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On the Asymptotic Behaviour of General Partition Functions, II

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Abstract. Let $\mathcal{A} = \{a_1, a_2, \ldots\}$ be a set of positive integers and let $p_{\mathcal{A}}(n)$ and $q_{\mathcal{A}}(n)$ denote the number of partitions of n into a's, resp. distinct a's. In an earlier paper the authors studied large values of $\frac{\log(\max(2, p_{\mathcal{A}}(n)))}{\log(\max(2, q_{\mathcal{A}}(n)))}$. In this paper the small values of the same quotient are studied.

Key words: partitions, generating functions, asymptotic estimate

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1. Introduction

$\mathbb{N}$ denotes the set of positive integers. If $\mathcal{A} = \{a_1, a_2, \ldots\}$ (with $a_1 < a_2 < \ldots$) is a set of positive integers, then $p_{\mathcal{A}}(n)$ denotes the number of partitions of n into a's, i.e., the number of solutions of the equation

$$x_1 a_1 + x_2 a_2 + \cdots = n \tag{1.1}$$

in non negative integers $x_1, x_2, \ldots$, while $q_{\mathcal{A}}(n)$ denotes the number of restricted partitions of n into a's; in other words, $q_{\mathcal{A}}(n)$ is the number of solutions of (1.1) with $x_i = 0$ or 1 for all i's.

The main result of [10] is that for any infinite set $\mathcal{A} \subset \mathbb{N}$, we have

$$\limsup_{n\to+\infty} \frac{\log(\max(2, p_{\mathcal{A}}(n)))}{\log(\max(2, q_{\mathcal{A}}(n)))} \geq \sqrt{2}. \tag{1.2}$$

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If $p(n) = p_{\mathbb{N}}(n)$ and $q(n) = q_{\mathbb{N}}(n)$ are the classical partition functions, it is well-known (cf. [8, 1]) that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right), \quad \pi\sqrt{\frac{2}{3}} = 2.56\ldots \tag{1.3}$$

and

$$q(n) \sim \frac{1}{4(3n^3)^{1/4}} \exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n}\right), \quad \frac{\pi}{\sqrt{3}} = 1.81\ldots \tag{1.4}$$

It follows from (1.3) and (1.4) that

$$\lim_{n\to\infty} \frac{\log p(n)}{\log q(n)} = \sqrt{2},$$

so that (1.2) is best possible. It was also proved in [10] that if $A(x) = \sum_{a_i \le x}^{i} 1$, the counting function of $\mathcal{A}$, satisfies

$$\liminf_{x\to\infty} \frac{\log A(x)}{\log x} = 0, \tag{1.5}$$

then we have

$$\limsup_{n\to+\infty} \frac{\log(\max(2, p_{\mathcal{A}}(n)))}{\log(\max(2, q_{\mathcal{A}}(n)))} = \infty. \tag{1.6}$$

In this paper, we shall deal with the inferior limit. In Section 2, we will prove

Theorem 1. *There exists a set $\mathcal{S} \subset \mathbb{N}$ with*

$$S(x) = \sum_{s\in\mathcal{S}, s\le x} 1 \ge x^{3/16} \tag{1.7}$$

and

$$\liminf_{n\to\infty} \frac{\log p_{\mathcal{S}}(n)}{\log q_{\mathcal{S}}(n)} = 1.$$

In Section 3, we shall prove:

Theorem 2. *Let $\mathcal{A}$ be a set of positive integers. Let us assume that*

$$\alpha = \liminf_{n\to\infty} \frac{A(n)}{n} \tag{1.8}$$

is positive. Then there exists $\eta = \eta(\alpha) > 0$ such that

$$p_{\mathcal{A}}(n) \ge (q_{\mathcal{A}}(n))^{1+\eta(\alpha)} \quad \textit{for } n \ge n_0. \tag{1.9}$$

The idea of the proof of Theorem 2 is to construct, from most of the restricted partitions of n into parts in $\mathcal{A}$, many unrestricted partitions of n.

In Section 4, we will prove the following theorem which shows that Theorem 2 is in some sense best possible:

Theorem 3. *Let $f(x)$ be any non-increasing function of $x > 0$ and tending to 0 as x tends to infinity. There is a set $\mathcal{A} \subset \mathbb{N}$ such that*

$$\frac{A(n)}{n} > f(n) \quad \textit{for } n > n_0 \tag{1.10}$$

and

$$\liminf_{n\to\infty} \frac{\log p_{\mathcal{A}}(n)}{\log q_{\mathcal{A}}(n)} = 1. \tag{1.11}$$

This result is much sharper than Theorem 1. The construction of the set $\mathcal{A}$ in Theorem 3 is similar to the construction of the set $\mathcal{S}$ in Theorem 1, however, here the construction is more complicated. The proof of Theorem 3 will be based mostly on Proposition 1 below. We will give only an outline of the proof of Proposition 1; a complete proof could be given, but it would be very lenghty and technical. Thus we have decided to give here (Section 2) a complete and precise proof of the weaker but much simpler version stated in Theorem 1.

Let $r(n, m)$ and $\varrho(n, m)$ denote the number of partitions of n into parts at least m, resp. into distinct parts at least m. (In other words, if $\mathcal{M} = \{n \in \mathbb{N}, n \geq m\}$, then $r(n, m) = p_{\mathcal{M}}(n)$ and $\varrho(n, m) = q_{\mathcal{M}}(n)$.)

It was proved in [5] and [11] that, for any $\lambda > 0$, we have

$$\lim_{n\to\infty} \frac{\log(r(n, \lambda\sqrt{n}))}{\sqrt{n}} = g(\lambda) \tag{1.12}$$

and

$$\lim_{n\to\infty} \frac{\log(\rho(n, \lambda\sqrt{n}))}{\sqrt{n}} = h(\lambda). \tag{1.13}$$

Moreover the two functions g and h have the same asymptotic expansion as $\lambda \to \infty$:

$$g(\lambda), h(\lambda) = \frac{2\log\lambda - \log\log\lambda + 1 - \log 2}{\lambda} + O\left(\frac{\log\log\lambda}{\lambda\log\lambda}\right). \tag{1.14}$$

Let us define, for $1 \leq x \leq y$, $r(n; x, y)$ and $\varrho(n; x, y)$ as the number of partitions of n into parts belonging to the interval $[x, y[$, resp. into distinct parts belonging to $[x, y[$.

Proposition 1. *There exist two continous functions $g_2(\lambda)$, $h_2(\lambda)$ defined for $\lambda > 0$ such that*

$$\lim_{n\to\infty} \frac{\log r(n; \lambda\sqrt{n}, 2\lambda\sqrt{n})}{\sqrt{n}} = g_2(\lambda) \tag{1.15}$$

and

$$\lim_{n\to\infty} \frac{\log \max\{1, \varrho(n; \lambda\sqrt{n}, 2\lambda\sqrt{n})\}}{\sqrt{n}} = h_2(\lambda). \tag{1.16}$$

Moreover, as $\lambda \to \infty$, *we have* $g_2(\lambda) \sim h_2(\lambda)$ *and both functions* g_2 *and* h_2 *satisfy the asymptotic expansion* (1.14).

The sketch of the proof of Proposition 1 will be given in Section 4. More precisely, we shall consider only (1.16); the proof of (1.15) would be similar, and we do not need (1.15) in the proof of Theorem 3. The proof of (1.16) follows the proof of (1.13) in [11] and consists of two parts, the upper bound for $\varrho(n; \lambda\sqrt{n}, 2\lambda\sqrt{n})$ and the lower bound. The upper bound is stated in Lemma 6 below. We have not given the proof of the lower bound which can be obtained by the methods used in [5] or [11] or by applying the saddle point method to the generating series.

2. An elementary counterexample

Lemma 1. *Let n be a positive integer and x a positive real number. Let us denote by* $p(n, x)$ *the number of partitions of n into parts* $\le x$ (*while* $r(n, x)$ *denotes the number of partitions of n into parts* $\ge x$, *as defined above*). *Then for* $n \ge 1$ *and* $\lambda > 0$ *we have*

$$\log p(n, \lambda\sqrt{n}) \le \left\{ \begin{matrix} (\lambda(3 - 2\log\lambda))\sqrt{n} & \textit{for } \lambda \le 1 \\ 3\sqrt{n} & \textit{for } \lambda > 1 \end{matrix} \right\} \le 3\sqrt{\lambda n} \tag{2.1}$$

and

$$\log r(n, \lambda\sqrt{n}) \le \left\{ \begin{matrix} \left(\frac{2\log\lambda + 3}{\lambda}\right)\sqrt{n} & \textit{for } \lambda \ge 1 \\ 3\sqrt{n} & \textit{for } \lambda < 1 \end{matrix} \right\} \le \frac{3}{\sqrt{\lambda}}\sqrt{n}. \tag{2.2}$$

Proof: The first inequality in (2.1), for $\lambda \le 1$, is proved in [6], Lemma 2, where it is deduced from the classical result

$$p(n, m) \le \frac{1}{m!} \binom{n + \frac{m(m+1)}{2} - 1}{m - 1}, \quad m \in \mathbb{N}$$

(see, e.g., [3]). For $\lambda > 1$ the second inequality in (2.1) follows from $p(n, \lambda\sqrt{n}) \le p(n)$ and from the upper bound $p(n) \le \exp(\pi\sqrt{\frac{2n}{3}})$ which holds for all $n \ge 1$ (cf. [12], Theorem 15.5). The inequality $\lambda(3 - 2\log\lambda) \le 3\sqrt{\lambda}$ for $\lambda \le 1$ is a simple analysis exercise. Finally, (2.2) follows from (2.1) and from the relation $r(n, x) \le p(n, n/x)$. □

Lemma 2. *Let* $\mathcal{A} = \{a_1, a_2, \ldots\}$ *be a set of positive integers, with* $a_1 = 1 < a_2 < \cdots$ *Let us denote by* $A(x)$ *the number of* a_i*'s not exceeding* x, *and by* $p_{\mathcal{A}}(n)$ *the number of partitions* *of n with parts in $\mathcal{A}$. Then, for $n \in \mathbb{N}$, we have*

$$p_{\mathcal{A}}(n) \le n^{A(n)-1}.$$

Proof: If $1 \le n < a_2$, this is obvious since $p_{\mathcal{A}}(n) = 1$ and $A(n) = 1$. If $n \ge a_2$, let us set $m = A(n) \ge 2$. Then $p_{\mathcal{A}}(n)$ is the number of solutions of

$$x_1 + x_2 a_2 + \cdots + x_m a_m = n.$$

The possible values for x_i are $0, 1, \ldots, \lfloor n/a_i \rfloor$, and, when $x_2, \ldots, x_m$ are fixed, there is only one possibility for x_1. Thus

$$p_{\mathcal{A}}(n) \le \prod_{i=2}^{m} \left(\left\lfloor \frac{n}{a_i} \right\rfloor + 1 \right) \le \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right)^{m-1} \le n^{m-1}.$$

□

Lemma 3. *Let $\mathcal{B} = \{b_1, \ldots, b_\beta\} \subset \mathbb{N}$ and t a positive integer. There is a $u \in [tb_1,\ tb_\beta]$ such that $q_{\mathcal{B}}(u)$, the number of partitions of u into distinct parts belonging to $\mathcal{B}$, satisfies:*

$$q_{\mathcal{B}}(u) \ge \frac{1}{t(b_\beta - b_1) + 1} \binom{\beta}{t}.$$

Proof: Let us consider the $\binom{\beta}{t}$ different choices $\beta_{i_1}, \ldots, \beta_{i_t}$; each of the sums $\beta_{i_1} + \cdots + \beta_{i_t}$ is between tb_1 and tb_β. Thus the most frequently occuring value will be obtained at least $\frac{1}{t(b_\beta - b_1)+1}\binom{\beta}{t}$ times. □

Proof of Theorem 1: For $k \ge 1$ set

$$t_k = 2^{4^k}, \quad \beta_k = \frac{t_{k+1}}{t_k} = t_k^3 = 2^{3 \cdot 4^k}$$

and

$$\mathcal{S}_k = \{t_{k+1} - \beta_k + 1,\ t_{k+1} - \beta_k + 2, \ldots, t_{k+1}\}.$$

Then

$$|\mathcal{S}_k| = \beta_k,$$

and since

$$t_{k+1} - \beta_k = t_k^4 - t_k^3 = t_k^3(t_k - 1) > t_k,$$

thus we have

$$\mathcal{S}_k \subset]t_k, \ldots t_{k+1}].$$

Now we define $\mathcal{S}$ by

$$\mathcal{S} = \{1\} \cup \left(\bigcup_{k \geq 1} \mathcal{S}_k \right).$$

Recalling that $S(x) = \sum_{s \in S,\ s \leq x} 1$, for $k \geq 2$ we have

$$\begin{aligned} \beta_{k-1} &\leq S(t_k) = 1 + \beta_1 + \cdots + \beta_{k-1} = 1 + 2^{12} + \cdots + 2^{3 \cdot 4^{k-1}} \\ &\leq 1 + 2 + 2^2 + \cdots + 2^{3 \cdot 4^{k-1}} < 2\beta_{k-1} = 2^{3 \cdot 4^{k-1}+1}. \end{aligned} \tag{2.3}$$

If $x > t_2 = 2^{16}$, then we define $l = l(x) \geq 2$ by $t_l < x \leq t_{l+1}$, which implies

$$4^l < \frac{\log x}{\log 2} \leq 4^{l+1}$$

and, from (2.3), we have

$$S(x) \geq S(t_l) \geq \beta_{l-1} = 2^{3 \cdot 4^{l-1}} = 2^{\frac{3}{16} 4^{l+1}} \geq x^{3/16}$$

which proves (1.7). (Similarly, it is not difficult to show that $S(x) \ll x^{3/4}$.)

Now we apply Lemma 3 with $\mathcal{B} = \mathcal{S}_k$ and $t = t_k$: there exist $u_k \in \mathbb{N}$ such that

$$(t_k - 1)t_{k+1} = (t_{k+1} - \beta_k)t_k < u_k \leq t_k t_{k+1} \tag{2.4}$$

and

$$q_{\mathcal{S}}(u_k) \geq q_{\mathcal{S}_k}(u_k) \geq \frac{1}{t_k(\beta_k - 1) + 1} \binom{\beta_k}{t_k} \geq \frac{1}{t_k \beta_k} \binom{\beta_k}{t_k}. \tag{2.5}$$

Now we will give an upper bound for $p_{\mathcal{S}}(u_k)$. Set $N = t_k t_{k+1} = t_k^5$. Since $1 \in \mathcal{S}$, thus $p_{\mathcal{S}}(n)$ is a non-decreasing function of n, so that from (2.4),

$$p_{\mathcal{S}}(u_k) \leq p_{\mathcal{S}}(t_k t_{k+1}) = p_{\mathcal{S}}(N). \tag{2.6}$$

The smallest element of $\mathcal{S}_{k+1}$ is

$$t_{k+2} - \beta_{k+1} + 1 > t_{k+2} - \beta_{k+1} = t_{k+2}\left(1 - \frac{1}{t_{k+1}}\right) > \frac{t_{k+2}}{2} = \frac{t_{k+1}^4}{2} > t_{k+1} t_k = N.$$

Thus if for $k \geq 2$ we set $\mathcal{C}_k = \{1\} \cup (\bigcup_{j \leq k-1} \mathcal{S}_j)$, then we have

$$p_{\mathcal{S}}(N) = p_{\mathcal{C}_k U \mathcal{S}_k}(N) = \sum_{j=0}^{N} p_{\mathcal{C}_k}(j) p_{\mathcal{S}_k}(N - j). \tag{2.7}$$

Now we apply Lemma 2 with $\mathcal{A} = \mathcal{C}_k$, $n = N$, $A(n) = S(t_k)$, which by (2.3) yields

$$p_{\mathcal{C}_k}(N) \leq N^{S(t_k)-1} \leq (t_k t_{k+1})^{2\beta_{k-1}}. \tag{2.8}$$

Since $1 \in \mathcal{C}_k$, thus $p_{\mathcal{C}_k}(j)$ is a non-decreasing function of j, and thus it follows from (2.7) that

$$p_{\mathcal{S}}(N) \le p_{\mathcal{C}_k}(N) \sum_{j=0}^{N} p_{\mathcal{S}_k}(N-j). \tag{2.9}$$

If we denote the elements of $\mathcal{S}_k$ by $s_1 < s_2 < \cdots < s_{\beta_k}$, then the sum above is the number of solutions of

$$x_1 s_1 + \cdots + x_{\beta_k} s_{\beta_k} \le N$$

which, by

$$x_1 s_1 + \cdots + x_{\beta_k} s_{\beta_k} \ge \big(x_1 + \cdots + x_{\beta_k}\big) s_1,$$

is smaller, than the number of solutions of

$$x_1 + \cdots + {}_{\beta_k} \le \lfloor N/s_1 \rfloor. \tag{2.10}$$

since

$$\frac{N}{s_1} = \frac{t_k t_{k+1}}{t_{k+1} - \beta_k + 1} < \frac{t_k t_{k+1}}{t_{k+1} - \beta_k} = \frac{t_k}{1 - 1/t_k} = t_k + 1 + \frac{1}{t_k} + \cdots < t_k + 2,$$

thus $\lfloor N/s_1 \rfloor \le t_k + 1$, so that the number of solutions of (2.10) is

$$\le \binom{t_k + 1 + \beta_k}{\beta_k} = \binom{\beta_k + t_k + 1}{t_k + 1}.$$

Thus we have

$$\sum_{j=0}^{N} p_{\mathcal{S}_k}(N-j) \le \binom{\beta_k + t_k + 1}{t_k + 1} = \frac{\beta_k + t_k + 1}{t_k + 1} \binom{\beta_k + t_k}{t_k} \le \beta_k \binom{\beta_k + t_k}{t_k}. \tag{2.11}$$

It follows from (2.6), (2.8), (2.9) and (2.11) that

$$p_{\mathcal{S}}(u_k) \le (t_k t_{k+1})^{2\beta_{k-1}} \beta_k \binom{\beta_k + t_k}{t_k}. \tag{2.12}$$

It remains to estimate $\binom{\beta_k}{t_k}$ and $\binom{\beta_k + t_k}{t_k}$. We have

$$\binom{\beta_k}{t_k} = \frac{\beta_k(\beta_k - 1)\ldots(\beta_k - t_k + 1)}{t_k!} \ge \frac{(\beta_k - t_k)^{t_k}}{t_k^{t_k}} = \left(\frac{\beta_k}{t_k}\right)^{t_k} \left(1 - \frac{t_k}{\beta_k}\right)^{t_k}$$

and

$$\left(1-\frac{t_k}{\beta_k}\right)^{t_k} = \exp\left(-t_k\log\left(1+\frac{t_k}{\beta_k-t_k}\right)\right) \geq \exp\left(-\frac{t_k^2}{\beta_k-t_k}\right)$$
$$\geq \exp\left(-\frac{2t_k^2}{\beta_k}\right) \quad \text{since } \beta_k \geq 2t_k$$
$$= \exp\left(-\frac{2}{t_k}\right) \quad \text{since } \beta_k = t_k^3$$

so that

$$\binom{\beta_k}{t_k} \geq \left(\frac{\beta_k}{t_k}\right)^{t_k}\exp\left(-\frac{2}{t_k}\right). \tag{2.13}$$

Similary, by using the weak form $n! \geq n^n e^{-n}$ of Stirling's formula:

$$\binom{\beta_k+t_k}{t_k} \leq \frac{(\beta_k+t_k)^{t_k}}{t_k!} \leq \frac{\beta_k^{t_k}}{t_k!}\exp\left(\frac{t_k^2}{\beta_k}\right) \leq \left(\frac{e\beta_k}{t_k}\right)^{t_k}\exp\left(\frac{1}{t_k}\right). \tag{2.14}$$

From (2.5) and (2.13), we get for $k \to \infty$:

$$\log q_{\mathcal{S}}(u_k) \geq t_k\log\left(\frac{\beta_k}{t_k}\right) - \frac{2}{t_k} - \log(\beta_k t_k) = (1+o(1))2t_k\log t_k, \tag{2.15}$$

and from (2.12) and (2.14)

$$\log p_{\mathcal{S}}(u_k) \leq t_k\log\left(\frac{e\beta_k}{t_k}\right) + \frac{1}{t_k} + 2\beta_{k-1}\log(t_k t_{k+1}) + \log\beta_k$$
$$= 2t_k\log t_k + t_k + \frac{1}{t_k} + 10t_k^{3/4}\log t_k + 3\log t_k = (1+o(1))2\,t_k\,\log t_k. \tag{2.16}$$

Since, obviously, $q_{\mathcal{S}}(u_k) \leq p_{\mathcal{S}}(u_k)$, Theorem 1 follows from (2.15) and (2.16). □

3. The case $\liminf A(n)/n = \alpha > 0$

First we shall prove (see [9], Theorem 16.1):

Lemma 4. *Let $\mathcal{A}$ be a set of coprime positive integers, α a positive real number such that $\liminf A(n)/n = \alpha$. Then for all ε, $0 < \varepsilon < \alpha$, there exist $n_0 = n_0(\varepsilon)$ such that for $n \geq n_0$ the following inequality holds:*

$$p_{\mathcal{A}}(n) \geq \exp(C\sqrt{(\alpha-\varepsilon)n}), \quad C = \pi\sqrt{\frac{2}{3}} = 2.56. \tag{3.1}$$

Proof: Let us call $\mathcal{P}(\mathcal{A})$ the property

$$\text{For all } a \in \mathcal{A}, \text{ the g.c.d. of the elements of } \mathcal{A} - \{a\} \text{ is } 1. \tag{3.2}$$

It follows from the Bateman-Erdős Theorem (cf. [2]) that, if $\mathcal{A}$ possesses property $\mathcal{P}(\mathcal{A})$, then $p_{\mathcal{A}}(n)$ is increasing from a certain point on. First we assume that $\mathcal{P}(\mathcal{A})$ holds. If we write $\mathcal{A} = \{a_1, a_2, \ldots\}$ with $a_1 < a_2 < \ldots$, then there exists $m_1 = m_1(\varepsilon)$ such that

$$a_m \leq \frac{m}{\alpha - \frac{\varepsilon}{2}}, \quad m \geq m_1. \tag{3.3}$$

Let us define $m = m(n)$ by $a_m \leq n < a_{m+1}$. Then $S(n) = \sum_{i=0}^{n} p_{\mathcal{A}}(i)$ is the number of solutions of

$$x_1 a_1 + \cdots + x_m a_m \leq n \tag{3.4}$$

and, for $m \geq m_1$, this is greater than the number of solutions of

$$x_{m_1} a_{m_1} + \cdots + x_m a_m \leq n.$$

But then from (3.3), $S(n)$ is greater, than the number of solutions S' of

$$m_1 x_{m1} + \cdots + m x_m \leq N = \left\lfloor \left(\alpha - \frac{\varepsilon}{2}\right) n \right\rfloor. \tag{3.5}$$

With any solutions of (3.5) we can associate at most $N^{m_1 - 1}$ solutions of

$$x_1 + 2x_2 + \cdots + m_1 x_{m1} + \cdots + m x_m \leq N. \tag{3.6}$$

By (3.3) (with $m+1$ in place of m) and $n < a_{m+1}$ we have $m + 1 > N$. Thus the number of solutions of (3.6) is $\sum_{i=0}^{N} p(i) \geq p(N)$, and we have from (1.3):

$$S(n) \geq S' \geq \frac{p(N)}{N^{m_1 - 1}} \geq \frac{1}{10} \exp(C\sqrt{N} - m_1 \log N). \tag{3.7}$$

Since $p_{\mathcal{A}}(n)$ is increasing, thus we have $p_{\mathcal{A}}(n) \geq S(n)/n$ which together with (3.7) and the value of N given in (3.5) proves Lemma 4 when $\mathcal{P}(\mathcal{A})$ holds.

Let us assume now that $\mathcal{P}(\mathcal{A})$ does not hold. Then there exists $a_{i\,1}$ such that the g.c.d. of the elements of $\mathcal{A}_1 = \mathcal{A} - \{a_{i\,1}\}$ is $g_1 \geq 2$. If $\mathcal{P}(\frac{1}{g_1}\mathcal{A}_1)$ does not hold, then there exists $a_{i\,2} \geq g_1$ such that the g.c.d. of the elements of $\mathcal{A}_2 = \mathcal{A}_1 \backslash \{a_{i\,2}\}$ is $g_2 \geq 4$, and so on. This process is finite, othervise for any k, we had a sequence $a_{i\,1}, \ldots a_{i\,k} \geq 2^{k-1}$, so that the elements of $\mathcal{A}_k = \mathcal{A} \backslash \{a_{i\,1}, \ldots a_{i\,k}\}$ have a g.c.d. $g_k \geq 2^k$. Then $A(2^{k-1}) \leq k$ for any k and $\lim_{n\to\infty} \frac{A(n)}{n} = 0$, which contradicts our hypothetis.

We may now assume that for some k, $\mathcal{P}(\mathcal{B}_k)$ holds, with $\mathcal{B}_k = \frac{1}{g_k}\mathcal{A}_k = \{b_1, b_2, \ldots\}$. We have $\liminf \frac{B_k(n)}{n} = \alpha g_k$. The numbers $a_{i\,1}, a_{i\,2}, \ldots, a_{i\,k}$ and g_k are coprime (any common divisor would divide all elements of $\mathcal{A}$). It is well-known that then there is n_0 such that any

$n \geq n_0$ can be written in the form

$$n = x_0\, g_k + x_1\, a_{i\,1} + \cdots + x_k\, a_{i\,k}, \quad x_j \geq 0.$$

For n large, let us write $n = n' + n_0 + g$ where $0 \leq g < g_k$ and n' is a multiple of g_k. We have

$$p_{\mathcal{A}}(n) \geq p_{\mathcal{B}_k}\left(\frac{n'}{g_k}\right).$$

But, from the first part of our proof, as $\mathcal{P}(\mathcal{B}_k)$ holds, we have:

$$p_{\mathcal{B}_k}\left(\frac{n'}{g_k}\right) \geq \exp\left(C\sqrt{\alpha g_k - \varepsilon}\sqrt{\frac{n'}{g_k}}\right)$$

and since $n - n' = O(1)$, this completes the proof of Lemma 4. □

Let us prove now:

Lemma 5. *Let $\mathcal{A} = \{a_1, a_2, \ldots\}$ be a set of positive integers and* $\beta = \limsup_{n\to\infty} \frac{A(n)}{n}$. *Then for all positive ε and n large enough, the following inequality holds*:

$$q_{\mathcal{A}}(n) \leq \exp\left(\frac{\pi}{\sqrt{3}}\sqrt{(\beta+\varepsilon)n}\right). \tag{3.8}$$

Proof: We shall follow the proof of Theorem 16.1 of [9]. First there exists $m_2 = m_2(\varepsilon)$ such that

$$m \geq m_2 \Rightarrow a_m \geq \frac{m}{\beta + \varepsilon/2}. \tag{3.9}$$

Let us set $\mathcal{A}_1 = \{a_1, a_2, \ldots, a_{m_2}\}$ and $\mathcal{A}_2 = \{a_{m_2+1}, a_{m_2+2}, \ldots\}$; we have $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$, $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$ so that

$$q_{\mathcal{A}}(n) = \sum_{m=0}^{n} q_{\mathcal{A}_2}(m) q_{\mathcal{A}_1}(n-m). \tag{3.10}$$

Further, $q_{\mathcal{A}_1}(n)$ is the number of solutions of $x_1a_1 + x_2a_2 + \cdots + x_{m_2}a_{m_2} = n$, with $x_i = 0, 1$; thus, for any $n \geq 0$,

$$q_{\mathcal{A}_1}(n) \leq 2^{m_2}. \tag{3.11}$$

Let

$$m = a_{k_1} + a_{k_2} + \cdots + a_{k_r}, \quad m_2 < k_1 < k_2 < \cdots < k_r$$

be a restricted partition of m with parts in $\mathcal{A}_2$; to this partition we associate the restricted partition

$$\nu = k_1 + k_2 + \cdots + k_r, \quad m_2 < k_1 < k_2 < \cdots < k_r,$$

and, from (3.9), $\nu \leq m(\beta + \varepsilon/2)$. This establishes a one-to-one mapping from restricted partitions of m with parts in $\mathcal{A}_2$ to restricted partitions of integers ν less than $m(\beta + \varepsilon/2)$. Since the restricted partition function $q(n)$ is non decreasing, we have

$$q_{\mathcal{A}_2}(m) \leq \sum_{0 \leq \nu \leq m(\beta + \frac{\varepsilon}{2})} q(\nu) \leq \left(1 + \left\lfloor m\left(\beta + \frac{\varepsilon}{2}\right)\right\rfloor\right) q\left(\left\lfloor m\left(\beta + \frac{\varepsilon}{2}\right)\right\rfloor\right).$$

It follows from (3.10) and (3.11) that

$$q_{\mathcal{A}}(n) \leq 2^{m_2} \sum_{m=0}^{n} q_{\mathcal{A}_2}(m) \leq 2^{m_2}(n+1)\left(1 + \left\lfloor n\left(\beta + \frac{\varepsilon}{2}\right)\right\rfloor\right) q\left(\left\lfloor n\left(\beta + \frac{\varepsilon}{2}\right)\right\rfloor\right),$$

which, with (1.4), implies (3.8) and the proof of Lemma 5 is completed. □

Proof of Theorem 2: If the greatest common divisor, say d, of the elements of $\mathcal{A}$ is greater than 1, then dividing every element of $\mathcal{A}$ by d we may reduce the problem to the case when the elements of $\mathcal{A}$ are coprime.

First we remark that, writing $\beta = \limsup_{n\to\infty} \frac{A(n)}{n}$, Lemma 5 implies

$$q_{\mathcal{A}}(n) \leq \exp\left(\frac{\pi}{\sqrt{3}}\sqrt{(\beta + \varepsilon)n}\right) \tag{3.12}$$

for any $\varepsilon > 0$ and n large enough. Since we have assumed that the elements of $\mathcal{A}$ are coprime, it follows from Lemma 4 that, for any $\varepsilon > 0$ and n large enough,

$$p_{\mathcal{A}}(n) \geq \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{(\alpha - \varepsilon)n}\right) \tag{3.13}$$

Inequalities (3.12) and (3.13) prove Theorem 2 when $\beta < 2\alpha$. However, this simple argument cannot be used for $\beta \geq 2\alpha$, so we need a different proof which covers all values of β. From Lemma 4, for n large enough we have.

$$p_{\mathcal{A}}(n) \geq \exp(2.5\sqrt{\alpha n}). \tag{3.14}$$

So, we may assume that, for n large enough,

$$q_{\mathcal{A}}(n) \geq \exp(2.4\sqrt{\alpha n}) \tag{3.15}$$

since othervise (1.9) holds with $\eta(\alpha) = 25/24$.

Now we claim that if (3.15) holds for some n, then there exist $c_2 = c_2(\alpha) > 0$ and $c_3 = c_3(\alpha) > 0$ such that for more than

$$\frac{1}{2}q_{\mathcal{A}}(n) \tag{3.16}$$

restricted $\mathcal{A}$-partitions π of n we have

$$\sum_{\substack{a\in\pi\\ a<c_3\sqrt{n}}} a > c_2 n. \tag{3.17}$$

Indeed, the number of exceptions is less than

$$\sum_{a=0}^{c_2 n} q(a)\varrho(n-a, c_3\sqrt{n}) \le q(c_2 n)\sum_{a=0}^{c_2 n} r(n-a, c_3\sqrt{n}),$$

and by (1.4) and Lemma 1, this is smaller than

$$n\exp\left(\frac{\pi}{\sqrt{3}}\sqrt{c_2 n}\right)\exp\left(\frac{3}{\sqrt{c_3\sqrt{\frac{n}{n-a}}}}\sqrt{n-a}\right) \le n\exp\left(\frac{\pi}{\sqrt{3}}\sqrt{c_2 n} + 3\sqrt{\frac{n}{c_3}}\right)$$

so that for c_3 large enough, and c_2 small enough, it is, in view of (3.15), smaller than $\frac{1}{2}q_{\mathcal{A}}(n)$. One can choose

$$c_2 = \frac{\alpha}{4} \quad \text{and} \quad c_3 = \frac{9}{\alpha}. \tag{3.18}$$

Now, consider all the restricted $\mathcal{A}$-partitions π of n satisfying (3.17). Let $\varepsilon = \varepsilon(\alpha)$ be small enough in terms of α and to be fixed later. Divide the interval $(0, c_3\sqrt{n}]$ into k equal parts where k is an integer which will be fixed later (in (3.25)). Then for $1 \le j \le k$, the length of each interval $I_j = ((j-1)\frac{c_3\sqrt{n}}{k}, j\frac{c_3\sqrt{n}}{k}]$ is $\frac{c_3\sqrt{n}}{k}$. For each of the partitions π satisfying (3.17) let $I(\pi)$ denote that interval I_j for which $\sum_{a\in I_j} a$ is maximal, so that

$$\sum_{a\in I(\pi)} a > \frac{c_2}{k}n. \tag{3.19}$$

By (3.16) and the pigeon hole principle, there is a $h \in \{1, 2, \dots, k\}$ so that

$$I(\pi) = I_h \tag{3.20}$$

holds for at least $\frac{1}{2k}q_{\mathcal{A}}(n)$ of the partitions π satisfying (3.17). Let P denote the set of the restricted $\mathcal{A}$-partitions satisfying (3.17) and (3.20), so that

$$|P| \ge \frac{1}{2k}q_{\mathcal{A}}(n). \tag{3.21}$$

To each $\pi \in P$ assign the partition

$$\pi' = \pi \backslash (\{a : a \in I_h\} \cup \{a : a \leq \varepsilon\sqrt{n}\}).$$

Since, for all j, I_j contains at most $1 + \frac{c_3\sqrt{n}}{k}$ integers, thus $\{a : a \in I_h\}$ can be chosen in at most

$$2^{1+c_3\sqrt{n}/k} < 2^{2c_3\sqrt{n}/k}$$

ways. It follows that writing

$$P' = \{\pi' : \pi \in P\}$$

we have

$$|P'| > |P|\, 2^{-2c_3\sqrt{n}/k}\, 2^{-\varepsilon\sqrt{n}} = |P|\, 2^{-(\varepsilon+2c_3/k)\sqrt{n}}. \tag{3.22}$$

Now, write

$$M = \left\lfloor \frac{\alpha}{2}\varepsilon\sqrt{n} \right\rfloor$$

so that, by (1.8), for n large enough

$$a_1 < a_2 < \cdots < a_M \leq \varepsilon\sqrt{n}. \tag{3.23}$$

Let

$$T = \left\lfloor \frac{2c_2}{c_3}\frac{1}{\varepsilon} \right\rfloor.$$

For some $\pi' \in P'$, consider all the sums

$$\sum_{a\in\pi'} a + \sum_{i=1}^{M} x_i a_i \quad \text{with} \quad 0 \leq x_1, \ldots, x_M \leq T. \tag{3.24}$$

It follows from (3.19), (3.23) and (3.24) that

$$\sum_{i=1}^{M} x_i a_i \leq TM\varepsilon\sqrt{n} \leq \frac{2c_2}{c_3} \cdot \frac{1}{\varepsilon} \cdot \frac{\alpha}{2}\varepsilon\sqrt{n} \cdot \varepsilon\sqrt{n} = \frac{c_2}{c_3}\alpha\varepsilon n < \sum_{a\in I(h)} a$$

by choosing k so that

$$k = \left\lfloor \frac{c_3}{\alpha\varepsilon} \right\rfloor \geq \frac{c_3}{2\alpha\varepsilon}. \tag{3.25}$$

It follows that the sum in (3.24) is smaller than n so that this sum forms an *unrestricted* partition of some m with $m < n$. Since for each $\pi' \in P'$ there are

$$|P'|\,(T+1)^M > |P'|\left(\frac{2c_2}{c_3}\cdot\frac{1}{\varepsilon}\right)^{\lfloor\frac{\alpha}{2}\varepsilon\sqrt{n}\rfloor} > |P'|\exp\left(\frac{\alpha}{4}\varepsilon\left(\log\frac{2c_2}{c_3\varepsilon}\right)\sqrt{n}\right)$$

partitions of form (3.24), we have from (3.21), (3.22) and (3.25):

$$\begin{aligned}\sum_{m\le n} p_{\mathcal{A}}(m) &\ge |P'|\exp\left(\frac{\alpha}{4}\varepsilon\left(\log\frac{2c_2}{c_3\varepsilon}\right)\sqrt{n}\right)\\ &\ge \frac{1}{2k}q_{\mathcal{A}}(n)\exp\left\{\left(\frac{\alpha}{4}\varepsilon\left(\log\frac{2c_2}{c_3\varepsilon}\right)-2\frac{c_3}{k}-\varepsilon\right)\sqrt{n}\right\}\\ &\ge \frac{1}{2k}q_{\mathcal{A}}(n)\exp\left\{\varepsilon\left(\frac{\alpha}{4}\log\frac{2c_2}{c_3\varepsilon}-(4\alpha+1)\right)\sqrt{n}\right\}.\end{aligned}$$

By choosing $\varepsilon = \frac{2c_2}{c_3}\exp(-17-\frac{4}{\alpha})$, for all large n it follows

$$\begin{aligned}\sum_{m\le n} p_{\mathcal{A}}(m) &> \frac{1}{2k}q_{\mathcal{A}}(n)\exp\left\{\left(\frac{2c_2}{c_3}\frac{\alpha}{4}\exp\left(-17-\frac{4}{\alpha}\right)\right)\sqrt{n}\right\}\\ &> q_{\mathcal{A}}(n)\exp\left\{\left(\frac{c_2}{c_3}\frac{\alpha}{4}\exp\left(-17-\frac{4}{\alpha}\right)\right)\sqrt{n}\right\}.\end{aligned} \tag{3.26}$$

It follows from (1.4) and (3.26) that

$$\sum_{m\le n} p_{\mathcal{A}}(m) > q_{\mathcal{A}}(n)q(n)^{2\eta} \ge q_{\mathcal{A}}(n)^{1+2\eta} \tag{3.27}$$

with, from (3.18),

$$\eta = \frac{c_2}{c_3}\frac{\alpha}{16}\exp\left(-17-\frac{4}{\alpha}\right) = \frac{\alpha^3}{576}\exp\left(-17-\frac{4}{\alpha}\right).$$

Since now property $\mathcal{P}(\mathcal{A})$ in (3.2) is assumed, thus we have $p_{\mathcal{A}}(n+1) > p_{\mathcal{A}}(n)$ for n large enough, whence

$$(n+1)p_{\mathcal{A}}(n) \ge \sum_{0\le m\le n} p_{\mathcal{A}}(m) \tag{3.28}$$

and (1.9) follows from (3.27) and (3.28). □

If $\mathcal{P}(\mathcal{A})$ does not hold, then we have seen in the proof of Lemma 4 that $\mathcal{A}$ can be written in the form $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}''$, $\mathcal{A}' \cap \mathcal{A}'' = \emptyset$, $\mathcal{A}'$ finite, $\mathcal{A}'' = g\mathcal{B}$, where g is the g.c.d. of the elements of $\mathcal{A}''$. In the constuction of π' we keep the parts belonging to $\mathcal{A}'$, we remove those parts from $\mathcal{A}''$ which are either smaller than $\varepsilon\sqrt{n}$ or belong to I_h, and we replace them by the elements $a_1, \dots, a_M$ belonging to $\mathcal{A}''$. All the sums obtained in (3.24) are congruent to n mod g, and since $\mathcal{P}(\mathcal{B})$ is true thus (3.28) follows, and we can conclude similarly.

4. Proof of Proposition 1

We will prove (1.16), the proof of (1.15) is similar. The proof follows the proof of (1.13) as given in [11]. We use the notation and the results of [11]:

$$F(x) = \int_x^\infty \frac{u}{1+e^u}\,du, \tag{4.1}$$

$$F(x) = \frac{\pi^2}{12} - \frac{x^2}{4} + O(x^3) \quad \text{as } x \to 0, \tag{4.2}$$

$$F(x) = (x+1)e^{-x} + O(xe^{-2x}) \quad \text{as } x \to \infty, \tag{4.3}$$

$$G(x) = \frac{x}{\sqrt{F(x)}} \text{ is increasing for } x \ge 0, \tag{4.4}$$

$$H \text{ is the inverse function of } G, \tag{4.5}$$

and for $\lambda \to \infty$, H satisfies

$$H(\lambda) = 2\log\lambda - \log\log\lambda - \log 2 + O\left(\frac{\log\log\lambda}{\log\lambda}\right). \tag{4.6}$$

Finally $h(\lambda)$, defined in (1.13), is equal to:

$$h(\lambda) = \frac{2H(\lambda)}{\lambda} - \lambda\log\left(1 + e^{-H(\lambda)}\right). \tag{4.7}$$

Here for $x \in \mathbb{R}$ we define

$$F_2(x) = F(x) - F(2x) = \int_x^{2x} \frac{u}{1+e^u}\,du \tag{4.8}$$

(note that, for $x > 0$, $F_2(-x) = 3x^2 - F_2(x)$ and $F_2(-x) \ge 0$) and

$$G_2(x) = \frac{x}{\sqrt{F_2(x)}}. \tag{4.9}$$

It follows from (4.2) and (4.8) that $G_2(0^+) = \frac{2}{\sqrt{3}}$. Now, we observe that if

$$\sum_{\lambda\sqrt{n} \le m < 2\lambda\sqrt{n}} m < n,$$

then we have $\varrho(n; \lambda\sqrt{n}, 2\lambda\sqrt{n}) = 0$. Hence, for $\lambda < \sqrt{\frac{2}{3}}$, (1.16) holds with $h_2(\lambda) = 0$. Further we set, for $s \in \mathbb{R}$,

$$F_2(x,s) = \int_x^{2x} \frac{u\,du}{1+e^{us}} = \frac{1}{s^2}F_2(sx). \tag{4.10}$$

Clearly, for x fixed, $F_2(x, s)$ is a decreasing function of s, and

$$\lim_{s\to-\infty} F_2(x, s) = \frac{3x^2}{2}, \quad F_2(x, 0) = \frac{3x^2}{4} \quad \text{and} \quad \lim_{s\to+\infty} F_2(x, s) = 0.$$

So, for $x \geq \sqrt{\frac{2}{3}}$, there is a unique value $s = s(x)$ such that $F_2(x, s(x)) = 1$. For $\lambda \geq \sqrt{2/3}$, we define

$$H_2(\lambda) = \lambda s(\lambda) \tag{4.11}$$

so that, from (4.10), we have

$$F_2(H_2(\lambda)) = \frac{H_2(\lambda)^2}{\lambda^2}. \tag{4.12}$$

It follows from (4.9) and (4.12) that, for $H_2(\lambda) > 0$ (i.e. for $\lambda > \frac{2}{\sqrt{3}}$) we have

$$G_2(H_2(\lambda)) = \lambda \tag{4.13}$$

and since $G_2(x)$, defined by (4.9), is increasing for x large enough, G_2 and H_2 are inverse in a neighborood of $+\infty$.

Since, from (4.3), for x large $F(2x)$ is much smaller than $F(x)$, $G_2(x)$ is close to $G(x)$, and it could be shown by a little computation (we leave the details to the reader) that $H_2(\lambda)$ satisfies the same asymptotic expansion as $H(\lambda)$ if $\lambda \to \infty$:

$$H_2(\lambda) = 2\log\lambda - \log\log\lambda - \log 2 + O\left(\frac{\log\log\lambda}{\log\lambda}\right). \tag{4.14}$$

Finally, for $\lambda > \sqrt{\frac{2}{3}}$ we set

$$h_2(\lambda) = \frac{2H_2(\lambda)}{\lambda} + 2\lambda\log\bigl(1 + e^{-2H_2(\lambda)}\bigr) - \lambda\log\bigl(1 + e^{-H_2(\lambda)}\bigr), \tag{4.15}$$

and, from (4.14), $h_2(\lambda)$ is asymptotic to (1.14) as $\lambda \to +\infty$. Note that expression (4.15) appears in formula (50) in [11]. When $\lambda \to \sqrt{2/3}$, with $\lambda > \sqrt{2/3}$, then $H_2(\lambda) \to -\infty$, and a simple calculation shows that $h_2(\lambda) \to 0$.

We now prove:

Lemma 6. *Let $\lambda > \sqrt{2/3}$, and $h_2(\lambda)$ defined by* (4.15). *For $n \geq 2$ we have*

$$\limsup_{n\to\infty} \frac{\log \varrho(n; \lambda\sqrt{n}, 2\lambda\sqrt{n})}{\sqrt{n}} \leq h_2(\lambda). \tag{4.16}$$

Proof: It is the same proof as the proof of Proposition 1 in [11]. We start from the generating function:

$$\sum_{n=0}^{\infty} \varrho(n; x, 2x) z^n = \prod_{x \le m < 2x} (1 + z^m)$$

which for real positive z yields

$$\varrho(n; \lambda\sqrt{n}, 2\lambda\sqrt{n}) \le z^{-n} \prod_{\lambda\sqrt{n} \le m < 2\lambda\sqrt{n}} (1 + z^m)$$

and

$$\log(\varrho(n; \lambda\sqrt{n}, 2\lambda\sqrt{n})) \le -n \log z + \sum_{\lambda\sqrt{n} \le m < 2\lambda\sqrt{n}} \log(1 + z^m).$$

Here we chose z as

$$z = \exp\left(-\frac{H_2(\lambda)}{\lambda\sqrt{n}}\right).$$

By comparing the above sum to the corresponding integral, we can show exactly in the same way as in [11] that, when $H_2(\lambda) \ge 0$, $\log \varrho(n; \lambda\sqrt{n}, 2\lambda\sqrt{n}) \le h_2(\lambda)\sqrt{n} + 1$, while, if $H_2(\lambda) < 0$ (i.e., for $\sqrt{\frac{2}{3}} < \lambda < \frac{2}{\sqrt{3}}$), by a slightly different argument it can be shown that $\log \varrho(n; \lambda\sqrt{n}, 2\lambda\sqrt{n}) \le h_2(\lambda)\sqrt{n} + 3\,|H_2(\lambda)| + 1$. In both cases, (4.16) follows, and the proof of Lemma 6 is completed. □

To prove (1.16), from Lemma 6 we need to show

$$\liminf_{n\to\infty} \frac{\log \varrho(n; \lambda\sqrt{n}, 2\lambda\sqrt{n})}{\sqrt{n}} \ge h_2(\lambda). \tag{4.17}$$

As mentionned in the introduction, (4.17) can be proved by the methods of [5] or [11], Section 3 or 5, or by analytical methods.

Finally, from (4.14) and (4.15), it follows that, for $\lambda \to \infty$, the asymptotic expansion of $h_2(\lambda)$ is (1.14).

5. Proof of Theorem 3

Let $(\lambda_k)_{k\ge1}$ be a non-decreasing sequence of integers satisfying $3 \le \lambda_1 \le \lambda_2 \le \dots$. With this sequence we associate the sequence $n_0 = 1,\ n_k = \lambda_k n_{k-1}$ for $k \ge 1$.

The set $\mathcal{A}$ is defined as

$$\mathcal{A} = \{1\} \cup \bigcup_{k\ge1} \{n_k, n_k + 1, \dots, 2n_k - 1\}.$$

In order to satisfy (1.11), we chose $\lambda_1, \lambda_2, \dots$ by induction so that for k large enough, $\lambda_{k+1} < \frac{1}{2f(n_k)}$. Indeed, then for $2n_k \le n < 2n_{k+1}$ we have

$$nf(n) \le 2n_{k+1} f(n_k) = 2\lambda_{k+1} n_k f(n_k) < n_k \le A(n)$$

whence (1.10) follows.

Let λ a fixed, but large, positive real number. We now choose, for $k \to \infty$, an integer $N = N_k$ defined as

$$N = N_k = \left\lfloor \frac{n_k^2}{\lambda^2} \right\rfloor. \tag{5.1}$$

A simple calculation shows that, for k large enough, $n_k - \frac{1}{2} < \lambda\sqrt{N} \le n_k$ holds, and we have

$$q_{\mathcal{A}}(N) \ge \varrho(N; n_k, 2n_k) = \varrho(N; \lambda\sqrt{N}, 2\lambda\sqrt{N})$$

and, from (5.1), Proposition 1 and (1.14), we can choose λ large enough so that, for k large enough, we have

$$\log q_{\mathcal{A}}(N_k) = \log q_{\mathcal{A}}(N) \ge \left(\frac{2\log\lambda - \log\log\lambda}{\lambda}\right)\sqrt{N}. \tag{5.2}$$

Further,

$$p_{\mathcal{A}}(N) = \sum_{N'+N''+N'''=N} P_1 P_2 P_3 \tag{5.3}$$

where

P_1 is the number of partitions of N' into parts in $\mathcal{A}$ and less than n_k,
P_2 is the number of partitions of N'' into parts in $\mathcal{A}$ and between n_k and $2n_k$,
P_3 is the number of partitions of N''' into parts greater than n_{k+1}.

From the definition of $\mathcal{A}$, we have

$$P_2 = r(N''; n_k, 2n_k) \le r(N'', n_k) = r(N'', \lambda\sqrt{N}) = r(N''; \lambda''\sqrt{N''})$$

with $\lambda'' = \lambda\sqrt{\frac{N}{N''}} \ge \lambda$, and thus from Lemma 1:

$$\log(P_2) \le \frac{2\log\lambda'' + 3}{\lambda''}\sqrt{N''} \le \frac{2\log\lambda + 3}{\lambda}\sqrt{N''} \le \frac{2\log\lambda + 3}{\lambda}\sqrt{N} \tag{5.4}$$

holds. Further, we have

$$P_1 \le p(N', 2n_{k-1}) = p\left(N', \frac{2n_k}{\lambda_k}\right) \le p\left(N', \frac{2\lambda\sqrt{N}}{\lambda_k}\right) = p(N', \lambda'\sqrt{N}')$$

with $\lambda' = \frac{2\lambda}{\lambda_k}\sqrt{\frac{N}{N'}}$. Therefore, from Lemma 1,

$$\log P_1 \le 3\sqrt{\lambda' N'} = 3\sqrt{\frac{2\lambda}{\lambda_k}\sqrt{NN'}} \le 3\sqrt{\frac{2\lambda}{\lambda_k}N} \le 5\sqrt{\frac{\lambda}{\lambda_k}}\sqrt{N}. \tag{5.5}$$

Finally, since $n_{k+1} = \lambda_{k+1} n_k \ge \lambda\lambda_k\sqrt{N}$, we have

$$P_3 \le r(N''', n_{k+1}) \le r(N''', \lambda\lambda_k\sqrt{N}) = r(N''', \lambda'''\sqrt{N'''})$$

with $\lambda''' = \lambda\lambda_k\sqrt{\frac{N}{N'''}} \ge \lambda\lambda_k$. So, from Lemma 1,

$$\log P_3 \le \frac{3}{\sqrt{\lambda'''}}\sqrt{N'''} \le \frac{3}{\sqrt{\lambda\lambda_k}}\sqrt{N'''} \le \frac{3}{\sqrt{\lambda\lambda_k}}\sqrt{N} \tag{5.6}$$

holds. Since the number of terms in the sum (5.3) is $\binom{N}{2} \le N^2$, it follows from (5.3), (5.4), (5.5) and (5.6) that

$$\log p_{\mathcal{A}}(N_k) = \log p_{\mathcal{A}}(N) \le \left(\frac{2\log\lambda + 3}{\lambda} + 5\sqrt{\frac{\lambda}{\lambda_k}} + \frac{3}{\sqrt{\lambda\lambda_k}}\right)\sqrt{N} + 2\log N \tag{5.7}$$

which together with (5.2) yields, for k large enough,

$$\frac{\log p_{\mathcal{A}}(N_k)}{\log q_{\mathcal{A}}(N_k)} \le \frac{\left(\frac{2\log\lambda+3}{\lambda} + 5\sqrt{\frac{\lambda}{\lambda_k}} + \frac{3}{\sqrt{\lambda\lambda_k}}\right)\sqrt{N_k}}{\left(\frac{2\log\lambda - \log\log\lambda}{\lambda}\right)\sqrt{N_k}} + \frac{2\log N_k}{\left(\frac{2\log\lambda - \log\log\lambda}{\lambda}\right)\sqrt{N_k}}.$$

When $k \to \infty$, $\lambda_k \to \infty$ and we have

$$\liminf_{n\to\infty} \frac{\log p_{\mathcal{A}}(n)}{\log q_{\mathcal{A}}(n)} \le \frac{2\log\lambda + 3}{2\log\lambda - \log\log\lambda}.$$

But λ can be choosen as large as we wish so that (1.11) holds, and the proof of Theorem 3 is completed.

References

1. G.E. Andrews, *The Theory of Partitions, Encyclopedia of Mathematics and its Applications*, Vol. 2, Addison Wesley, 1976.
2. P.T. Bateman and P. Erdős, "Monotonicity of partition functions," *Mathematika* **3** (1956), 1–14.
3. L. Comtet, *Analyse Combinatoire*, Vol. 1, PUF, Paris, 1970.
4. J. Dixmier and J.-L. Nicolas, "Partitions without small parts," *Coll. Math. Soc. J. Bolyai* **51** (1987), 9–33.
5. J. Dixmier and J.-L. Nicolas, "Partitions sans petits sommants," in *A Tribute to Paul Erdős* (A. Baker, B. Bollobás, and A. Hajnal, eds.), Cambridge University Press, 1990, pp. 121–152.
6. P. Erdős, J.-L. Nicolas, and A. Sárközy, "On the number of partitions without a given subsum II," in *Analytic Number Theory* (B. Berndt, H. Diamond, H. Halberstam, and A. Hildebrand, eds.), Birkhäuser, 1990, pp. 205–234.

7. G.H. Hardy and S. Ramanujan, "Asymptotic formulae for the distribution of integers of various types," *Proc. London Math. Soc.* **16**(2) (1917), 112–132 and Collected Papers of S. Ramanujan 245–261.
8. G.H. Hardy and S. Ramanujan, "Asymptotic formulae in combinatory analysis," *Proc. London Math. Soc.* **17**(2) (1918), 75–115 and Collected Papers of S. Ramanujan, 276–309, Cambridge University Press, 1927, reprinted by Chelsea, New-York, 1962.
9. M.B. Nathanson, *Elementary Methods in Number Theory, Graduate Texts in Mathematics*, No. 195, Springer-Verlag, 2000.
10. J.-L. Nicolas and A. Sárközy, "On the asymptotic behaviour of general partition functions," *The Ramanujan Journal* **4** (2000), 29–39.
11. J.-L. Nicolas and A. Sárközy, "On two partition problems," *Acta Math. Hungar*. **77** (1997), 95–121.
12. J.H. van Lint and R.M. Wilson, *A Course in Combinatorics*, Cambridge University Press, 1975.