Large values of $n/\varphi(n)$ and $\sigma(n)/n$

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Dedicated to the memory of the great
mathematician Andrzej Schinzel

Abstract

Let n be a positive integer, $\varphi(n)$ the Euler totient function, and $\sigma(n) = \sum_{d|n} d$ the sum of the divisors of n. It is easy to prove that $\sigma(n)/n \leqslant n/\varphi(n)$ holds. When $n \to \infty$, Landau proved that, $\limsup n/\big(\varphi(n)\log\log n\big) = e^{\gamma}$, where $\gamma = 0.577\ldots$ is the Euler constant, while, a few years later, Gronwall proved that $\limsup \sigma(n)/\big(n \log\log n\big)$ is also equal to e^{γ}. Afterwards, several authors have given effective upper bounds for $n/\varphi(n)$ and $\sigma(n)/n$, either under the Riemann hypothesis or without assuming it. Let $X \geqslant 4$ be a real number and $\Phi(X)$ the maximum of $n/\varphi(n)$ for $n \leqslant X$. Similarly, we denote by $\Sigma(X)$ the maximum of $\sigma(n)/n$ for $n \leqslant X$. Our first result gives effective upper and lower bounds for the quotient $\Phi(X)/\Sigma(X)$. Next, we give new effective upper bounds for $n/\varphi(n)$ and for $\sigma(n)/n$.

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1 Introduction

Let n be a positive integer. In this paper, we consider Euler's totient function φ which represents the number of positive integers up to n that are relatively prime to n, and the arithmetic function σ which is defined by

$$\sigma(n) = \sum_{d|n} d$$

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and denotes the sum of the divisors of n. The two arithmetic functions $n/\varphi(n)$ and $\sigma(n)/n$ are multiplicative with

$$\frac{\sigma(p^a)}{p^a} = \frac{1+p+\ldots+p^a}{p^a} = \frac{p^{a+1}-1}{p^a(p-1)} < \frac{p}{p-1} = \frac{p^a}{\varphi(p^a)}. \tag{1.1}$$

Hence $\sigma(n)/n < n/\varphi(n)$ for every integer $n > 1$. In 1903, Landau (cf. [18] and [16, Theorem 328]) proved that

$$\limsup_{n\to\infty} \frac{n}{\varphi(n)\log\log n} = e^{\gamma}, \tag{1.2}$$

where $\gamma = 0.577\ldots$ denotes the Euler constant, while in 1913, Gronwall (cf. [14, p. 119] and [16, Theorem 323]) found the maximal order of σ by showing that

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n\log\log n} = e^{\gamma}. \tag{1.3}$$

In the proof of (1.3), Gronwall used the asymptotic formula

$$\prod_{p\leqslant x} \frac{p}{p-1} \sim e^{\gamma}\log x \quad \text{as} \quad x\to\infty, \tag{1.4}$$

where p runs over primes not exceeding x, which is due to Mertens [19].

In view of (1.2), Rosser and Schoenfeld [30, p. 72] raised the question whether there are infinitely many integers n such that $n/\varphi(n) > e^{\gamma}\log\log n$. In [21, 22], this question was proved in the affirmative. With regard to (1.3), it is natural to ask the same question for the function $\sigma(n)/n$. In turns out that the answer depends on the truth of the Riemann hypothesis on the nontrivial zeros of the Riemann zeta function. Under the assumption that the Riemann hypothesis is true, Ramanujan [27, p. 143] gave the asymptotic upper bound

$$\frac{\sigma(n)}{n} \leqslant e^{\gamma}\left(\log\log n - \frac{2(\sqrt{2}-1)}{\sqrt{\log n}} + S_1(\log n) + \frac{O(1)}{\sqrt{\log n}\log\log n}\right) \tag{1.5}$$

with $S_1(x) = \sum_{\rho} x^{\rho-1}/|\rho|^2$, where ρ runs over the nontrivial zeros of the Riemann zeta function. If the Riemann hypothesis is true, one has (cf. [26, eq. (226)])

$$|S_1(x)| \leqslant \frac{1}{\sqrt{x}}\sum_{\rho}\frac{1}{\rho(1-\rho)} = \frac{1}{\sqrt{x}}\sum_{\rho}\left(\frac{1}{\rho}+\frac{1}{1-\rho}\right) = \frac{2}{\sqrt{x}}\sum_{\rho}\frac{1}{\rho} = \frac{\tau}{\sqrt{x}}, \tag{1.6}$$

where

$$\tau = 2 + \gamma - \log 4\pi = 0.0461914179322420\ldots.$$

So, if the Riemann hypothesis is true, the asymptotic upper bound (1.5) implies that there is a positive integer n_0 so that

$$\frac{\sigma(n)}{n} < e^{\gamma} \log\log n \quad \text{for} \quad n \geqslant n_0. \tag{1.7}$$

In 1983, Robin [28, Théorème 1] was able to show that (1.7) is even a sufficient criterion for the truth of the Riemann hypothesis. Only one year later, Robin [29, Théorème 1] gave an explicit version of his result. He found that the Riemann hypothesis is true if and only if

$$\frac{\sigma(n)}{n} < e^{\gamma} \log\log n \quad \text{for} \quad n > 5040. \tag{1.8}$$

Hence, there are infinitely many integers n satisfying $\sigma(n)/n > e^{\gamma} \log\log n$ only if the Riemann hypothesis fails.

The equivalent criterion (1.8) for the Riemann hypothesis is called *Robin's criterion* and the inequality (1.8) is called *Robin's inequality*. Robin's inequality (1.8) has been slightly improved in [24, Corollary 1.2]. Even Robin's inequality remains open in general: so far it is proved to hold unconditionally in many cases (see, for instance, [11], [15], and [5]). In particular, Robin's inequality has been proven for several m-free integers (cf. [11], [32], [9], [20], and [3]). Here a positive integer n is called *m-free* if n is not divisible by the mth power of any prime number.

Let $f(n)$ be a positive arithmetical function, i.e. a function defined on the positive integers with positive values. A positive integer n is said to be a f-champion if $1 \leqslant m < n$ implies $f(m) < f(n)$. The champions for $\sigma(n)/n$ are said to be *superabundant* (SA for short), i.e. the number n is SA if

$$m < n \quad \text{implies} \quad \frac{\sigma(m)}{m} < \frac{\sigma(n)}{n}. \tag{1.9}$$

The SA numbers have been introduced and studied by Alaoglu and Erdős (cf. [1, Sect. 4]). They also were defined and studied by Ramanujan (cf. [27, Sect. 59]) who called them *generalised highly composite*. It is possible to adapt the algorithm described in [24, Sect. 3.4] to compute a table of SA numbers (cf. [34]). Let p_k denote the kth prime and

$$M_{p_k} = p_1 p_2 \ldots p_k$$

the kth primorial, i.e. the product of the first k primes. If $n < M_{p_k}$ then the standard factorization of n can be written $n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_j^{\alpha_j}$ with $q_1 < q_2 < \dots < q_j$, $j < k$ and $q_i \geqslant p_i$ for $1 \leqslant i \leqslant j$. Therefore,

$$\frac{n}{\varphi(n)} = \prod_{i=1}^{j} \frac{q_i}{q_i - 1} \leqslant \prod_{i=1}^{j} \frac{p_i}{p_i - 1} < \prod_{i=1}^{k} \frac{p_i}{p_i - 1} = \frac{M_{p_k}}{\varphi(M_{p_k})} \tag{1.10}$$

and we can see that if $f(n) = n/\varphi(n)$ then the f-champions are the numbers M_{p_k} for $k \geqslant 1$.

The aim of this paper is to study and compare the large values taken by the two functions $n/\varphi(n)$ and $\sigma(n)/n$. Let X be a positive real number. It is convenient to introduce

$$\Sigma(X) = \max_{n \leqslant X} \frac{\sigma(n)}{n} \quad \text{and} \quad \Phi(X) = \max_{n \leqslant X} \frac{n}{\varphi(n)}, \tag{1.11}$$

so that, $\Phi(X)/\Sigma(X) = 1$ for $1 \leqslant X < 2$ and $\Phi(X)/\Sigma(X) = 4/3$ for $2 \leqslant X < 4$. We prove

Theorem 1.1. *For every real $X \geqslant 4$, we have*

$$1 + \frac{2\sqrt{2}}{\sqrt{\log X} \log\log X} - \frac{4.143}{\sqrt{\log X}(\log\log X)^2} \leqslant \frac{\Phi(X)}{\Sigma(X)}$$
$$\leqslant 1 + \frac{2\sqrt{2}}{\sqrt{\log X} \log\log X} + \frac{3.17}{\sqrt{\log X}(\log\log X)^2}. \tag{1.12}$$

However, the ratios $\sigma(n)/n$ and $n/\varphi(n)$ cannot be too large. Rosser and Schoenfeld [30, Theorem 15] showed that the inequality

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} \leqslant e^{\gamma} \log\log n + \frac{2.50636\dots}{\log\log n} \tag{1.13}$$

holds unconditionally for every integer $n \geqslant 3$ with equality for $n = 223\,092\,870 = \prod_{2 \leqslant p \leqslant 23} p$. The advantage of the inequality (1.13) compared to Robin's inequality (1.8) is that it holds for every positive integer n where $\log\log n$ is positive. Robin (cf. [29, Théorème 2] and [8, eq. (7.83), p. 212]) used a lower bound for Chebyshev's θ-function $\theta(x) = \sum_{p \leqslant x} \log p$, where p runs over primes not exceeding x, to improve (1.13) by showing

$$\frac{n}{\varphi(n)} < e^{\gamma} \log\log n + \frac{0.6}{\log\log n}$$

holds unconditionally for every integer $n \geqslant \prod_{2\leqslant p<20000} p$ and that the inequality

$$\frac{\sigma(n)}{n} < e^\gamma \log\log n + \frac{0.648\ldots}{\log\log n}$$

holds unconditionally for every integer $n \geqslant 3$ with equality for $n = 12$. In [2, Theorem 1.1], the same method with improved effective estimates for Chebyshev's θ-function is used to see that

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} < e^\gamma \log\log n + \frac{0.1209}{(\log\log n)^2} \quad \text{for} \quad n \geqslant 10^{10^{10}}. \tag{1.14}$$

Robin [28] proved that if Robin's inequality (1.8) is satisfied for two consecutive colossally abundant numbers N and N' (cf. Sect. 3), then it is also satisfied for all integers n between N and N'. By computing all CA numbers up to $10^{10^{10}}$, Briggs [6] proved that

$$\frac{\sigma(n)}{n} < e^\gamma \log\log n \quad \text{for} \quad 5040 < n \leqslant 10^{10^{10}}. \tag{1.15}$$

Combined with (1.14), it turns out that

$$\frac{\sigma(n)}{n} < e^\gamma \log\log n + \frac{0.1209}{(\log\log n)^2} \quad \text{for} \quad n > 5040.$$

For $k = 999\,999\,476\,056$ and $p_k = 29\,996\,208\,012\,611$, we define

$$M^{(0)} = M_{p_k} = \exp(29\,996\,203\,625\,537.226167\ldots) = 10^{10^{13.114850604\ldots}}. \tag{1.16}$$

Morrill and Platt [20] improved Briggs' result (1.15) by showing that

$$\frac{\sigma(n)}{n} < e^\gamma \log\log n \quad \text{for } 5040 < n \leqslant M^{(0)}. \tag{1.17}$$

In [3], it has been shown that

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} < e^\gamma \log\log n + \frac{\alpha_0}{(\log\log n)^2} \quad \text{for} \quad n \geqslant M^{(0)}, \tag{1.18}$$

where

$$\alpha_0 = 0.0094243 \times e^\gamma = 0.0167853\ldots. \tag{1.19}$$

In the proof of (1.18), improved effective estimates for the product in (1.4) were utilized. Now, let $i = 564\,397\,542$. Then,

$$\begin{aligned} M_{p_{i-1}} &= \exp(\theta(p_{i-1})) = \exp(12\,530\,479\,255.595893\ldots), \\ M_{p_i} &= \exp(\theta(p_i)) = \exp(12\,530\,479\,278.847331\ldots). \end{aligned}$$

Further, let

$$X^{(0)} = \exp(12\,530\,479\,255.595931). \tag{1.20}$$

Note that $X^{(0)}$ satisfies $M_{p_{i-1}} < X^{(0)} < M_{p_i}$ and is much smaller than $M^{(0)}$. In the following theorem, we use Theorem 1.1 to see that the right-hand side inequality in (1.18) also holds with the same constant 0.0094243 for every integer $n \geqslant X^{(0)}$.

Theorem 1.2. *Let α_0 and $X^{(0)}$ be defined as in* (1.19) *and* (1.20)*, respectively. Then, for every integer $n \geqslant X^{(0)}$, we have*

$$\frac{n}{\varphi(n)} \leqslant e^{\gamma} \log\log n + \frac{\alpha_0}{(\log\log n)^2} \tag{1.21}$$

while the inequality (1.21) *does not hold for $n = M_{p_{i-1}}$.*

If we combine (1.17) with (1.18), we can see that

$$\frac{\sigma(n)}{n} < e^{\gamma} \log\log n + \frac{\alpha_0}{(\log\log n)^2} \quad \text{for} \quad n > 5040. \tag{1.22}$$

Under the assumption that the Riemann hypothesis is true, an effective form of the asymptotic upper bound (1.5) was shown in [24], namely

$$\begin{aligned} \frac{\sigma(n)}{n} \leqslant e^{\gamma} \bigg(\log\log n - \frac{2(\sqrt{2}-1)}{\sqrt{\log n}} &+ S_1(\log n) \\ &+ \frac{3.789}{\sqrt{\log n}\log\log n} + \frac{0.026\log\log n}{\log^{2/3} n} \bigg) \end{aligned} \tag{1.23}$$

for every integer $n \geqslant 3$. It turns out that the validity of this inequality for every $n \geqslant 3$ even provides an equivalent criterion for the Riemann hypothesis. Under the assumption that the Riemann hypothesis is true, inequality (1.23) combined with (1.6) yields that

$$\frac{\sigma(n)}{n} \leqslant e^{\gamma} \left(\log\log n + \frac{-2(\sqrt{2}-1)+\tau}{\sqrt{\log n}} + \frac{3.789}{\sqrt{\log n}\log\log n} + \frac{0.026\log\log n}{\log^{2/3} n} \right) \tag{1.24}$$

for every integer $n \geqslant 3$. Finally, we can utilize Theorem 1.1 and Theorem 1.2 to give the following unconditional result concerning (1.24) which simultaneously provides an improvement of (1.22).

Theorem 1.3. *Let α_0 be defined as in* (1.19). *For every $n \geqslant \exp(26\,318\,064\,420)$, we have*

$$\frac{\sigma(n)}{n} < e^\gamma \log\log n + \frac{\alpha_0}{(\log\log n)^2} - \frac{2\sqrt{2}e^\gamma}{\sqrt{\log n}} + \frac{4.143e^\gamma}{\sqrt{\log n}\log\log n}. \tag{1.25}$$

1.1 Notation

- $\theta(x) = \sum_{p\leqslant x} \log p$ is the Chebyshev θ-function.
- $\pi(x) = \sum_{p\leqslant x} 1$ is the prime counting function.
- p_j is the jth prime with $p_1 = 2, p_2 = 3, \ldots$. For p prime and $n \in \mathbb{N}$, $v_p(n)$ denotes the largest exponent such that $p^{v_p(n)}$ divides n.
- $M_{p_k} = p_1 p_2 \ldots p_k$ is the kth primorial. If p is the kth prime then $M_p = M_{p_k}$.
- CA numbers are defined in Sect. 3, SA numbers in (1.9) and HR numbers in Sect. 3.1.

We use the following constants:

- $\alpha_0 = 0.0094243 \times e^\gamma = 0.0167853\ldots$ (cf. (1.19)).
- $\xi^{(0)} = 10^9 + 7$ is the smallest prime exceeding 10^9, $\log \xi^{(0)} = 20.723265\ldots$, cf. (3.15) and (3.17).
- $M^{(0)}$ is defined as in (1.16).
- $N^{(0)}$ is defined in (3.16) and the numbers $\left(\xi_k^{(0)}\right)_{2\leqslant k\leqslant 33}$ in (3.20).

All the computation have been carried out in Maple, see [34].

2 Useful results

We shall use the following results: for $u > 0, v > 0, w \in \mathbb{R}$,

$$t \mapsto \frac{(\log t - w)^u}{t^v} \text{ is decreasing for } t > \exp(w + u/v), \tag{2.1}$$

$$-\log(1-w) \leqslant w\Big(1 + \frac{w}{2(1-w_0)}\Big) \leqslant \frac{w}{1-w_0} \quad \text{for} \quad 0 \leqslant w \leqslant w_0 < 1, \tag{2.2}$$

$$1 + u \leqslant \exp(u) \quad \text{for} \quad u \in \mathbb{R}, \tag{2.3}$$

and

$$\exp(u) \leqslant 1 + u + \frac{u^2}{2(1-u_0)} \quad \text{for} \quad 0 \leqslant u \leqslant u_0 < 1. \tag{2.4}$$

Remark 2.1. *Note that the inequality* (2.4) *is sharp if u_0 is close to zero, but it is useless if u_0 is close to 1. This inequality does appear in the proof of Theorem 1.1 with* $u_0 = 0.000005$.

Lemma 2.2. *Let $\theta(x) = \sum_{p \leqslant x} \log p$ be the Chebyshev function. For every $x \geqslant 0$, one has*

$$\theta(x) < \big(1 + 1.93378 \times 10^{-8}\big)x \tag{2.5}$$

and for every $x \geqslant 41113$, *one has*

$$|\theta(x) - x| < \frac{0.0806\,x}{\log x}. \tag{2.6}$$

Further, for every $x \geqslant 10^9$, *one has*

$$|\theta(x) - x| \leqslant \frac{0.42065x}{\log^3 x}. \tag{2.7}$$

Proof. The inequality (2.5) is a result of Broadbent et al. [7, Corollary 2.1]. From [7, Table 15], we know that the inequality (2.6) holds for every $x \geqslant 10^5$. A direct computer check provides that the inequality (2.6) also holds for every x with $41113 \leqslant x < 10^5$. According to [7, Table 15], we already know that the inequality (2.7) is fulfilled for every $x \geqslant 10^9$. □

Remark 2.3. *It should be noted that, compared to the estimates given in Lemma 2.2, there are asymptotically stronger, and still explicit, estimates for Chebyshev's*

θ-function that improve the inequalities (2.5)–(2.7) *for very large x. For instance, Fiori, Kadiri, and Swidinsky [13, Corollary 14] found that*

$$|\theta(x) - x| \leqslant 121.0961 \left(\frac{\log x}{R}\right)^{3/2} \exp\left(-2\sqrt{\frac{\log x}{R}}\right)$$

for every $x \geqslant 2$, where $R = 5.5666305$.

Remark 2.4. *Under the assumption that the Riemann hypothesis is true, von Koch [33] deduced the stronger asymptotic formula $\theta(x) = x + O(\sqrt{x}\log^2 x)$ as $x \to \infty$. An explicit version of the last result was given by Schoenfeld [31, Theorem 10]. Under the assumption that the Riemann hypothesis is true, he has found that*

$$|\theta(x) - x| < \frac{\sqrt{x}}{8\pi} \log^2 x \quad \text{for} \quad x \geqslant 599. \tag{2.8}$$

Büthe [10, Theorem 2] investigated a method to show that the inequality (2.8) *holds unconditionally for every x such that $599 \leqslant x \leqslant 1.4 \times 10^{25}$. Büthe's result was improved by Platt and Trudgian [25, Corollary 1]. They proved that the inequality* (2.8) *holds unconditionally for every x satisfying $599 \leqslant x \leqslant 2.169 \times 10^{25}$. Recently, Johnston [17, Corollary 3.3] extended the last result by showing that the inequality* (2.8) *holds unconditionally for every x with $599 \leqslant x \leqslant 1.101 \times 10^{26}$.*

Lemma 2.5. *For $x \geqslant x_0 = 10^9$, one has*

$$\pi\big(x(1 + 0.069/\log^2 x)\big) - \pi(x) \geqslant 2.083\frac{\sqrt{x}}{\log x}. \tag{2.9}$$

Proof. Applying Lemma 2.3 of [12] (that is a consequence of (2.7)) with $K = 2$, $\alpha = 0.42065$, $x_0 = 10^9$, and $a = 0.069$, we get that

$$\pi(x(1 + 0.069/\log^2 x)) - \pi(x) \geqslant b\frac{x}{\log^3 x} \quad \text{for} \quad x \geqslant 10^9$$

where

$$b = \left(1 - \frac{a}{\log^{K+1} x_0}\right)\left(a - \frac{2\alpha}{\log x_0} - \frac{\alpha\, a}{\log^{K+1} x_0}\right) \geqslant 0.0283.$$

Finally, for $x \geqslant x_0$,

$$0.0283\frac{x}{\log^3 x} \geqslant 0.0283\frac{\sqrt{x_0}}{\log^2 x_0}\frac{\sqrt{x}}{\log x} = 2.083\ldots\frac{\sqrt{x}}{\log x}$$

which completes the proof. □

Lemma 2.6. *Let a be a positive real number and let*

$$h = h(a,t) = ae^{t/2}t^2 - 2t\sqrt{2}. \tag{2.10}$$

Then

(i) For $t > 2\sqrt{2} - 4 = -1.17\ldots$, the mapping $t \mapsto h(a,t)$ is convex and $h' = \partial h/\partial t$ is increasing on a and on t.

(ii) If t_0 is a number > 2.54557 such that $h(a,t_0) > -5$, then $h(a,t)$ is increasing on t for $t > t_0$.

Proof. One has

$$h' = \frac{\partial h}{\partial t} = \frac{a}{2}e^{t/2}t^2 + 2ae^{t/2}t - 2\sqrt{2}$$

and

$$h'' = \frac{\partial^2 h}{\partial t^2} = ae^{t/2}\Big(\frac{t^2}{4} + 2t + 2\Big).$$

The trinomial $t^2/4 + 2t + 2$ is positive for $t > 2\sqrt{2} - 4$, which proves (i).

If $h(a,t_0) > -5$ then $ae^{t_0/2}t_0^2 > 2\sqrt{2}t_0 - 5$ holds, which implies

$$\begin{aligned} h'(a,t_0) &= \frac{a}{2}e^{t_0/2}t_0^2\Big(1 + \frac{4}{t_0}\Big) - 2\sqrt{2} > \Big(\sqrt{2}t_0 - \frac{5}{2}\Big)\Big(1 + \frac{4}{t_0}\Big) - 2\sqrt{2} \\ &= \frac{1}{t_0}\Big(\sqrt{2}t_0^2 - \Big(\frac{5}{2} - 2\sqrt{2}\Big)t_0 - 10\Big) \end{aligned}$$

and the above trinomial is positive for $t_0 > 2.545565\ldots$ Since $h(a,t)$ is convex on t, the derivative $h'(a,t)$ is increasing on t and therefore, as it is positive for $t = t_0$, it is also positive for $t > t_0$, which proves (ii). □

3 Colossally abundant (CA) numbers

A positive integer N is said to be *colossally abundant* (or a *CA number*) if there exists a real number $\varepsilon > 0$ such that

$$\frac{\sigma(n)}{n^{1+\varepsilon}} \leqslant \frac{\sigma(N)}{N^{1+\varepsilon}}$$

for every positive integers n. The number ε is called a *parameter* of the CA number N. The colossally abundant numbers were introduced in 1944 by Alaoglu and Erdős (cf. [1, Sect. 3], [27, Sect. 59], [8, Sect. 6.3], and [24, Sect. 4]). Below, some properties of CA numbers are recalled.

If t is a real number with $t > 1$ and k is a positive integer, one defines

$$F(t,k) = \frac{\log(1 + 1/(t^k + t^{k-1} + \ldots + t))}{\log t} = \frac{\log(1 + (t-1)/(t^{k+1} - t))}{\log t}. \quad (3.1)$$

Note that the second formula allows us to calculate $F(t,u)$ for u real positive and that $F(t,u)$ is decreasing on t for u fixed and on u for t fixed.

We consider the set

$$\mathcal{E} = \{F(p,k), p \text{ prime}, k \text{ integer } \geqslant 1\}. \quad (3.2)$$

It is convenient to order the elements of $\mathcal{E} \cup \{\infty\}$ defined in (3.2) in the decreasing sequence

$$\varepsilon_1 = \infty > \varepsilon_2 = F(2,1) = \frac{\log(3/2)}{\log 2} = 0.58\ldots$$
$$> \varepsilon_3 = \frac{\log(4/3)}{\log 3} = 0.26\ldots > \ldots > \varepsilon_i = F(q_i,k_i) > \ldots. \quad (3.3)$$

In the set $\mathcal{E}$ defined by (3.2), there could exist elements admitting two representations (cf. [24, Sect. 4.1])

$$\varepsilon_i = F(q_i,k_i) = F(q_i',k_i') \quad (3.4)$$

with $k_i > k_i' \geqslant 1$ and $q_i < q_i'$. An element $\varepsilon_i \in \mathcal{E}$ satisfying (3.4) is said to be *extraordinary*, but none is known. If ε_i is not extraordinary, it is said to be *ordinary* and satisfies in only one way

$$\varepsilon_i = F(q_i,k_i). \quad (3.5)$$

To $\varepsilon_i \in \mathcal{E}$, we attach the number $\xi = \xi_1$ defined by $F(\xi,1) = \varepsilon_i$ and, for $k \geqslant 1$, the numbers ξ_k defined by

$$F(\xi_k,k) = \frac{\log(1 + 1/(\xi_k + \xi_k^2 + \ldots + \xi_k^k))}{\log \xi_k} = F(\xi,1) = \varepsilon_i, \quad (3.6)$$

$$K = \text{ the largest } k \text{ such that } \xi_k \geqslant 2 \quad (3.7)$$

and the CA number

$$N_{\varepsilon_i} = \prod_{k=1}^{K} \prod_{\xi_{k+1} < p \leqslant \xi_k} p^k = \prod_{k=1}^{K} \prod_{p \leqslant \xi_k} p, \quad \frac{\sigma(N_{\varepsilon_i})}{N_{\varepsilon_i}} = \prod_{k=1}^{K} \prod_{\xi_{k+1} < p \leqslant \xi_k} \frac{1 - 1/p^{k+1}}{1 - 1/p}. \quad (3.8)$$

Remark 3.1. *Note that $\xi = \xi_1 > \xi_2 > \ldots > \xi_K$. Also, ξ and, as well, ξ_k (for k fixed) and K do not decrease when ε_i decreases, i.e., when N_{ε_i} increases.*

If ε_i is ordinary and satisfies (3.5), then $v_{q_i}\big(N_{\varepsilon_i}\big) = k_i$, $q_i = \xi_{k_i}\big(N_{\varepsilon_i}\big)$,

$$N_{\varepsilon_{i-1}} = \frac{N_{\varepsilon_i}}{q_i}, \quad \text{and} \quad \frac{\sigma\big(N_{\varepsilon_{i-1}}\big)}{N_{\varepsilon_{i-1}}} = \left(\frac{1 - 1/\xi_{k_i}^{k_i}}{1 - 1/\xi_{k_i}^{k_i+1}}\right)\frac{\sigma\big(N_{\varepsilon_i}\big)}{N_{\varepsilon_i}}. \tag{3.9}$$

If ε_i is extraordinary and satisfies (3.4), we have $v_{q_i}\big(N_{\varepsilon_i}\big) = k_i$, $q_i = \xi_{k_i}\big(N_{\varepsilon_i}\big)$, $v_{q_i'}\big(N_{\varepsilon_i}\big) = k_i'$, $q_i' = \xi_{k_i'}\big(N_{\varepsilon_i}\big)$,

$$N_{\varepsilon_{i-1}} = \frac{N_{\varepsilon_i}}{q_i q_i'}, \quad \text{and} \quad \frac{\sigma\big(N_{\varepsilon_{i-1}}\big)}{N_{\varepsilon_{i-1}}} = \left(\frac{1 - 1/\xi_{k_i}^{k_i}}{1 - 1/\xi_{k_i}^{k_i+1}}\right)\left(\frac{1 - 1/\xi_{k_i'}^{k_i'}}{1 - 1/\xi_{k_i'}^{k_i'+1}}\right)\frac{\sigma\big(N_{\varepsilon_i}\big)}{N_{\varepsilon_i}}. \tag{3.10}$$

In both cases, we get that

$$N_{\varepsilon_i}/\xi^2 < N_{\varepsilon_{i-1}}. \tag{3.11}$$

Moreover, for $k \geqslant 2$, (cf. [24, Proposition 4.7]), one has

$$\xi^{1/k} \leqslant \xi_k \leqslant (k\xi)^{1/k}. \tag{3.12}$$

For $k = 2$, more precise estimates are given:

$$\xi_2 \geqslant \sqrt{2\,\xi}\Big(1 - \frac{\log 2}{2\log\xi}\Big) > \sqrt{2\,\xi}\Big(1 - \frac{0.347}{\log\xi}\Big) \quad \text{for} \quad \xi \geqslant 1530 \tag{3.13}$$

and

$$\xi_2 \leqslant \sqrt{2\,\xi}\Big(1 - \frac{0.323}{\log\xi}\Big) \quad \text{for} \quad \xi \geqslant 10^9. \tag{3.14}$$

The following data (cf. [24, Sect. 4.4]) will be used in Lemma 3.2 below:

$$\xi = \xi^{(0)} = \xi_1^{(0)} = 10^9 + 7, \tag{3.15}$$

(the smallest prime exceeding 10^9) and $\varepsilon_i = F(\xi, 1)$, the CA number N_{ε_i} is

$$N^{(0)} = 2^{33}3^{21}5^{14}7^{11}11^9 13^8 \prod_{p=17}^{23} p^7 \prod_{p=29}^{41} p^6 \prod_{p=43}^{83} p^5 \prod_{p=89}^{241} p^4 \prod_{p=251}^{1409} p^3 \prod_{p=1423}^{44021} p^2 \prod_{p=44027}^{1000000007} p, \tag{3.16}$$

$$\log N^{(0)} = 1000014552.11\ldots, \quad \log\log N^{(0)} = 20.7232\ldots, \tag{3.17}$$

$$\sigma(N^{(0)})/N^{(0)} = 36.909618566\ldots, \tag{3.18}$$

$$K = K(N^{(0)}) = 33, \tag{3.19}$$

$$\xi_2 = \xi_2^{(0)} = 44023.5\ldots, \quad \xi_3 = \xi_3^{(0)} = 1418.3\ldots, \quad \xi_4 = \xi_4^{(0)} = 247.3\ldots,$$
$$\xi_5 = \xi_5^{(0)} = 85.6\ldots, \quad \ldots, \quad \xi_{33} = \xi_{33}^{(0)} = 2.033\ldots, \quad \xi_{34} = \xi_{34}^{(0)} = 1.991\ldots \tag{3.20}$$

and, for $N_{\varepsilon_i} \geqslant N^{(0)}$ (cf. [24, Lemma 4.10]), one has

$$T_3 = \sum_{k=3}^{K} \xi_k \leqslant 1.9769\,\xi^{1/3}. \tag{3.21}$$

Lemma 3.2. *Let X be a real number satisfying $X \geqslant N^{(0)}$ (cf.* (3.16)*). For simplicity, we write L for* $\log X$*, λ for* $\log\log X$*, N' for $N_{\varepsilon_{i-1}}$, and N for N_{ε_i} (cf.* (3.8)*). One defines ε_i and ε_{i-1} belonging to $\mathcal{E}$ (cf.* (3.2) *and* (3.3)*) such that*

$$N' = N_{\varepsilon_{i-1}} \leqslant X < N_{\varepsilon_i} = N \tag{3.22}$$

To ε_i, one associates ξ, ξ_k and K (cf. (3.6) *and* (3.7)*). Then*

$$0.999952\,\xi < \xi - \frac{0.42065\xi}{\log^3 \xi} \leqslant L = \log X < \xi + \frac{0.8302\,\xi}{\log^3 \xi} < 1.000094\,\xi, \tag{3.23}$$

$$0.9999L < L/1.000094 < \xi < L/0.999952 < 1.000049L, \tag{3.24}$$

$$0.999995\lambda \leqslant \log \xi \leqslant 1.0000024\lambda, \tag{3.25}$$

$$|L - \xi| \leqslant \frac{0.8302\,\xi}{\log^3 \xi} \leqslant \frac{0.8303\,L}{\lambda^3}, \tag{3.26}$$

$$\left| \frac{1}{\sqrt{L}\lambda} - \frac{1}{\sqrt{\xi}\log \xi} \right| \leqslant \frac{0.456}{\sqrt{L}\lambda^4} \leqslant \frac{0.0011}{\sqrt{L}\lambda^2}, \tag{3.27}$$

$$\left| \frac{1}{\sqrt{L}\lambda^2} - \frac{1}{\sqrt{\xi}\log^2 \xi} \right| \leqslant \frac{0.496}{\sqrt{L}\lambda^5} \leqslant \frac{0.000056}{\sqrt{L}\lambda^2}. \tag{3.28}$$

Proof. Since ξ is non-decreasing on N (cf. Remark 3.1) and $N > X \geqslant N^{(0)}$ is assumed, we get that $\xi \geqslant \xi^{(0)} > 10^9$ holds (cf. (3.15)). Using (3.21), (3.14), and (2.1), it turns out that

$$\begin{aligned} T_2 &= \sum_{k=2}^{K} \xi_k = \xi_2 + T_3 \leqslant \sqrt{2\xi}\Big(1 - \frac{0.323}{\log \xi}\Big) + 1.9769\,\xi^{1/3} \\ &= \sqrt{2\xi}\left(1 + \frac{1}{\log \xi}\Big(\frac{1.9769 \log \xi}{\sqrt{2}\xi^{1/6}} - 0.323\Big)\right) \\ &\leqslant \sqrt{2\xi}\left(1 + \frac{1}{\log \xi}\Big(\frac{1.9769 \log 10^9}{\sqrt{2}10^{9/6}} - 0.323\Big)\right) \leqslant \sqrt{2\xi} + 0.5931\frac{\sqrt{2\xi}}{\log \xi} \\ &\leqslant \sqrt{2\xi} + \frac{0.8388\sqrt{\xi}}{\log \xi} \leqslant \sqrt{\xi}\Big(\sqrt{2} + \frac{0.8388}{\log 10^9}\Big) \leqslant 1.4547\sqrt{\xi}. \end{aligned} \tag{3.29}$$

From (3.22), (3.8), (2.5), (2.7), (3.29), and (2.1), we may write

$$\begin{aligned} L < \log N &= \sum_{k=1}^{K} \theta\big(\xi_k\big) \leqslant \theta(\xi) + (1 + 1.93378 \times 10^{-8})T_2 \\ &\leqslant \xi + \frac{0.42065\xi}{\log^3 \xi} + 1.455\sqrt{\xi} = \xi\left(1 + \frac{1}{\log^3 \xi}\Big(0.42065 + \frac{1.455 \log^3 \xi}{\sqrt{\xi}}\Big)\right) \\ &\leqslant \xi\left(1 + \frac{1}{\log^3 \xi}\Big(0.42065 + \frac{1.455 \log^3 10^9}{\sqrt{10^9}}\Big)\right) \\ &\leqslant \xi\Big(1 + \frac{0.8302}{\log^3 \xi}\Big) \leqslant \xi\Big(1 + \frac{0.8302}{\log^3 10^9}\Big) \leqslant 1.000094\xi, \end{aligned} \tag{3.30}$$

which proves the upper bound of (3.23). From (3.22), (3.11), and (3.8), we deduce that $L \geqslant \log N_{\varepsilon_{i-1}} \geqslant \log N - 2\log \xi \geqslant \theta(\xi) + \theta(\xi_2) - 2\log \xi$. But, from (3.20), $\xi_2 \geqslant 44023$, so that from (2.6) and (3.12),

$$\theta(\xi_2) \geqslant \xi_2\Big(1 - \frac{0.0806}{\log \xi_2}\Big) \geqslant \xi_2\Big(1 - \frac{0.0806}{\log 44023}\Big) \geqslant 0.9924\xi_2 \geqslant 0.9924\xi^{1/2},$$

which is $> 2\log \xi$ for $\xi > 10^9$. Therefore, from (2.7),

$$L \geqslant \theta(\xi) \geqslant \xi\Big(1 - \frac{0.42065}{\log^3 \xi}\Big) \geqslant \xi\Big(1 - \frac{0.42065}{\log^3 10^9}\Big) \geqslant 0.999952\,\xi, \tag{3.31}$$

which completes the proof of (3.23).

The proof of (3.24) follows from (3.23).

From (3.17), we obtain that $\log L = \log\log X \geqslant \log\log N^{(0)} > 20.72$. Now we can use (3.24) to see that

$$\log \xi \leqslant \log 1.000049 + \log L \leqslant (\log L)(1 + 0.000049/20.72) < 1.0000024\lambda$$

and, similarly,

$$\log \xi \geqslant (\log L)(1 + \log(0.9999)/20.72) > 0.999995\lambda,$$

which prove (3.25).

In order to prove (3.26), we utilize (3.23), (3.24), and (3.25) to see that

$$|\xi - L| \leqslant \frac{0.8302\,\xi}{\log^3 \xi} \leqslant \frac{0.8302 \times 1.000049\,L}{(0.999995)^3\,\lambda^3} < \frac{0.8303\,L}{\lambda^3}.$$

Next, we show that the inequalities given in (3.27) hold. For this purpose, it is convenient to introduce

$$\rho = \min(\xi, L) \geqslant \min\left(\xi^{(0)}, \log N^{(0)}\right) > 10^9.$$

From (3.24) and (3.25), it follows that

$$L \geqslant \rho \geqslant L/1.000094,\ \log \rho \geqslant 0.999995\lambda,\ \text{and}\ L \geqslant \rho \geqslant 10^9. \tag{3.32}$$

One may write

$$\left|\frac{1}{\sqrt{L}\,\lambda} - \frac{1}{\sqrt{\xi}\log \xi}\right| = \left|\int_{\xi}^{L} \frac{\log t + 2}{2t^{3/2}\log^2 t}\,\mathrm{d}t\right| \leqslant \frac{|L - \xi|}{2\rho^{3/2}\log \rho}\left(1 + \frac{2}{\log \rho}\right)$$

and, from (3.26) and (3.32), this is

$$\leqslant \frac{0.8303\,L}{\log^3 L}\,\frac{1.000094^{3/2}}{2L^{3/2} \times 0.999995\,\lambda}\left(1 + \frac{2}{\log 10^9}\right) \leqslant \frac{0.456}{\sqrt{L}\lambda^4} \leqslant \frac{0.0011}{\sqrt{L}\lambda^2}.$$

which proves (3.27).

The proof of (3.28) is similar to that of (3.27). One has

$$\left|\frac{1}{\sqrt{L}\,\lambda^2}-\frac{1}{\sqrt{\xi}\log^2\xi}\right| = \left|\int_\xi^L \frac{\log t+4}{2t^{3/2}\log^3 t}\,\mathrm{d}t\right|$$
$$\leqslant \frac{0.8303\,L}{\log^3 L}\,\frac{1.000094^{3/2}}{2L^{3/2}\times 0.999995^2\,\lambda^2}\left(1+\frac{4}{\log 10^9}\right)$$
$$\leqslant \frac{0.496}{\sqrt{L}\lambda^5} \leqslant \frac{0.000056}{\sqrt{L}\lambda^2},$$

which completes the proof of Lemma 3.2. □

Lemma 3.3. *(i) The only numbers that are both SA and primorial are* 2 *and* 6.
(ii) There is at most one primorial between two consecutive CA numbers > 6.

Proof. Let $p \geqslant 7$ be a prime and M_p its primorial. We set $n = 2M_p/p < M_p$. Then, one has

$$\frac{\sigma(n)/n}{\sigma(M_p)/M_p} = \frac{\sigma(4)/4}{\sigma(2p)/(2p)} = \frac{7/4}{3(p+1)/(2p)} = \frac{7/6}{1+1/p} > 1.$$

So, M_p is not a SA number. If $p = 5$, then $M_p = 30$ is also not a SA number because $\sigma(24)/24 = 5/2 > \sigma(30)/30 = 12/5$, which proves (i).

Let N and N' be two consecutive CA numbers with $6 < N < N'$. Further, let p the largest prime factor of N and p' the prime following p. Then, $N' \leqslant p'N$ holds. Let us assume that there are two primes $q < q'$ such that

$$N < M_q < M_{q'} < N' \leqslant p'N.$$

Then,

$$q' \leqslant M_{q'}/M_q < Np'/N = p'$$

so that $q' \leqslant p$. But all primes $\leqslant p$ divides N, whence $M_{q'}$ divides N and $M_{q'} \leqslant N$, a contradiction, which completes the proof of Lemma 3.3. □

3.1 An algorithm to compute SA numbers

Let ε be a positive real number and let N be a CA number of parameter ε. For a positive integer n, the *benefit* of n is defined by

$$\mathrm{ben}_\varepsilon(n) = \log\left(\frac{\sigma(N)}{N^{1+\varepsilon}}\right) - \log\left(\frac{\sigma(n)}{n^{1+\varepsilon}}\right) = \log\left(\frac{\sigma(N)/N}{\sigma(n)/n}\right) + \varepsilon\log\left(\frac{n}{N}\right). \quad (3.33)$$

If B is a given positive real number, then the set of integers n satisfying $\mathrm{ben}_\varepsilon(n) \leqslant B$ is finite (cf. [24, Proposition 4.14]). In [24, Sect. 4.6], an algorithm is described to compute all integers n such that $\mathrm{ben}_\varepsilon(n) \leqslant B$ holds.

A Hardy-Ramanujan number (HR number for short) is an integer n such that, if $p < p'$ are two primes, then $v_p(n) \geqslant v_{p'}(n)$ holds. In [1, Theorem 1], it is proved that every SA number is a HR number. Let N' be the CA number following N. It is easy to adapt the above algorithm to compute all HR numbers $n_1, n_2, \ldots, n_r$ satisfying $N \leqslant n_1 < \ldots < n_r \leqslant N'$ and $\mathrm{ben}_\varepsilon \leqslant B$, and to prune them. The number n_i should be pruned if there exists $j < i$ such that $\sigma(n_j)/n_j \geqslant \sigma(n_i)/n_i$. Let us denote by $S_1 < S_2 < \ldots < S_s$ the pruned list and let us show that these s numbers S_i are SA. Ad absurdum, let us assume that S_i is not SA. Let S be the largest SA number $< S_i$. Since N is SA, it turns out that $S \geqslant N$ holds and we would have $\sigma(S_i)/S_i < \sigma(S)/S$, which would imply

$$\begin{aligned}\mathrm{ben}_\varepsilon(S) &= \log\left(\frac{\sigma(N)/N}{\sigma(S)/S}\right) + \varepsilon \log\left(\frac{S}{N}\right)\\ &< \log\left(\frac{\sigma(N)/N}{\sigma(S_i)/S_i}\right) + \varepsilon \log\left(\frac{S_i}{N}\right) = \mathrm{ben}_\varepsilon(S_i) \leqslant B.\end{aligned}$$

So, S would be equal to some $n_j < S_i$ and S_i would have been pruned. Therefore, all the elements of the pruned list are SA. But, do we get all the SA numbers between N and N'? If there exists an SA number S between S_i and S_{i+1} with $1 \leqslant i \leqslant s-1$, then $\sigma(S_i)/S_i < \sigma(S)/S$ and

$$\begin{aligned}\mathrm{ben}_\varepsilon(S) &= \log\left(\frac{\sigma(N)/N}{\sigma(S)/S}\right) + \varepsilon \log\left(\frac{S}{N}\right) < \log\left(\frac{\sigma(N)/N}{\sigma(S_i)/S_i}\right) + \varepsilon \log\left(\frac{S_{i+1}}{N}\right)\\ &= \mathrm{ben}_\varepsilon(S_i) + \varepsilon \log\left(\frac{S_{i+1}}{S_i}\right). \qquad (3.34)\end{aligned}$$

Let us set

$$B' = \max_{1 \leqslant i \leqslant s-1} \mathrm{ben}_\varepsilon(S_i) + \varepsilon \log(S_{i+1}/S_i).$$

If $B' \leqslant B$, then (3.34) provides that S would belong to the pruned list which leads to a contradiction to our hypothesis. If $B' > B$, we start the algorithm again with B' instead of B and (3.34) proves that S would belong to the new pruned list.

In the proof of Theorem 1.1, this algorithm has been used with $B = 2\varepsilon$ and B' was always smaller than B.

Let us say that an integer n is largely superabundant if n not a SA number and if m is the largest SA number not exceeding n, we have $\sigma(n)/n = \sigma(m)/m$. Between 1 and 10^{50} there is only one such number, namely $n = 360360$ with $\sigma(n)/n = 48/11$. The preceding SA number is $m = 332640$ with $\sigma(m)/m = 48/11$.

4 Proof of Theorem 1.1

First, we observe that for $X \geqslant 4$, one has

$$\frac{\Phi(X)}{\Sigma(X)} > 1. \tag{4.1}$$

Indeed, $\Sigma(X)$ is equal to $\sigma(n)/n$ where n is the largest SA number not exceeding X. So, n is a HR number (cf. Sect. 3.1). If P denotes the largest prime factor of n, then n can be written as

$$n = \prod_{p \leqslant P} p^{a_p} \quad \text{with} \quad a_p \geqslant 1 \quad \text{for all} \quad p \leqslant P,$$

so that $M_P \leqslant n \leqslant X$ holds. Now we can use (1.1) to see that

$$\Sigma(X) = \frac{\sigma(n)}{n} = \prod_{p \leqslant P} \left(\frac{\sigma(p^{a_p})}{p^{a_p}} \right) < \prod_{p \leqslant P} \left(\frac{p}{p-1} \right) = \frac{M_P}{\varphi(M_P)} \leqslant \Phi(X),$$

which proves (4.1). It is convenient to set

$$g(X) = \frac{\Phi(X)}{\Sigma(X)} - 1 > 0 \tag{4.2}$$

while $\rho(X)$ is defined by

$$g(X) = \frac{2\sqrt{2}}{\sqrt{\log X} \log\log X} + \frac{\rho(X)}{\sqrt{\log X} (\log\log X)^2}.$$

Using (2.10), we get that $\rho(X)$ can be expressed as

$$\rho(X) = h(g(X), t) = g(X) e^{t/2} t^2 - 2t\sqrt{2} \quad \text{with} \quad t = \log\log X. \tag{4.3}$$

4.1 Proof of Theorem 1.1 for $X > N^{(0)}$

We first consider the case $X > N^{(0)}$, where $N^{(0)}$ is defined by (3.16). If m is the largest SA number $\leqslant X$ then (1.11) and (1.9) imply that $\Sigma(X)$ is equal to $\sigma(m)/m$.

Similarly, as the champions for $n/\varphi(n)$ are the numbers M_{p_k} (cf. (1.10)), $\Phi(X) = M_{p_k}/\varphi(M_{p_k})$ with M_{p_k} defined by $M_{p_k} \leqslant X < M_{p_{k+1}}$. We determine the two CA numbers N' and N that are chosen such that $N' = N_{\varepsilon_{i-1}} \leqslant X < N = N_{\varepsilon_i}$. Then, we have $N > N^{(0)}$, $\xi = \xi(N) \geqslant \xi^{(0)} > 10^9$, and $\xi_k \geqslant \xi_k^{(0)}$. Since every CA number is a SA number, we obtain that

$$\frac{\sigma(N')}{N'} \leqslant \Sigma(X) < \frac{\sigma(N)}{N}. \tag{4.4}$$

Let p_r be the largest prime factor of N. From (3.8), it follows that

$$p_r \leqslant \xi < p_{r+1} \tag{4.5}$$

and $N \geqslant M_{p_r}$. We define u to be the smallest positive integer such that

$$M_{p_{r+u-1}} < N \leqslant M_{p_{r+u}} \tag{4.6}$$

and let $M = M_{p_{r+u}}$. Next, we give some effective estimates for u. In order to do this, we first note that (3.8) implies

$$\log N = \log N_{\varepsilon_i} = \theta(\xi) + E \quad \text{with} \quad E = \sum_{k=2}^{K} \theta(\xi_k). \tag{4.7}$$

This notion of excess has already been used in [12, Sect. 3.5]. Since $\xi_2 \geqslant \xi_2^{(0)}$, we can utilize (3.20) to get that $\xi_2 \geqslant 44023$. If we apply successively (3.8), (2.6), (3.12), and (3.13), it turns out that

$$\begin{aligned} E &\geqslant \theta(\xi_2) \geqslant \xi_2\Big(1 - \frac{0.0806}{\log \xi_2}\Big) \geqslant \xi_2\Big(1 - \frac{0.0806}{\log \sqrt{\xi}}\Big) \\ &\geqslant \sqrt{2\xi}\Big(1 - \frac{0.347}{\log \xi}\Big)\Big(1 - \frac{0.1612}{\log \xi}\Big) \geqslant \sqrt{2\xi}\Big(1 - \frac{0.5082}{\log \xi}\Big). \end{aligned} \tag{4.8}$$

On the other hand, we can apply successively (2.5), (3.21), (2.6), (3.12), and (3.14) to see that

$$\begin{aligned} E &\leqslant \theta(\xi_2) + (1 + 1.93378 \times 10^{-8}) \sum_{k=3}^{K} \xi_k \leqslant \theta(\xi_2) + 1.977\, \xi^{1/3} \\ &\leqslant \xi_2 \Big(1 + \frac{0.0806}{\log \xi_2}\Big) + 1.977\, \xi^{1/3} \\ &\leqslant \sqrt{2\xi}\Big(1 - \frac{0.323}{\log \xi}\Big)\Big(1 + \frac{0.1612}{\log \xi}\Big) + \frac{1.977 \log \xi}{\sqrt{2}\, \xi^{1/6}} \frac{\sqrt{2\xi}}{\log \xi} \\ &\leqslant \sqrt{2\xi}\Big(1 - \frac{0.1618}{\log \xi}\Big) + \frac{1.977 \log\left(10^9\right)}{\left(\sqrt{2}\right) 10^{9/6}} \frac{\sqrt{2\xi}}{\log \xi} \leqslant \sqrt{2\xi}\Big(1 + \frac{0.755}{\log \xi}\Big). \end{aligned} \tag{4.9}$$

From (4.5), (4.6), and (4.7), it follows that

$$\log M = \theta(\xi) + \sum_{i=1}^{u} \log p_{r+i} \geqslant \log N = \theta(\xi) + E$$

and

$$\log M / p_{r+u} = \theta(\xi) + \sum_{i=1}^{u-1} \log p_{r+i} \leqslant \log N = \theta(\xi) + E.$$

Using (4.5), these inequalities imply that

$$(u - 1) \log \xi \leqslant E \leqslant u \log p_{r+u}. \tag{4.10}$$

If we combine (4.9) with the left-hand side inequality of (4.10), it turns out that

$$u \leqslant 1 + \frac{\sqrt{2\xi}}{\log \xi}\Big(1 + \frac{0.755}{\log \xi}\Big) \leqslant \frac{\sqrt{2\xi}}{\log \xi}\Big(1 + \frac{0.765}{\log \xi}\Big) \leqslant 1.467 \frac{\sqrt{\xi}}{\log \xi}. \tag{4.11}$$

Note that Lemma 2.5 and (4.11) yield

$$\pi\left(\xi\Big(1 + \frac{0.069}{\log^2 \xi}\Big)\right) - \pi(\xi) \geqslant 2.083 \frac{\sqrt{\xi}}{\log \xi} \geqslant u. \tag{4.12}$$

Since $\pi(\xi) = r$ (see (4.5)), (4.12) implies

$$\xi \leqslant p_{r+u} \leqslant \xi\Big(1 + \frac{0.069}{\log^2 \xi}\Big) \leqslant \xi\Big(1 + \frac{0.0034}{\log \xi}\Big) \leqslant 1.00017\, \xi. \tag{4.13}$$

Hence $\log p_{r+u} \leqslant \log\xi + 0.00017$. Together with (4.8) and the right-hand side inequality of (4.10), we obtain that the inequality

$$\begin{aligned} u &\geqslant \frac{E}{\log p_{r+u}} \geqslant \frac{\sqrt{2\xi}}{\log\xi}\Big(1-\frac{0.522}{\log\xi}\Big)\Big(\frac{1}{1+0.00017/\log\xi}\Big) \\ &\geqslant \frac{\sqrt{2\xi}}{\log\xi}\Big(1-\frac{0.522}{\log\xi}\Big)\Big(1-\frac{0.00017}{\log\xi}\Big) \geqslant \frac{\sqrt{2\xi}}{\log\xi}\Big(1-\frac{0.523}{\log\xi}\Big) \geqslant 1.377\frac{\sqrt{\xi}}{\log\xi} \end{aligned} \tag{4.14}$$

holds. Hence $u \geqslant 3$ and we set $M' = M/\big(p_{r+u}p_{r+u-1}p_{r+u-2}\big)$. From (4.5), (4.6), and (3.11), it follows that

$$M' < \frac{M}{p_{r+u}p_{r+1}^2} < \frac{M}{p_{r+u}\xi^2} < \frac{N}{\xi^2} \leqslant N_{\varepsilon_{i-1}} = N' \leqslant X < N = N_{\varepsilon_i} \leqslant M \tag{4.15}$$

and consequently, $M'/\varphi(M') \leqslant \Phi(X) < M/\varphi(M)$. Combined with (4.4), we get

$$\frac{M'N}{\varphi(M')\sigma(N)} < \frac{\Phi(X)}{\Sigma(X)} < \frac{MN'}{\varphi(M)\sigma(N')}. \tag{4.16}$$

Using the definition of M', one gets that

$$\begin{aligned} \frac{M'}{\varphi(M')} &= \frac{M}{\varphi(M)}\Big(1-\frac{1}{p_{r+u}}\Big)\Big(1-\frac{1}{p_{r+u-1}}\Big)\Big(1-\frac{1}{p_{r+u-2}}\Big) \\ &> \frac{M}{\varphi(M)}\Big(1-\frac{1}{\xi}\Big)^3 > \frac{M}{\varphi(M)}\Big(1-\frac{3}{\xi}\Big). \end{aligned}$$

If ε_i is ordinary, we can use (3.9) and (3.12) to get

$$\frac{\sigma(N')}{N'} > \frac{\sigma(N)}{N}\Big(1-\frac{1}{\xi_{k_i}^{k_i}}\Big) \geqslant \Big(1-\frac{1}{\xi}\Big)\frac{\sigma(N)}{N}$$

while in the case where ε_i is extraordinary, (3.10) and (3.12) can be used to see that

$$\frac{\sigma(N')}{N'} > \Big(1-\frac{1}{\xi}\Big)^2\frac{\sigma(N)}{N} > \Big(1-\frac{2}{\xi}\Big)\frac{\sigma(N)}{N}.$$

Therefore, (4.16) yields

$$\Big(1-\frac{3}{\xi}\Big)\frac{M}{\varphi(M)}\frac{N}{\sigma(N)} < \frac{\Phi(X)}{\Sigma(X)} < \Big(\frac{1}{1-2/\xi}\Big)\frac{M}{\varphi(M)}\frac{N}{\sigma(N)}. \tag{4.17}$$

4.1.1 Estimates of $\log\left(\frac{M}{\varphi(M)}\frac{N}{\sigma(N)}\right)$

From (3.8), it follows that

$$\log\left(\frac{\sigma(N)}{N}\right) = \log\left(\prod_{k=1}^{K}\prod_{\xi_{k+1} < p \leqslant \xi_k}\frac{1-1/p^{k+1}}{1-1/p}\right) = U_1+U_2+U_3-U_4-U_5 \quad (4.18)$$

with

$$U_1 = \sum_{k=3}^{K}\sum_{\xi_{k+1} < p \leqslant \xi_k}\log\left(1-\frac{1}{p^{k+1}}\right),\ U_2 = \sum_{\xi_3 < p \leqslant \xi_2}\log\left(1-\frac{1}{p^3}\right),$$

$$U_3 = \sum_{p > \xi_2}\log\left(1-\frac{1}{p^2}\right),\ U_4 = \sum_{p > \xi}\log\left(1-\frac{1}{p^2}\right),\ \text{ and }\ U_5 = \sum_{p \leqslant \xi}\log\left(1-\frac{1}{p}\right).$$

Without the assumption that the Riemann hypothesis is true, the following estimates were found in [24, (5.5)–(5.8)]:

$$-\frac{0.251}{\xi^{2/3}} \leqslant U_1 \leqslant 0, \qquad -\frac{0.087}{\xi^{2/3}} \leqslant U_2 \leqslant 0, \quad (4.19)$$

and

$$-\frac{\sqrt{2}}{\sqrt{\xi}\log\xi} + \frac{2.5627}{\sqrt{\xi}\log^2\xi} \leqslant U_3 - U_4 \leqslant -\frac{\sqrt{2}}{\sqrt{\xi}\log\xi} + \frac{3.353}{\sqrt{\xi}\log^2\xi}. \quad (4.20)$$

We can utilize (4.6) and (4.5) to get that

$$\log\left(\frac{M}{\varphi(M)}\right) = -U_5 + U_6 \quad \text{where} \quad U_6 = -\sum_{i=1}^{u}\log\left(1-\frac{1}{p_{r+i}}\right). \quad (4.21)$$

Now we apply (4.14) and (4.13) to the definition of U_6 to see that

$$\begin{aligned} U_6 &\geqslant \sum_{i=1}^{u}\frac{1}{p_{r+i}} \geqslant \frac{u}{p_{r+u}} \geqslant \frac{u}{\xi(1+0.0034/\log\xi)} \\ &\geqslant \frac{\sqrt{2}}{\sqrt{\xi}\log\xi}\left(1-\frac{0.523}{\log\xi}\right)\left(1-\frac{0.0034}{\log\xi}\right) \\ &\geqslant \frac{\sqrt{2}}{\sqrt{\xi}\log\xi}\left(1-\frac{0.5264}{\log\xi}\right) \geqslant \frac{\sqrt{2}}{\sqrt{\xi}\log\xi} - \frac{0.745}{\sqrt{\xi}\log^2\xi}. \end{aligned} \quad (4.22)$$

On the other hand, we can use the inequality (2.2) with $w = 1/\xi$ and $w_0 = 10^{-9}$ to obtain that the inequality

$$U_6 \leqslant -u\log\left(1-\frac{1}{\xi}\right) \leqslant \frac{u}{\xi}\left(1+\frac{1}{2\xi(1-10^{-9})}\right) \leqslant \frac{u}{\xi}\left(1+\frac{\log 10^9}{2(\log\xi)(10^9-1)}\right)$$

holds. Substituting (4.11) into the last inequality, we get that

$$\begin{aligned} U_6 &\leqslant \frac{\sqrt{2}}{\sqrt{\xi}\log\xi}\left(1+\frac{0.779}{\log\xi}\right)\left(1+\frac{0.0001}{\log\xi}\right) \leqslant \frac{\sqrt{2}}{\sqrt{\xi}\log\xi}\left(1+\frac{0.77911}{\log\xi}\right) \\ &\leqslant \frac{\sqrt{2}}{\sqrt{\xi}\log\xi} + \frac{1.102}{\sqrt{\xi}\log^2\xi}. \end{aligned} \tag{4.23}$$

If we combine (4.18) with (4.21), it turns out that (note that U_5 disappears)

$$\log\left(\frac{M}{\varphi(M)}\frac{N}{\sigma(N)}\right) = -(U_1+U_2) - (U_3-U_4) + U_6. \tag{4.24}$$

The inequalities given in (4.19) imply that

$$0 \leqslant -(U_1+U_2) \leqslant \frac{0.338}{\xi^{(2/3)}} \leqslant \frac{0.338\log^2 10^9}{10^{9/6}\sqrt{\xi}\log^2\xi} \leqslant \frac{4.6}{\sqrt{\xi}\log^2\xi}. \tag{4.25}$$

Now we can substitute (4.25), (4.20), and (4.23) into (4.24) to see that

$$\log\left(\frac{M}{\varphi(M)}\frac{N}{\sigma(N)}\right) \leqslant \frac{2\sqrt{2}}{\sqrt{\xi}\log\xi} + \frac{4.6-2.5627+1.102}{\sqrt{\xi}\log^2\xi} = \frac{2\sqrt{2}}{\sqrt{\xi}\log\xi} + \frac{3.1393}{\sqrt{\xi}\log^2\xi}. \tag{4.26}$$

On the other hand, if we substitute (4.25), (4.20), and (4.22) into (4.24), we get that

$$\log\left(\frac{M}{\varphi(M)}\frac{N}{\sigma(N)}\right) \geqslant \frac{2\sqrt{2}}{\sqrt{\xi}\log\xi} - \frac{3.353+0.745}{\sqrt{\xi}\log^2\xi} = \frac{2\sqrt{2}}{\sqrt{\xi}\log\xi} - \frac{4.098}{\sqrt{\xi}\log^2\xi}. \tag{4.27}$$

Applying (2.2) with $w = 2/\xi$ and $w_0 = 2\times 10^{-9}$, we obtain that

$$\begin{aligned} -\log\left(1-\frac{2}{\xi}\right) &\leqslant \frac{2}{\xi\left(1-2\times 10^{-9}\right)} \leqslant \frac{2.0001}{\xi} = \frac{2.0001}{\sqrt{\xi}\log^2\xi}\frac{\log^2\xi}{\sqrt{\xi}} \\ &\leqslant \frac{2.0001}{\sqrt{\xi}\log^2\xi}\frac{\log^2 10^9}{\sqrt{10^9}} \leqslant \frac{0.0272}{\sqrt{\xi}\log^2\xi} \end{aligned} \tag{4.28}$$

and, similarly, with $w = 3/\xi$ and $w_0 = 3 \times 10^{-9}$,

$$\log\left(1 - \frac{3}{\xi}\right) \geqslant -\frac{3}{\xi\left(1 - 3 \times 10^{-9}\right)} \geqslant -\frac{0.0408}{\sqrt{\xi}\log^2 \xi}. \tag{4.29}$$

Consequently, (4.17), (4.26), (4.28), (4.27), and (4.29) allow us to write

$$\frac{2\sqrt{2}}{\sqrt{\xi}\log \xi} - \frac{4.1388}{\sqrt{\xi}\log^2 \xi} \leqslant \log\left(\frac{\Phi(X)}{\Sigma(X)}\right) \leqslant \frac{2\sqrt{2}}{\sqrt{\xi}\log \xi} + \frac{3.1665}{\sqrt{\xi}\log^2 \xi}. \tag{4.30}$$

By using (3.27), (3.28), and the notation $L = \log X$, $\lambda = \log\log X$, (4.30) yields

$$\frac{2\sqrt{2}}{\sqrt{L}\lambda} - \frac{4.143}{\sqrt{L}\lambda^2} \leqslant \log\left(\frac{\Phi(X)}{\Sigma(X)}\right) \leqslant \frac{2\sqrt{2}}{\sqrt{L}\lambda} + \frac{3.1698}{\sqrt{L}\lambda^2} \leqslant \frac{2.9814}{\sqrt{L}\lambda} \leqslant 0.000005. \tag{4.31}$$

The lower bound of (1.12) follows from (2.3) while the upper bound is obtained by applying (2.4) with $u = 2\sqrt{2}/\left(\sqrt{L}\lambda\right) + 3.1698/\left(\sqrt{L}\lambda^2\right)$, $u_0 = 0.000005$, and

$$\frac{u^2}{2(1-u_0)} \leqslant \frac{2.9814^2}{1.99999\left(\sqrt{L}\lambda\right)^2} \leqslant \frac{2.9814^2}{1.99999\sqrt{10^9 L}\lambda^2} \leqslant \frac{0.000141}{\sqrt{L}\lambda^2}.$$

This completes the proof of Theorem 1.1 for every real $X > N^{(0)}$.

4.2 Proof of Theorem 1.1 for $X \leqslant N^{(0)}$

4.2.1 Proof of Theorem 1.1 for $4 \leqslant X \leqslant 10^{49}$

We can use the algorithm described in [24, Sect. 4.1] (see also [23, Sect. 3.4]), to compute all SA numbers up to 10^{50}. We also compute the primorials in the range $[4, 10^{50}]$. We sort these 351 numbers in ascending order in a sequence $n_1 = 4$, $n_2 = 6$, up to n_{351}. The largest one is the SA number

$$n_{351} = 2^8 3^5 5^3 7^3 11^2 13^2 \prod_{17 \leqslant p \leqslant 107} p = 5.12\ldots \times 10^{49}.$$

For $X \in [n_j, n_{j+1})$, $g(X)$ is constant and equal to $a_j = m_j s_j/(\varphi(m_j)\sigma(s_j)) - 1$, where s_j (resp. m_j) is the largest SA number (resp. primorial) $\leqslant n_j$ and $g(X)$ is given by (4.2). Let us assume $1 \leqslant j \leqslant 350$ and $n_j \leqslant X < n_{j+1}$. Then, $n_j \geqslant 4 > e$ and $\rho(X)$ is given by (4.3). Now, we have to find the maximum and the minimum of $\rho(X)$. In order to do this, we make use of the convexity of h (cf. Lemma 2.6, note that $\log\log X \geqslant \log\log 4 > 0 > 2\sqrt{2} - 4$). We set $t_j = \log\log n_j$, $t_{j+1} = \log\log n_{j+1}$ and consider the following cases:

- If $h'(g(X), t_j) \geqslant 0$ then $\rho(X) = h(g(X), t)$ is increasing on $[t_j, t_{j+1})$ (case 1).
- If $h'(g(X), t_j) < 0$ and $h'(g(X), t_{j+1}) \leqslant 0$ then $\rho(X) = h(g(X), t)$ is decreasing on $[t_j, t_{j+1})$ (case 2).
- If $h'(g(X), t_j) < 0$ and $h'(g(X), t_{j+1}) > 0$ then $\rho(X) = h(g(X), t)$ has its minimum on (t_j, t_{j+1}) (case 3).

Note that case 2 occurs for X in $[4, 30)$ and $[120, 210)$ while case 3 occurs just once, namely for X in $[60, 120)$. The smallest value of $\rho(X)$ is $\rho(27720) = -3.3308\ldots$ and the largest one is $1.5566\ldots$ in the interval $[M_{127}, N^{(1)})$ when X tends to

$$N^{(1)} = 2^9 3^4 5^3 7^2 11^2 13^2 \prod_{17 \leqslant p \leqslant 103} p = 4.14\ldots \times 10^{48},$$

(cf. [34]).

4.2.2 Proof of Theorem 1.1 for $10^{49} \leqslant X \leqslant 10^{7648}$

Let $12 \leqslant N' < N$ be two consecutive CA numbers. By the algorithm described above in Sect. 3.1, one determines all SA numbers n satisfying $N' \leqslant n \leqslant N$ and also, from a precomputed table of primorials, the largest primorial $M' < N'$ and the largest primorial $M < N$. From Lemma 3.3, we have either $M = M'$ or M is the primorial following M'. We order these SA numbers n and M (if $M > N'$) in a sequence

$$M' < N' = n_1 < n_2 < \ldots < n_r = N.$$

For $X \in [n_j, n_{j+1})$ with $1 \leqslant j \leqslant r-1$, $g(X)$ is equal to $mn/(\varphi(m)\sigma(n)) - 1$ where m is the largest of M' and M satisfying $m \leqslant n_j$ and n is the largest SA number $\leqslant n_j$. Then, we apply the algorithm of Sect. 4.2.1. As explained in [24, Sect. 4.2], from a precomputed table of CA2 numbers, one generates the CA numbers $> 5.98 \times 10^{44}$ up to $1.19\ldots \times 10^{7648}$. For $X < 10^{49}$, we have checked that the results coincide with those of Sect. 4.2.1 and for $X \geqslant 10^{49}$ all intervals are of case 1. The minimal value $-0.977\ldots$ of $\rho(X)$ is obtained for (cf. [34])

$$N^{(2)} = 2^8 3^5 5^3 7^2 11^2 13^2 \prod_{17 \leqslant p \leqslant 113} p = 8.201\ldots \times 10^{54}.$$

The maximal value $2.419\ldots$ is obtained for $X < N^{(3)}$ and tending to

$$N^{(3)} = 2^{12} 3^7 5^5 7^4 11^3 13^3 \prod_{17 \leqslant p \leqslant 43} p^2 \prod_{47 \leqslant p \leqslant 1091} p = 1.036\ldots \times 10^{485}.$$

4.2.3 Proof of Theorem 1.1 for $10^{7648} < X \leqslant N^{(0)}$

For X very large, the algorithm exposed in Sect. 4.2.2 would be too long to carry out. So, from our precomputed table of CA2 numbers, we generate all the CA numbers up to $N^{(0)}$. If $N' < N$ are two consecutive CA numbers and $N' \leqslant X \leqslant N$ then $\sigma(N')/N' \leqslant \Sigma(X) \leqslant \sigma(N)/N$. If M' is the largest primorial $< N'$ and M the largest primorial $< N$, we see that $M'/\varphi(M') \leqslant \Phi(X) \leqslant M/\varphi(M)$. Note that N' and N are chosen in the same way in terms of X as in Sect. 4.1. Therefore, for $N' \leqslant X \leqslant N$, we obtain that

$$a_1 = \frac{M'N}{\varphi(M')\sigma(N)} - 1 \leqslant g(X) = \frac{\Phi(X)}{\Sigma(X)} - 1 \leqslant a_2 = \frac{MN'}{\varphi(M)\sigma(N')} - 1, \tag{4.32}$$

and from (4.3), we get that

$$h(a_1, t) \leqslant \rho(X) = h\big(g(X), t\big) \leqslant h(a_2, t) \quad \text{with} \quad t = \log\log X. \tag{4.33}$$

In view of applying Lemma 2.6 (ii), one checks that $h(a_1, \log\log N') > -5$. This implies that $h(a_2, \log\log N') > -5$. The two mappings $t \mapsto h(a_1, t)$ and $t \mapsto h(a_2, t)$ are increasing on the interval $[\log\log N', \log\log N]$ which provides

$$h(a_1, \log\log N') \leqslant \rho(X) \leqslant h(a_2, \log\log N).$$

For $10^{7648} < X \leqslant N^{(0)}$ this yields $-1.91 \leqslant \rho(X) < 2.39$ (cf. [34]).

Conclusion. *By gathering the results of Sect. 4.2.1, 4.2.2 and 4.2.3, we see that*

$$-3.34 \leqslant \rho(X) \leqslant 2.42 \quad \textit{for} \quad 4 \leqslant X \leqslant N^{(0)},$$

which completes the proof of Theorem 1.1.

5 Proof of Theorem 1.2

In order to prove that the inequality (1.21) holds for every $n \geqslant X^{(0)}$ (defined by (1.20)), we introduce the following notation. Let $M^{(0)}$ be defined by (1.16) and let $i = 564\,397\,542$. Then $p_i = 12\,530\,577\,161$ and

$$M_{p_{i-1}} < X^{(0)} < M_{p_i} = \exp(12\,530\,479\,278.847331\ldots).$$

Similarly, let $m = 1\,469\,923\,277$. Then, $p_m = 34\,110\,324\,851$ and

$$M_{p_m} = \exp(34\,110\,069\,410.651478\ldots).$$

5.1 Proof of Theorem 1.2 for $n \geqslant M^{(0)}$

For every integer $n \geqslant M^{(0)}$, the required inequality (1.21) was already proven in [3] (cf. (1.18)). So it suffices to deal with the case where n satisfies $X^{(0)} \leqslant n < M^{(0)}$.

5.2 Proof of Theorem 1.2 for $M_{p_m} \leqslant n < M^{(0)}$

In this case, we can utilize Theorem 1.1 to see that

$$\frac{n}{\varphi(n)} \leqslant \Sigma(n) \left(1 + \frac{2\sqrt{2}}{\sqrt{\log n} \log\log n} + \frac{3.17}{\sqrt{\log n}(\log\log n)^2} \right). \tag{5.1}$$

Let N be the largest superabundant number not exceeding n. Then $5040 < N < M^{(0)}$ and $\Sigma(n) = \sigma(N)/N$. Applying (1.17), we get

$$\Sigma(n) = \frac{\sigma(N)}{N} < e^\gamma \log\log N \leqslant e^\gamma \log\log n.$$

Now we substitute this inequality into (5.1) to obtain that

$$\frac{n}{\varphi(n)} \leqslant e^\gamma \log\log n + \frac{2\sqrt{2}e^\gamma}{\sqrt{\log n}} + \frac{3.17e^\gamma}{\sqrt{\log n} \log\log n}. \tag{5.2}$$

Note that

$$\frac{2\sqrt{2}}{\sqrt{\log x}} + \frac{3.17}{\sqrt{\log x} \log\log x} < \frac{0.0094243}{(\log\log x)^2} \tag{5.3}$$

for every $x \geqslant M_{p_m}$. If we combine (5.2) with (5.3), it turns out that the inequality (1.21) holds for every integer n with $M_{p_m} \leqslant n < M^{(0)}$.

5.3 Proof of Theorem 1.2 for $M_{p_i} \leqslant n < M_{p_m}$

For the sake of readability, we introduce the function

$$H(x) = e^\gamma \log\log x + \frac{\alpha_0}{(\log\log x)^2}. \tag{5.4}$$

Note that H is an increasing function on the interval $[3.69, \infty)$. We need to show that $n/\varphi(n) \leqslant H(n)$ for every integer n with $M_{p_i} \leqslant n < M_{p_m}$. For this purpose, we check with a computer that the inequality

$$\prod_{p \leqslant p_r} \frac{p}{p-1} \leqslant H(\exp(\theta(p_r))) \tag{5.5}$$

holds for every integer r with $i \leqslant r \leqslant m$ while the inequality (5.5) is violated for $r = i - 1$. Let n be an integer with $n \in [M_{p_i}, M_{p_m})$ and let k be the unique positive integer with $M_{p_k} \leqslant n < M_{p_{k+1}}$. Then $i \leqslant k < m$. Now we can use (1.10), (5.5), and the identity $\log M_{p_k} = \theta(p_k)$ to get

$$\frac{n}{\varphi(n)} \leqslant \frac{M_{p_k}}{\varphi(M_{p_k})} = \prod_{p \leqslant p_k} \frac{p}{p-1} \leqslant H(M_{p_k}) \leqslant H(n).$$

Hence, the required inequality (1.21) holds for every integer n with $M_{p_i} \leqslant n < M_{p_m}$.

Remark 5.1. *In order to check the inequality* (5.5) *for every integer r with $i \leqslant r \leqslant m$, we first use PARI/GP, and independently Maple, to compute the value*

$$\frac{M_{p_m}}{\varphi(M_{p_m})} = \prod_{p \leqslant p_m} \frac{p}{p-1} = 43.19611605653021721681\ldots \tag{5.6}$$

with an accuracy of 70 digits. After this, we successively check the inequality (5.5) *with Maple for every integer r with $i \leqslant r \leqslant m$ with an accuracy of 70 digits which resulted in a run time of 200 hours on a standard desktop.*

Remark 5.2. *It is natural to ask whether it is sufficient to utilize Rosser and Schoenfeld type bounds (cf. [30]) for the computation of the product given in* (5.6)*. If we use, for instance, [4, Proposition 7.1], we see that*

$$43.196082\ldots \leqslant \frac{M_{p_m}}{\varphi(M_{p_m})} = \prod_{p \leqslant p_m} \frac{p}{p-1} \leqslant 43.196138\ldots,$$

while

$$H(M_{p_m}) = 43.196125\ldots.$$

Thus, we cannot conclude that $M_{p_m}/\varphi(M_{p_m}) \leqslant H(M_{p_m})$. For this reason, we need the exact value of the product given in (5.6)*.*

5.4 Proof of Theorem 1.2 for $X^{(0)} \leqslant n < M_{p_i}$

Finally, let n be an integer with $X^{(0)} \leqslant n < M_{p_i}$. Since $X^{(0)} \in (M_{p_{i-1}}, M_{p_i})$, it turns out that

$$\begin{aligned}\frac{n}{\varphi(n)} \leqslant \frac{M_{p_{i-1}}}{\varphi(M_{p_{i-1}})} &= 41.41251143922748882925 8\ldots \\ &< 41.412511439227488829267\ldots = H(X^{(0)}) \leqslant H(n),\end{aligned}$$

and we arrive at the end of the proof of Theorem 1.2.

6 Proof of Theorem 1.3

For better readability, we write $Y^{(0)} = \exp(26\,318\,064\,420)$ and $Y^{(1)} = \exp(35\,528\,457\,899)$.

6.1 Proof of Theorem 1.3 for $n \geqslant Y^{(1)}$

Let n be an integer with $n \geqslant Y^{(1)}$ and let

$$f(n) = 1 + \omega \quad \text{with} \quad \omega = \frac{2\sqrt{2}}{\sqrt{\log n}\log\log n} - \frac{4.143}{\sqrt{\log n}(\log\log n)^2}.$$

Note that $0 \leqslant \omega \leqslant 2\sqrt{2}/(\sqrt{\log n}\log\log n)$. Using Theorem 1.1, we get

$$\begin{aligned}\frac{\sigma(n)}{n} &\leqslant \Sigma(n) \leqslant \frac{\Phi(n)}{f(n)} = \frac{\Phi(n)}{1+\omega} \leqslant \Phi(n)(1 - \omega + \omega^2)\\ &\leqslant \Phi(n)\left(1 - \frac{2\sqrt{2}}{\sqrt{\log n}\log\log n} + \frac{4.143}{\sqrt{\log n}(\log\log n)^2} + \frac{8}{\log n(\log\log n)^2}\right).\end{aligned} \tag{6.1}$$

Define M_{p_r} to be the largest primorial not exceeding n. Then $\Phi(n) = M_{p_r}/\varphi(M_{p_r})$. Now we can utilize Theorem 1.2 to get that

$$\Phi(n) = \frac{M_{p_r}}{\varphi(M_{p_r})} \leqslant H(M_{p_r}) \leqslant H(n),$$

where $H(x)$ is defined as in (5.4). If we substitute this inequality into (6.1), we see that

$$\frac{\sigma(n)}{n} \leqslant e^\gamma \log\log n + \frac{\alpha_0}{(\log\log n)^2} - \frac{2\sqrt{2}e^\gamma}{\sqrt{\log n}} + \frac{4.143e^\gamma}{\sqrt{\log n}\log\log n} + \frac{r(n)}{\sqrt{\log n}\log\log n},$$

where α_0 is defined as in (1.19) and

$$r(x) = -\frac{2\sqrt{2}\alpha_0}{(\log\log x)^2} + \frac{4.143\alpha_0}{(\log\log x)^3} + \frac{8\alpha_0}{\sqrt{\log x}(\log\log x)^3} + \frac{8e^\gamma}{\sqrt{\log x}}.$$

It suffices to note that $r(x) < 0$ for every $x \geqslant Y^{(1)}$ to conclude that the inequality (1.25) holds for every integer $n \geqslant Y^{(1)}$.

6.2 Proof of Theorem 1.3 for $Y^{(0)} \leqslant n < Y^{(1)}$

In order to prove the inequality (1.25) for every integer n satisfying $Y^{(0)} \leqslant n < Y^{(1)}$, we note that

$$\frac{a_0 e^\gamma}{(\log\log x)^2} - \frac{2\sqrt{2}e^\gamma}{\sqrt{\log x}} + \frac{4.143 e^\gamma}{\sqrt{\log x}\log\log x} > 0$$

for every $x \geqslant Y^{(0)}$. If we combine this inequality with (1.17), we get the required inequality (1.25) for every integer n with $Y^{(0)} \leqslant n < Y^{(1)}$ which completes the proof of Theorem 1.3.

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References

[1] Alaoglu, L., Erdős, P.: On highly composite and similar numbers. Trans. Amer. Math. Soc. **56**, 448–469 (1944)

[2] Axler, C.: A new upper bound for the sum of divisors function. Bull. Aust. Math. Soc. **96**(3), 374–379 (2017)

[3] Axler, C.: On Robin's inequality. To appear in the Ramanujan J.

[4] Axler, C.: Effective estimates for some functions defined over primes. Preprint. Available at `arxiv.org/abs/2203.05917` (2022)

[5] Banks, W. D., Hart, D. N., Moree, P., Nevans, C. W.: The Nicolas and Robin inequalities with sums of two squares. Monatsh. Math. **157**, 303–322 (2009)

[6] Briggs, B.: Abundant numbers and the Riemann hypothesis. Experiment. Math. **15**(2), 251–256 (2006)

[7] Broadbent, S., Kadiri, H., Lumley, A., Ng, N., Wilk, K.: Sharper bounds for the Chebyshev function $\theta(x)$. Math. Comp. **90** (331), 2281–2315 (2021)

[8] Broughan, K.: Equivalents of the Riemann Hypothesis, vol. 1. Encyclopedia of Mathematics and its Applications, 164, Cambridge University Press, (2017)

[9] Broughan, K., Trudgian, T.: Robin's inequality for 11-free integers. Integers **15**, Paper No. A12, 5 pp. (2015)

[10] Büthe, J.: Estimating $\pi(x)$ and related functions under partial RH assumptions. Math. Comp. **85**, no. 301, 2483–2498 (2016)

[11] Choie, Y.-J., Lichiardopol, N., Moree, P., Solé, P.: On Robin's criterion for the Riemann hypothesis. J. Théor. Nombres Bordeaux **19**, 357–372 (2007)

[12] Deléglise, M., Nicolas, J.-L.: The Landau function and the Riemann hypothesis. J. Comb. Number Theory **11**(2), 45–95 (2019)

[13] Fiori, A., Kadiri, H., Swidinsky, J.: Sharper bounds for the error term in the Prime Number Theorem. Preprint. Available at `arxiv.org/abs/2206.12557` (2022)

[14] Gronwall, T. H.: Some asymptotic expressions in the theory of numbers. Trans. Amer. Math. Soc. **14**, 113–122 (1913)

[15] Grytczuk, A.: Upper bound for sum of divisors function and the Riemann hypothesis. Tsukuba J. Math. **31**, 67–75 (2007)

[16] Hardy, G.H., Wright, E.M.: An introduction to the theory of numbers. 4th edition, Oxford, Clarendon Press, (1960)

[17] Johnston, D. R.: Improving bounds on prime counting functions by partial verification of the Riemann hypothesis. Preprint. Available at `arxiv.org/abs/2109.02249` (2021)

[18] Landau, E.: Über den Verlauf der zahlentheoretischen Funktion $\varphi(x)$. Archiv. der Mathematik und Physik, Ser. 3, **5**, 86–91 (1903) (see also: Handbuch der Lehre von der Verteilung der Primzahlen, I, 2nd ed., Chelsea, 216–219 (1953))

[19] Mertens, F.: Ein Beitrag zur analytischen Zahlentheorie. J. Reine Angew. Math. **78**, 42–62 (1874)

[20] Morrill, T., Platt, D. J.: Robin's inequality for 20-free integers. Integers **21**, Paper No. A28, 7 pp. (2021)

[21] Nicolas, J.-L.: Petites valeurs de la fonction d'Euler. J. Number Theory **17**, 375–388 (1983)

[22] Nicolas, J.-L.: Small values of the Euler function and the Riemann hypothesis. Acta Arith. **155**(3), 311–321 (2012)

[23] Nicolas, J.-L.: Highly Composite Numbers and the Riemann hypothesis, Ramanujan J. **57**, no. 2, 507–550 (2022)

[24] Nicolas, J.-L.: The sum of divisors function and the Riemann hypothesis. Ramanujan J. **58**, no. 4, 1113–1157 (2022)

[25] Platt, D. J., Trudgian, T.: The Riemann hypothesis is true up to $3 \cdot 10^{12}$. Bull. Lond. Math. Soc. **53**, no. 3, 792–797 (2021)

[26] Ramanujan, S.: Highly composite numbers. Proc. London Math. Soc., Serie 2, **14**, 347–409 (1915). In: Collected papers of Srinivasa Ramanujan, 78–128. Cambridge University Press (1927)

[27] Ramanujan, S.: Highly composite numbers, annotated and with a foreword by J.-L. Nicolas and G. Robin. Ramanujan J. **1**, 119–153 (1997)

[28] Robin, G.: Sur l'ordre maximum de la fonction somme des diviseurs. Seminar on number theory, Paris (1981–82) 233–244. Progr. Math., 38, Birkhäuser, Boston, (1983)

[29] Robin, G.: Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. J. Math. Pures Appl. **63**, 187–213 (1984)

[30] Rosser, J. B., Schoenfeld, L.: Approximate formulas for some functions of prime numbers. Illinois J. Math. **6**, 64–94 (1962)

[31] Schoenfeld, L.: Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$ II. Math. Comp. **30**, no. 134, 337–360 (1976)

[32] Solé, P., Planat, M.: The Robin inequality for 7-free integers. Integers **11**, Paper No. A65, 8 pp. (2011)

[33] von Koch, H.: Sur la distribution des nombres premiers. Acta Math. **24**, no. 1, 159–182 (1901)

[34] `http://math.univ-lyon1.fr/homes-www/nicolas/calculSigPhiX.html`

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