```
> read "phisig2.mpl": read "lispithetaphi.m":
>  # phisig2.mpl contains the procedures
   # lispithetaphi, list of 6698 primorials [p,pi(p),log(M)=theta
  (p),M/phi(M)]
   # allowing to compute all primorials with largest prime factor
  <=  2000530391
```

```
> Digits:=50:delta:=evalf(exp(gamma)*(4+gamma-log(4*Pi))); # proof
  of (1.5)
  egam:=exp(evalf(gamma));
```

$\delta := 3.6444150964073701410651161928351481600522602466432$

$egam := 1.7810724179901979852365041031071795491696452143034$ **(1)**

```
> k:=120568: pk:=ithprime(k): lispk:=thetansphidex(pk):
  # lispk is equal to [pk,theta(pk),pi(pk),M_{pk}/phi(M_{pk})]
  logMpk:=lispk[2]: "k=",k,"p_k=",pk,"log M_{p_k}=", logMpk; #
  proof of (1.7)
  MpksphiMpk:=lispk[4];
```

"k=", 120568, "p_k=", 1591873, "log M_{p_k}=",

$1.5901716359732498491839002117855197255307208199851\ 10^{6}$

$MpksphiMpk :=$ **(2)**

$25.435450967660792417740307516130753659522786573757$

```
> pkp1:=ithprime(k+1): pkp2:=ithprime(k+2): pkm10:=ithprime(k-10):
  "pkp1=",pkp1,"pkp2=",pkp2,"pkm10=",pkm10;
  logA:=logMpk+log(evalf(pkp1*pkp2/pk/pkm10)); # proof of (1.8)
  AsphiA:=MpksphiMpk*evalf((pkp1/(pkp1-1))*(pkp2/(pkp2-1))/(pk/
  (pk-1))/(pkm10/(pkm10-1)));
  cA:=(AsphiA-egam*log(logA))*sqrt(logA); # proof of (1.9)
```

"pkp1=", 1591883, "pkp2=", 1591901, "pkm10=", 1591697

$logA := 1.5901716361076886219247158578229223864019648609209\ 10^{6}$

$AsphiA := 25.435450965512590892512283387263655114247428990011$

$cA := 3.6444151157099992864981182052257513679328189664401$ **(3)**

```
> ######### Sect. 2.1 super champions for n/phi(n) #########
  ##########################################################

  Digits:=7: M:=1: MsphiM:=1: # computation of (2.2) and of Fig. 1
  for i from 1 to 7 do
     p:=ithprime(i): epshat_i:= evalf(log(p/(p-1))/log(p)):
     M:=M*p; MsphiM:=MsphiM*p/numtheory[phi](p):
     print("i=",i,"p=",p,"epshat_i=",epshat_i,"M=",M,"M/phi(M)=",
  MsphiM,
           "[eps_",i+1,"eps_",i,"]");
  od: M:='M': MsphiM:='MsphiM':
```

"i=", 1, "p=", 2, "epshat_i=", 1., "M=", 2, "M/phi(M)=", 2, "[eps_", 2, "eps_", 1, "]" "i=", 2, "p=", 3, "epshat_i=", 0.3690703, "M=", 6, "M/phi(M)=", 3, "[eps_", 3, "eps_", 2, "]"

"i=", 3, "p=", 5, "epshat_i=", 0.1386469, "M=", 30, "M/phi(M)=", $\frac{15}{4}$, "[eps_", 4, "eps_", 3, "]"

"i=", 4, "p=", 7, "epshat_i=", 0.07921795, "M=", 210, "M/phi(M)=", $\frac{35}{8}$, "[eps_", 5, "eps_", 4, "]"

"i=", 5, "p=", 11, "epshat_i=", 0.03974744, "M=", 2310, "M/phi(M)=", $\frac{77}{16}$, "[eps_", 6, "eps_", 5, "]"

"i=", 6, "p=", 13, "epshat_i=", 0.03120623, "M=", 30030, "M/phi(M)=", $\frac{1001}{192}$, "[eps_", 7, "eps_", 6, "]"

"i=", 7, "p=", 17, "epshat_i=", 0.02139783, "M=", 510510, "M/phi(M)=", $\frac{17017}{3072}$, "[eps_", 8, "eps_", 7, "]" **(4)**

```
> ################## Sect. 3 Proof of Theorem 1.1 ###############
  ################################################################
```

```
> Digits:=50:
  eps:=evalf(log(pkp1/(pkp1-1))/log(pkp1)); # proof of (3.1)
  evalf(log(pk/(pk-1))/log(pk));
```

$eps := 4.3989372112566484371147188052990163267941828644178\ 10^{-8}$

$4.3989667800682590602775943520186467188570631945715\ 10^{-8}$ **(5)**

```
> lispk:=thetansphidex(pk); lispkp1:=thetansphidex(pkp1);
  #[p,theta(p)=log M_p,pi(p),M_p/phi(M_p)]
```

$lispk := [1591873, 1.5901716359732498491839002117855197255307208199851\ 10^{6}, 120568, 25.435450967660792417740307516130753659522786573757]$

$lispkp1 := [1591883, 1.5901859164014000726343881285507153047254008941605\ 10^{6}, 120569, 25.435466945887173305766189924693395947552715628038]$ **(6)**

```
> logMpk:=lispk[2];MpksphiMpk:=lispk[4];cMpk:=(MpksphiMpk-egam*log
  (logMpk))*sqrt(logMpk);
```

$logMpk := 1.5901716359732498491839002117855197255307208199851\ 10^{6}$

$MpksphiMpk := 25.435450967660792417740307516130753659522786573757$

(7)

$cMpk := 3.644418014363336599268320540433345402065457899 6841$ **(7)**

```
> logMpkp1:=lispkp1[2];Mpkp1sphiMpkp1:=lispkp1[4];
  cMpkp1:=(Mpkp1sphiMpkp1-egam*log(logMpkp1))*sqrt(logMpkp1);
```

$logMpkp1 :=$

$1.5901859164014000726343881285507153047254008941605\ 10^6$

$Mpkp1sphiMpkp1 :=$

$25.435466945887173305766189924693395947552715628038$

$cMpkp1 := 3.6444135698227070808123288480569082249632313486421$ **(8)**

```
> lam:=evalf(4+gamma-log(4*Pi));delta:=lam*evalf(exp(gamma)); # cf.
  (3.2)
```

$lam := 2.0461914179322420676286204958129905832438642543041$

$\delta := 3.6444150964073701410651161928351481600522602466432$ **(9)**

```
> ##################### proof of Lemma 3.1 ##############

  lambda:='lambda': g:=ep*t-log(log(t)+lambda/sqrt(t)); # cf. 3.3
  g1:=diff(g,t); g2:=diff(g1,t); # cf. (3.4 and (3.5)
  "9*lambda^2/16=",evalf(9*lam^2/16,10);
```

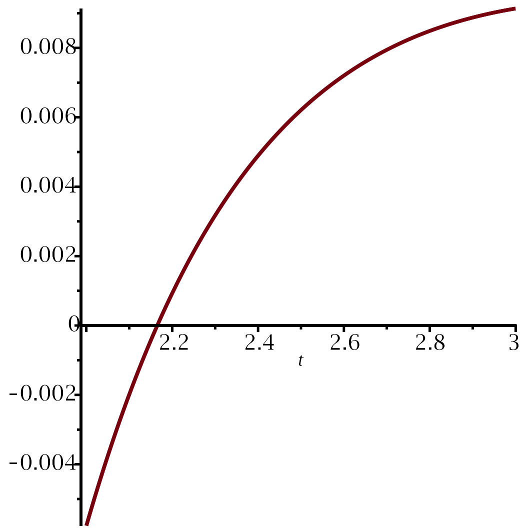
$$g := ep\,t - \ln\left(\ln(t) + \frac{\lambda}{\sqrt{t}}\right)$$

$$g1 := ep - \frac{\frac{1}{t} - \frac{\lambda}{2\,t^{3/2}}}{\ln(t) + \frac{\lambda}{\sqrt{t}}}$$

$$g2 := -\frac{-\frac{1}{t^2} + \frac{3\,\lambda}{4\,t^{5/2}}}{\ln(t) + \frac{\lambda}{\sqrt{t}}} + \frac{\left(\frac{1}{t} - \frac{\lambda}{2\,t^{3/2}}\right)^2}{\left(\ln(t) + \frac{\lambda}{\sqrt{t}}\right)^2}$$

"9*lambda^2/16=", 2.355130867 **(10)**

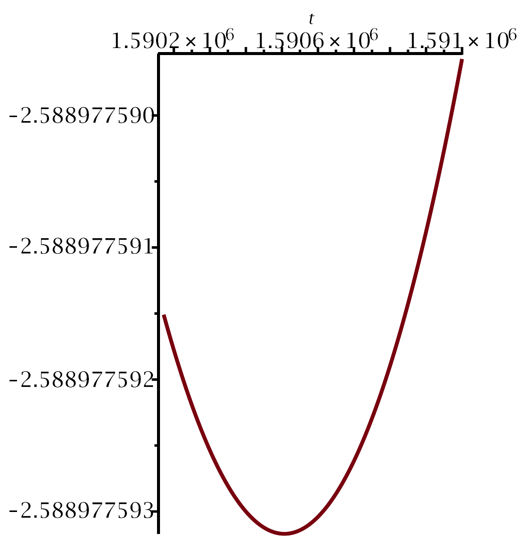
```
> plot(subs(lambda=lam,g2),t=2..3);
```




```
> evalf(fsolve(subs(lambda=lam,g2)=0,t=2..2.4),10);
```

$$2.165252580 \tag{11}$$

```
> plot(subs(lambda=lam,ep=eps,g),t=logMpk..1591000);
  "g'(log M_{p_k})=", evalf(subs(ep=eps,lambda=lam,t=logMpk,g1),20)
  ;
  "g'(log M_{p_{k+1}})=", evalf(subs(ep=eps,lambda=lam,t=logMpkp1,
  g1),20);
  "The minimum of g is attained on t=",
  evalf(fsolve(subs(ep=eps,lambda=lam,g1)=0,t=1590000..1591000),20)
  ;
```



$$\text{"g'(log M\_\{p\_k\})=", } -9.914696854038474\,10^{-12}$$

$$\text{"g'(log M\_\{p\_\{k+1\}\})=", } -9.492082596714804\,10^{-12}$$

$$\text{"The minimum of g is attained on t=", } 1.5905067305279091792\,10^{6} \tag{12}$$

```
> ######################## proof of Lemma 3.2
  #######################

  maxg:=evalf(subs(ep=eps,lambda=lam,t=logMpk,g));
  # maxg = log(M_{p_k}) is the max of g on the interval [log(M_
```

```
{p_k}),log(M_{p_{k+1}})]
majbenn:=maxg-evalf(gamma)+log(MpksphiMpk)-eps*logMpk; # Proof of
(3.8)
# majbenn is
```

maxg := −2.5889775915083028465124934706885910104330884534497

majbenn := 9.0974000017586434290536712028474719100935 10^−11 **(13)**

```
> ####################### proof of Lemma 3.3
#######################
```

```
> for i from 0 to 15 do print("i=",i,"p_{k+i}=",ithprime(k+i),"p_
{k-i}=",ithprime(k-i)); od;
```

"i=", 0, "p_{k+i}=", 1591873, "p_{k-i}=", 1591873
"i=", 1, "p_{k+i}=", 1591883, "p_{k-i}=", 1591871
"i=", 2, "p_{k+i}=", 1591901, "p_{k-i}=", 1591859
"i=", 3, "p_{k+i}=", 1591913, "p_{k-i}=", 1591841
"i=", 4, "p_{k+i}=", 1591921, "p_{k-i}=", 1591813
"i=", 5, "p_{k+i}=", 1591927, "p_{k-i}=", 1591787
"i=", 6, "p_{k+i}=", 1591949, "p_{k-i}=", 1591783
"i=", 7, "p_{k+i}=", 1591969, "p_{k-i}=", 1591753
"i=", 8, "p_{k+i}=", 1591973, "p_{k-i}=", 1591729
"i=", 9, "p_{k+i}=", 1591981, "p_{k-i}=", 1591721
"i=", 10, "p_{k+i}=", 1592027, "p_{k-i}=", 1591697
"i=", 11, "p_{k+i}=", 1592047, "p_{k-i}=", 1591663
"i=", 12, "p_{k+i}=", 1592051, "p_{k-i}=", 1591637
"i=", 13, "p_{k+i}=", 1592069, "p_{k-i}=", 1591631
"i=", 14, "p_{k+i}=", 1592081, "p_{k-i}=", 1591621
"i=", 15, "p_{k+i}=", 1592099, "p_{k-i}=", 1591589 **(14)**

```
> Digits:=50:
Benp(eps,1591663,1,0); #  computation of ben_eps(M_{p_k}/p_{k-11}
) cf. (3.12)
```

9.2907978283197872287384991276240349401149957 88 10^−11 **(15)**

```
> Benp(eps,2,1,2); # computation of ben_eps{2M_{p_k}), cf. (3.13)
```

3.0491109254427743783015548686244769709745065734238 10^−8 **(16)**

```
> Benp(eps,ithprime(k+1),0,2); # computation of ben_eps(M_{p_k}*p_
{k+1}^2), cf. (3.14)
```

6.2818706782694883882197047714605813242611470198405 10^−7 **(17)**

```
> Benp(eps,ithprime(k+15),0,1); # computation of ben_eps(M_{p_k}*p_
{k+15}) cf. (3.15)
Benp(eps,ithprime(k+14),0,1); # computation of ben_eps(M_{p_k}*p_
{k+14})
```

$9.1194579134978370483906384767351889183533132 56\ 10^{-11}$

$8.3595945068602688396062894950730204807012912 05\ 10^{-11}$ **(18)**

```
> pkp14:=ithprime(k+14): evalf(pkp14/pk)^14; # proof of r=s in
  (3.16)
```

1.001830846113614343368329642358705338754168435 7934 **(19)**

```
> Bnumden:=proc(r) local Bnum, Bden,i; global eps; global k;
  Bnum:=0.; Bden:=0.;
  for i from 1 to r do Bnum:=Bnum+Benp(eps,ithprime(k+i),0,1); od;
  for i from 1 to r do Bden:=Bden+Benp(eps,ithprime(k-i+1),1,0);
  od;
  Bnum+Bden;
  end:
> Bnumden(4),Bnumden(5); # proof of r <= 4 in Lemma 3.3
```

$7.3471920120990702147539365341420070327755203 39\ 10^{-11}$, **(20)**

$1.2160947002823435477748103147693764876887235 993\ 10^{-10}$

```
>  ##################  computation of Theorem 1.1
  ############################
> Digits:=30: ti:=time[real](): lisresuo30:=benef(1591873,Float
  (9.1,-11),4,0): time[real]()-ti;
  # lissupo[r] is the ordered list of n/M_{p_k} =q_1..q_r
  # lisinfpo[s] is the ordered list of n/M_{p_k} =1/(q'_1..q'_s)
```

"r=", 1, "nops(lissupo[r])=", 14

"r=", 2, "nops(lissupo[r])=", 55

"r=", 3, "nops(lissupo[r])=", 93

"r=", 4, "nops(lissupo[r])=", 74

"s=", 1, "nops(lisinfo[s])=", 11

"s=", 2, "nops(lisinfo[s])=", 37

"s=", 3, "nops(lisinfo[s])=", 50

"s=", 4, "nops(lisinfo[s])=", 25

"number of numbers with ben < benmax =", 882

"number of numbers n satisfying M_{p_k} < n < M_{p_{k+1}} and c(n) > delta =", 882

5.841 **(21)**

```
> for ter in lisresuo30[1..10] do print(ter[1],evalf(ter[2..5],20))
  ; od;
  # [1, 2, "/", 0, 10]    means p_{k+1}*p_{k+2}/(p_{k-0}*p_{k-10})
```

[1, 2, "/", 0, 10], $[1.5901716361076886219\ 10^{6}$, 25.435450965512590893, 3.6444151157099992865, $9.0370863162683380770\ 10^{-11}]$

[1, 2, "/", 3, 8], $[1.5901716361076868033\ 10^{6}$, 25.435450965512649013, 3.6444151157858569183, $9.0368498144271909394\ 10^{-11}]$

[2, 3, "/", 1, 8], $\left[1.59017163610768630 61\ 10^{6}, 25.435450965512664903,\right.$
$\left.3.644415115806595 6839, 9.0367851570404228982\ 10^{-11}\right]$
[1, 2, "/", 5, 6], $\left[1.5901716361076855672\ 10^{6}, 25.435450965512688517,\right.$
$\left.3.6444151158374164023, 9.0366890670825189787\ 10^{-11}\right]$
[1, 2, 3, "/", 1, 2, 7], $\left[1.5901716361076850747\ 10^{6}, 25.435450965512704253,\right.$
$\left.3.6444151158579556122, 9.0366250318517055054\ 10^{-11}\right]$
[1, 2, 5, "/", 0, 1, 7], $\left[1.5901716361076847764\ 10^{6}, 25.435450965512713787,\right.$
$\left.3.6444151158703985006, 9.0365862385761998023\ 10^{-11}\right]$
[3, 5, "/", 0, 7], $\left[1.5901716361076846344\ 10^{6}, 25.435450965512718327,\right.$
$\left.3.6444151158763236966, 9.0365677655534504245\ 10^{-11}\right]$
[1, 5, "/", 4, 6], $\left[1.5901716361076843975\ 10^{6}, 25.435450965512725896,\right.$
$\left.3.6444151158862036976, 9.0365369626088208926\ 10^{-11}\right]$
[2, 3, "/", 4, 5], $\left[1.5901716361076843833\ 10^{6}, 25.435450965512726351,\right.$
$\left.3.6444151158867970839, 9.0365351126043327888\ 10^{-11}\right]$
[1, 2, 3, "/", 2, 3, 6], $\left[1.5901716361076840329\ 10^{6}, 25.435450965512737550,\right.$ **(22)**
$\left.3.6444151159014135272, 9.0364895428224668139\ 10^{-11}\right]$

```
> nops(lisresuo30),lisresuo30[882];
```

882, [[1, "/", 0], $1.590171635979531737620094998 79\ 10^{6}$, **(23)**
25.435450967560418666939 2823969,
3.6444178789246427314892284 5059,
$4.2225510394502129082731 03\ 10^{-12}$]

```
> ######################### Study of precision
  ########################
> Digits:=50: lisresuo50:=benef(1591873,Float(9.1,-11),4,0): # same
  computation with Digits=50
```

"r=", 1, "nops(lissupo[r])=", 14
"r=", 2, "nops(lissupo[r])=", 55
"r=", 3, "nops(lissupo[r])=", 93
"r=", 4, "nops(lissupo[r])=", 74
"s=", 1, "nops(lisinfo[s])=", 11
"s=", 2, "nops(lisinfo[s])=", 37
"s=", 3, "nops(lisinfo[s])=", 50
"s=", 4, "nops(lisinfo[s])=", 25
"number of numbers with ben < benmax =", 882
(24)

"number of numbers n satisfying M_{p_k} < n < M_{p_{k+1}} and c(n) > delta =", 882 **(24)**

```
> maxilogn:=-infinity: maxinsphin:=-infinity: maxicn:=-infinity:
  maxibenn:=-infinity:
  # study of precision
  for i from 1 to 882 do
     ter30:=lisresuo30[i]; ter50:=lisresuo50[i]:
     maxilogn:=max(maxilogn,abs(ter30[2]-ter50[2]));
     maxinsphin:=max(maxinsphin,abs(ter30[3]-ter50[3]));
     maxicn:=max(maxicn,abs(ter30[4]-ter50[4]));
     maxibenn:=max(maxibenn,abs(ter30[5]-ter50[5]));
  od:print("maxilog=",maxilogn,"maxinsphin=",maxinsphin);
  print("maxicn=",maxicn,"maxibenn=",maxibenn);
```

"maxilog=", $3.4258397081652116146010^{-23}$, "maxinsphin=", $9.978976050243693572910^{-28}$

"maxicn=", $1.46018147502928775514082910^{-24}$, "maxibenn=", $1.519345425872356850512377391 0^{-30}$ **(25)**

```
> comlogn:=0: comnsphin:=0: comcn:=0: combenn:=0:
  # study of increasingness or decreasingness
  for j from 1 to 881 do
     if lisresuo30[j+1][2] < lisresuo30[j][2] then comlogn:=
  comlogn+1: fi;
     if lisresuo30[j+1][3] > lisresuo50[j][3] then comnsphin:=
  comnsphin+1: fi;
     if lisresuo30[j+1][4] > lisresuo30[j][4] then comcn:=comcn+1:
  fi;
     if lisresuo30[j+1][5] < lisresuo30[j][5] then combenn:=
  combenn+1: fi;
  od;
  print("comlogn=",comlogn,"comnsphin=",comnsphin);
  print("comcn=",comcn,"combenn=",combenn);
```

"comlogn=", 881, "comnsphin=", 881

"comcn=", 881, "combenn=", 881 **(26)**

```
> ################################ the end
  ####################################
>
```