\left(\sqrt{2} + \frac{\varepsilon}{2}\right)^{k-1}\left(2 + \left(\sqrt{2} + \frac{1}{2\sqrt{2}}\right)\varepsilon + \frac{1}{2}\varepsilon^2\right) \\
&> b\left(\sqrt{2} + \frac{\varepsilon}{2}\right)^{k-1}\left(2 + \sqrt{2}\,\varepsilon + \frac{1}{4}\varepsilon^2\right) = b\left(\sqrt{2} + \frac{\varepsilon}{2}\right)^{k+1},
\end{aligned}$$

so that (2.12) also holds with $k + 1$ in place of k. This completes the proof of (2.12).

By (2.11), it follows from (2.12) that for $k \to \infty$ we have

$$\log\log p_A(n_3 2^k) \geq (1 + o(1))k \log\left(\sqrt{2} + \frac{\varepsilon}{2}\right). \tag{2.13}$$

On the other hand, clearly we have $p_A(n) \leq p(n)$, and thus it follows from (1.2) that for $k \to +\infty$ we have

$$\log\log p_A(n_3 2^k) \leq \log\log p(n_3 2^k) = (1 + o(1)) \log(n_3 2^k)^{1/2} = (1 + o(1))k \ \log \sqrt{2}$$

which contradicts (2.13). This completes the proof of Theorem 1.

3. Proof of Theorem 2.

We will prove the Theorem by contradiction. Assume that an infinite set $A \subset \mathbf{N}$ satisfies (1.3), but (1.4) does not hold, i.e., there are numbers $M, n_4 \in \mathbf{N}$ such that

$$p_A(n) \leq (q_A(n))^M \quad (n \geq n_4). \tag{3.1}$$

Using again (2.8), with $t = 3n$ and keeping only the term with $r = s = n$ on the left hand side, we obtain

$$p_A(n) q_A(n) \leq p_A(3n) \quad (n \in \mathbf{N}). \tag{3.2}$$

Writing $\delta = 1/M$, it follows from (3.1) and (3.2) that

$$p_A(3n) \geq p_A(n)(p_A(n))^{1/M} = (p_A(n))^{1+\delta} \quad (n \geq n_4). \tag{3.3}$$

As in the proof of Theorem 1, we may assume that (2.1) and then also (2.3) holds. Write $n_5 = \max\{n_1, n_4\}$. Then it follows from (2.3) and (3.3) by induction on k that

$$p_A(3^k n_5) \geq (p_A(n_5))^{(1+\delta)^k} \geq 2^{(1+\delta)^k} \quad (k = 0, 1, 2, \ldots). \tag{3.4}$$

Consider a large integer n, and define the integer $k = k(n)$ by

$$3^k n_5 + n_1 \leq n < 3^{k+1} n_5 + n_1 \tag{3.5}$$

so that

$$k = \frac{\log n}{\log 3} + O(1) \quad (n \to +\infty). \tag{3.6}$$

Define the integer $m = m(n)$ by

$$3^k n_5 + m = n$$

so that $m \geq n_1$ by (3.5). Thus by (2.3) (which we have assumed) m has at least one partition into a's. Fixing such a partition of m and combining it with distinct partitions of $n - m = 3^k n_5$ into a's, we obtain distinct partitions of n into a's and thus, by (3.4),

$$p_A(n) \geq p_A(n - m) = p_A(3^k n_5) \geq 2^{(1+\delta)^k} \tag{3.7}$$

for n large enough. It follows from (3.6) and (3.7) that

$$\frac{\log\log p_A(n)}{\log n} \geq \frac{k \log(1+\delta) + O(1)}{k \log 3 + O(1)} = \frac{\log(1+\delta)}{\log 3} + o(1) \quad (n \to +\infty). \tag{3.8}$$

On the other hand, if we write $A = \{\alpha_1, \alpha_2, \ldots\}$ with $\alpha_1 < \alpha_2 < \cdots$, and call $A(n)$ the number of elements of A up to n then $p_A(n)$ denotes the number of solutions of

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_{A(n)}\alpha_{A(n)} = n \tag{3.9}$$

in non-negative integers $x_1, x_2, \ldots, x_{A(n)}$ (for all $n \in \mathbf{N}$). Here each x_i $(i = 1, 2, \ldots, A(n))$ is one of the $(n+1)$ integers $0, 1, \ldots, n$. It follows that the number of solutions of (3.9) is

$$p_A(n) \leq (n+1)^{A(n)} \leq (2n)^{A(n)}$$

whence

$$\log\log p_A(n) \leq \log A(n) + \log\log(2n)$$

so that, by (1.3),

$$\liminf_{n\to+\infty} \frac{\log\log p_A(n)}{\log n} \leq \liminf_{n\to+\infty} \left(\frac{\log A(n)}{\log n} + \frac{\log\log(2n)}{\log n} \right) = 0.$$

This contradicts (3.8), and the proof of Theorem 2 is complete.

4. Proof of Theorem 3.

Let us denote by $f_r(x)$ the generating function:

$$f_r(x) = \sum_{n=0}^{\infty} p_{A_r}(n)x^n = \prod_{a \in A}(1 - x^a)^{-1}.$$

At the end of their famous paper [5], Hardy and Ramanujan have written an asymptotic estimation for $p_{A_r}(n)$, without giving a complete proof, just saying that their method used to estimate $p(n)$ can be extended. A complete proof was given later by Wright in [9]. As far as we know, no asymptotic estimation for $q_{A_r}(n, m)$ has been published, though it is doable by using the generating function

$$F(x) = \sum_{n=0}^{\infty} q_{A_r}(n, m)x^n = \prod_{a \in A}(1 + x^a + x^{2a} + \cdots + x^{ma}) = \frac{f_r(x^{m+1})}{f_r(x)}.$$

One can get an asymptotic estimate for $q_{A_r}(n, m)$ by using the estimate of $f_r(x)$ when $x \to 1^-$ given in [5, Section 7.3], or in [9], and then applying the Tauberian theorem of Ingham (cf. [6]).

Here, it is enough to have an asymptotic estimate for the logarithms of $p_{A_r}(n)$ and $q_{A_r}(n, m)$ and we shall use the Tauberian theorem of Hardy and Ramanujan [4]. It is proved in [4] that

$$\log f_r(x) \sim \Gamma\left(\frac{1}{r} + 1\right)\zeta\left(\frac{1}{r} + 1\right)\left(\log \frac{1}{x}\right)^{-1/r} \tag{4.1}$$

and

$$\log p_{A_r}(n) \sim (r + 1)\left(\frac{1}{r}\,\Gamma\left(\frac{1}{r} + 1\right)\zeta\left(\frac{1}{r} + 1\right)\right)^{r/(r+1)} n^{1/(r+1)}. \tag{4.2}$$

Thus, by (4.1), it follows that the generating function of $q_{A_r}(n, m)$ verifies

$$\log F(x) \sim \Gamma\left(\frac{1}{r} + 1\right)\zeta\left(\frac{1}{r} + 1\right)\left(1 - \frac{1}{(m + 1)^{1/r}}\right)\left(\log \frac{1}{x}\right)^{-1/r}.$$

The Tauberian theorem of Hardy and Ramanujan says that, if $\log F(x) \sim D(\log \frac{1}{x})^{-\alpha}$, then

$$\log\left(\sum_{n=0}^{N} q_{A_r}(n, m)\right) \sim BN^{\alpha/(1+\alpha)} \tag{4.3}$$

with $B = D^{1/(1+\alpha)}\alpha^{-\alpha/(1+\alpha)}(1+\alpha)$. It follows easily from (4.3) and the fact that $q_{A_r}(n, m)$ is an increasing function of n that

$$\log q_{A_r}(n, m) \sim (r+1)\left(\frac{1}{r}\,\Gamma\left(\frac{1}{r}+1\right)\zeta\left(\frac{1}{r}+1\right)\left(1-\frac{1}{(m+1)^{1/r}}\right)\right)^{r/(r+1)} n^{1/(r+1)}.$$

This, together with (4.2), yields (1.5).

5. Sketch of the Proof of Theorem 4.

It follows from (1.7) that (2.1) and (2.3) hold. Again we proceed by contradiction: assume that for some $\varepsilon > 0$ and $n \geq n_6$ we have

$$\log q_A(n, m) > \left(\sqrt{\frac{m}{m+1}}+\varepsilon\right)\log p_A(n) \quad (n \geq n_6). \tag{5.1}$$

Denote the generating function of $q_A(n, m)$ by $G_A(x, m)$:

$$G_A(x, m) = \sum_{n=0}^{\infty} q_A(n, m)x^n = \prod_{a\in A}\left(1+\sum_{j=1}^{m} x^{ja}\right) = \prod_{a\in A}\frac{1-x^{(m+1)a}}{1-x^a} \quad (\text{for } |x| < 1).$$

Then we have

$$F_A(x^{m+1})G_A(x) = F_A(x)$$

so that

$$\left(\sum_{r=0}^{+\infty} p_A(r)x^{(m+1)r}\right)\left(\sum_{s=0}^{+\infty} q_A(s, m)x^s\right) = \sum_{t=0}^{+\infty} p_A(t)x^t$$

whence

$$\sum_{\substack{(m+1)r+s=t\\ r,s\geq 0}} p_A(r)q_A(s, m) = p_A(t).$$

Substituting $t = (m+1)^2 n$, and keeping only the term with $r = n$, $s = m(m+1)n$ on the left hand side, we obtain that

$$p_A(n)q_A(m(m+1)n, m) \leq p_A((m+1)^2 n) \quad (\text{for all } n \in \mathbf{N}). \tag{5.2}$$

By (2.3), (5.1) and (5.2) we have for large n

$$\log p_A((m+1)^2 n) > \log p_A(n) + \left(\sqrt{\frac{m}{m+1}} + \varepsilon\right) \log p_A(m(m+1)n) \quad (n \geq n_7). \quad (5.3)$$

By a result of Bateman and Erdős [1] it follows from (1.7) that, for n large enough, $p_A(n)$ is increasing:

$$p_A(n) < p_A(n+1) \quad (n \geq n_8). \quad (5.4)$$

Now it follows from (2.3), (5.3) and (5.4) by induction on N that if δ, ε' (> 0) are small enough and N_0 is large enough in terms of $m, \varepsilon, n_1, n_7, n_8$, then we have

$$\log p_A(N) > \delta N^{(1/2)+\varepsilon'} \quad (N \geq N_0). \quad (5.5)$$

Indeed, observe first that if $N_0 \geq n_1$ then, by (2.3), (5.5) holds for $N = N_0, N_0+1, \ldots, (m+1)^2 N_0$, provided δ is small enough. Next we assume that $N > (m+1)^2 N_0$ and that (5.5) holds for all N' with $N_0 \leq N' \leq N-1$. Our goal is to show that (5.5) also holds for $N' = N$. To prove this, define the positive integer n by

$$(m+1)^2 n \leq N < (m+1)^2(n+1) \quad (5.6)$$

so that

$$n \geq N_0 \quad (5.7)$$

and, by (5.4) and (5.6),

$$p_A(N) \geq p_A((m+1)^2 n). \quad (5.8)$$

We can obtain a lower bound for the right hand side of (5.3) by using the induction hypothesis in both terms; by (5.8), this is also a lower bound for $\log p_A(N)$. A simple computation shows that if ε' is small enough and N_0 is large enough, then this lower bound for $\log p_A(N)$ is greater than the right hand side of (5.5), and this completes the proof.

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