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On the Eulerian Numbers $M_n = \max_{1\le k\le n} A(n, k)$

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1. INTRODUCTION AND SUMMARY

The eulerian numbers $A(n, k)$ have been the subject of many studies since Euler's time to the present [3, 7, 14]. They can be defined and computed for every k and n, $1 \le k \le n$, by means of the triangular recurrence relation

$$A(n+1, k) = (n-k+2)A(n, k-1) + kA(n, k) \tag{1}$$

with the starting conditions $A(n, 1) = A(n, n) = 1$. These numbers satisfy the symmetrical relation

$$A(n, k) = A(n, n-k+1). \tag{2}$$

We give below the table of [5], for $n \le 12$.

	k					
n	1	2	3	4	5	6
1	1					
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	1	26	66	26	1	
6	1	57	302	302	57	1
7	1	120	1191	2416	1191	120
8	1	247	4293	15619	15619	4293
9	1	502	14608	88234	156190	88234
10	1	1013	47840	455192	1310354	1310354
11	1	2036	152637	2203488	9738114	15724248
12	1	4083	478271	10187685	66318474	162512286

On each line n, $n = 2p-1$, the $A(n, k)$'s increase from 1 for $k = 1$ to the maximum $M_{2p-1} = A(2p-1, p)$ for $k = p$, whereas for $n = 2p$, the $A(n, k)$'s increase from $1\ (k=1)$ to the maximum $M_{2p} = A(2p, p) = A(2p, p+1)$. The maximum on the line n is therefore equal to:

$$M_n = \max_{k=1,2,\dots,n} A(n, k) = A\left(n, \left[\frac{n}{2}\right] + 1\right). \tag{3}$$

The $A(n, k)$'s have a well-known expression, [3, t. 2, p. 85],

$$A(n, k) = \sum_{0\le j\le k} (-1)^j \binom{n+1}{j} (k-j)^n \tag{4}$$

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so that

$$M_{2p-1}=\sum_{0\leq j\leq p}(-1)^j\binom{2p}{j}(p-j)^{2p-1},\qquad M_{2p}=\sum_{0\leq j\leq p}(-1)^j\binom{2p+1}{j}(p-j)^{2p}.$$

Now, it happens that we meet those same numbers in a very different form in a question with an algebraic origin† related to the theory of modules [4], that is to say:

$$s(n)=\sum_{j=0}^{n}\binom{n+2}{j}(-1)^{j+1}\sum_{q=\max(0,[n/2]+2-j)}^{n+1-j}q^n. \tag{5}$$

The table of values of $s(n)$, given in Appendix 1, shows that $s(n)=M_n$. This remark led us to study the new properties of eulerian number $M_n=\max_{k=1,\ldots,n}A(n,k)$ in a more systematic manner. Note that M_n is a 'peak' on the line $n=2p-1$, and a 'plateau' on the line $n=2p$.

In Section 2.1 we begin by proving the equality $M_n=s(n)$ (Theorem 1). Then we recall some immediate properties of the M_n's resulting from known properties of the $A(n,k)$'s (Theorem 2). Next we prove in Section 3.1 that the sequences $M_{2p}/(2p)!$ and $M_{2p-1}/(2p-1)!$ are decreasing, and this is more difficult. In order to obtain that result we have proved the following inequality which compares $A(n,k-1)$ and $A(n,k)$ on the line n:

$$A(n,k-1)<\left(\frac{n-k}{n-k+2}\right)^{n-2k+2}A(n,k),\qquad n\geq 2k-1,\quad k\geq 2,$$

and we apply this inequaltiy near the maximum. However, it is true for all $k\leq(n+1)/2$ (Theorem 3). The above inequality also allows the study of the variation of $A(n,k)/n!$ on the fixed column k. $A(n,k)/n!$ increases from $1/k!\,(n=k)$ to a maximum of $M_{2k-1}/(2k-1)!\,(n=2k-1)$ and decreases afterwards towards zero (Theorem 4).

Up to this point, the methods are chiefly combinatorial. In Section 3 we obtain an asymptotic expansion of $M_{2k-1}/(2k-1)!$ using analytical methods. Some authors have given approximate expressions of $A(n,k)/n!$, [1, 2, 13], mainly by means of probabilistic methods. Let us mention, for instance, Sirăzdinov's formula [13], which reduces for $n=2p-1$, $k=p$ to‡

$$\frac{M_{2p-1}}{(2p-1)!}=\sqrt{\frac{3}{\pi p}}\left(1-\frac{3}{40p}\right)+O\left(\frac{1}{p^{\frac{5}{2}}}\right).$$

The remainder here is not precise enough for the questions we are looking at, such as the decreasing property of $M_{2p-1}/(2p-1)!$. We propose a deeper analysis using Cauchy's integral formula and the Laplace method to calculate the coefficients of the generating functions. After several technical and useful explanations given in Section 3.2, we arrive at the following inequalities:

$$\forall p\geq 3,\qquad \frac{M_{2p-1}}{(2p-1)!}\leq\sqrt{\frac{3}{\pi p}}\left(1-\frac{3}{40p}\right)\qquad\text{(Section 3.3, Lemma 1)}$$

$$\forall p\geq 1,\qquad \frac{M_{2p-1}}{(2p-1)!}\geq\sqrt{\frac{3}{\pi p}}\left(1-\frac{3}{40p}-\frac{13}{4480p^2}\right)\qquad\text{(Section 3.4, Lemma 1)}$$

† For other recent applications of eulerian numbers to algebra, see also J.-L. Loday [9] (Section 3.3, Lemma 1).

‡ We are very grateful to Mrs V. Glaymann for translating this paper from Russian.

They allow us to prove the following theorem which gives the best possible inequalities between the maxima eulerian numbers M_n:

THEOREM 3 (*Section* 3.5). (i) *The sequence* $M_{2p-1}/(2p-1)!$ *is decreasing.*
(ii) *The sequence* $M_{2p}/(2p)!$ *is decreasing.*
(iii) *For all* $p \geqslant 1$, *we have*:

$$\frac{M_{2p+5}}{(2p+5)!} \leqslant \frac{M_{2p}}{(2p)!} \leqslant \frac{M_{2p+3}}{(2p+3)!}$$

2. Combinatorial Results

2.1. *Equality between the expression $s(n)$ and the eulerian number M_n*

The numbers M_n and $s(n)$ have been defined by formulas (3) and (5) of Section 1. Now we will prove their equality.

THEOREM 1. *We have* $s(n) = M_n$.

PROOF. *Case* $n = 2p$.

$$s(2p) = \sum_{j=0}^{2p} \binom{2p+2}{j} (-1)^{j+1} \sum_{q=\max(0,p+2-j)}^{2p+1-j} q^{2p}.$$

Let us transpose both sums:

$$s(2p) = \sum_{q=1}^{2p+1} q^{2p} \sum_{j=\max(0,p+2-q)}^{2p+1-q} (-1)^{j+1} \binom{2p+2}{j}.$$

Replacing the binomial coefficient $\binom{2p+2}{j}$ by $\binom{2p+1}{j} + \binom{2p+1}{j-1}$, we obtain for the coefficient C_q of q^{2p}:

$$C_1 = (-1)^p \left[\binom{2p+1}{p} + \binom{2p+1}{2p} \right] \qquad C_{p+2} = -\binom{2p+1}{p-1}$$

$$C_2 = (-1)^{p+1} \left[\binom{2p+1}{p-1} + \binom{2p+1}{2p-1} \right] \qquad C_{p+3} = +\binom{2p+1}{p-2}$$

$$C_3 = (-1)^{p+2} \left[\binom{2p+1}{p-2} + \binom{2p+1}{2p-2} \right] \qquad C_{p+4} = -\binom{2p+1}{p-3}$$

$$\vdots \qquad\qquad \vdots$$

$$C_{p+1} = (+1) \left[1 + \binom{2p+1}{p} \right] \qquad C_{2p+1} = (-1)^p$$

Hence,

$$s(2p) = (-1)^p \binom{2p+1}{p} + (-1)^{p+1} \binom{2p+1}{p-1} 2^{2p} + (-1)^{p+2} \binom{2p+1}{p-2} 3^{2p}$$
$$+ \cdots + (p+1)^{2p} + (-1)^p \left[\binom{2p+1}{2p} 1^{2p} - \binom{2p+1}{2p-1} 2^{2p} + \binom{2p+1}{2p-2} 3^{2p} \right.$$
$$\left. - \cdots + (2p+1)^{2p} \right].$$

But we have for every x (see Lemma 1 further on):

$$x^{2p} - \binom{2p+1}{1}(x+1)^{2p} + \binom{2p+1}{2}(x+2)^{2p} - \cdots + (-1)(x+2p+1)^{2p} = 0.$$

Putting $x = 0$, we see that the term between brackets in the expression of $s(2p)$ is zero. Therefore, what remains, because of formula (4) of Section 1, is:

$$\begin{aligned} s(2p) &= (-1)^p\binom{2p+1}{p} + (-1)^{p+1}\binom{2p+1}{p-1}2^{2p} + (-1)^{p+2}\binom{2p+1}{p-2}3^{2p} \\ &\quad + \cdots + (p+1)^{2p} \\ &= \sum_{0\leqslant j\leqslant p} (-1)^j\binom{2p+1}{j}(p-j)^{2p} = A(2p, p) = A(2p, p+1). \end{aligned}$$

We know that this is $\max_{1\leqslant k\leqslant 2p} A(2p, k) = M_{2p}$.

Case $n = 2p - 1$. The calculations are similar. We find:

$$s(2p-1) = \sum_{0\leqslant j\leqslant p} (-1)^j\binom{2p}{j}(p-j)^{2p-1} = A(2p-1, p) = M_{2p-1}.$$

The following lemma completes the proof of Theorem 1.

Lemma 1. *The polynomial*:

$$f(x) = x^k - \binom{n}{1}(x+1)^k + \binom{n}{2}(x+2)^k - \cdots + (-1)^n(x+n)^k$$

is zero for all integers k *and* n, $0 \leqslant k < n$.

Proof. For every polynomial $P \in \mathbb{R}[X]$, we define the difference operator Δ: $\mathbb{R}[X] \to \mathbb{R}[X]$ by:

$$\Delta P(X) = P(X) - P(X+1) = \Delta_1 P(X).$$

and, when $j \geqslant 2$, $\Delta_j P(X) = \Delta(\Delta_{j-1}P(X))$. We can easily prove by induction that:

$$\Delta_j P(X) = P(X) - \binom{j}{1}P(X+1) + \binom{j}{2}P(X+2) - \cdots + (-1)^j P(X+j),$$

and, if $\deg P \geqslant j$, $\deg \Delta_j P = \deg P - j$. Taking $P(X) = X^k$, the polynomial f in the lemma is $f(X) = \Delta_n P(X)$. Then $\Delta_k P(X)$ has degree 0, and, if $n > k$, $\Delta_n P(X) = 0$. □

2.2. *First Properties of* M_n *and* $M_n/n!$

These properties result from known properties of the eulerian numbers $A(n, k)$.

Theorem 2. *We have*:
(i) $(n-1)! \leqslant M_n \leqslant n!$, $n = 1, 2, \ldots$;
(ii) $M_{2p+1} = (2p+2)M_{2p}$, $p = 1, 2, \ldots$;
(iii) $M_{2p} \leqslant (2p)!/2$, $p = 1, 2, \ldots$;
(iv) $M_{2p+1} \leqslant (0.55) \times (2p+1)!$, $p = 2, 3, \ldots$;
(v) $M_n/n! \sim \sqrt{6/[\pi(n+1)]}$ *when* $n \to +\infty$, *and therefore* $\lim_{n\to+\infty} M_n/n! = 0$.

Proof. (i) follows from the equality $\sum_{k=1}^{n} A(n, k) = n!$ and the definition $M_n = \max_{1\leqslant k\leqslant n} A(n, k)$ which implies:

$$n! \leqslant nM_n \leqslant n \cdot (n!).$$

(ii) is a consequence of the triangular recurrence relation $A(n+1, k) = (n-k+2)A(n, k-1) + kA(n, k)$, where $n = 2p$, $k = p+1$. It gives:

$$A(2p+1, p+1) = (p+1)A(2p, p) + (p+1)A(2p, p+1) = (2p+2)A(2p, p),$$

because $A(2p, p) = A(2p, p+1)$ by the symmetrical relation (2) of Section 1. Thus we have $M_{2p+1} = (2p+2)M_{2p}$.

(iii) In the line $2p$ there is a symmetry with respect to the middle. The sum of the first p terms is therefore $(2p)!/2$ and it is greater than the pth term M_{2p}.

(iv) From (ii) and (iii) we have:

$$\frac{M_{2p+1}}{(2p+1)!} = \frac{(2p+2)M_{2p}}{(2p+1)(2p)!} \leqslant \frac{2p+2}{2p+1} \times \frac{1}{2}.$$

The function $p \mapsto (2p+2)/(2p+1) = 1 + 1/(2p+1)$ is decreasing. Therefore, when $p > 19$ $(n > 39)$, we have:

$$\frac{2p+2}{2p+1} < \frac{40}{39} < 1.03 \quad \text{and} \quad \frac{M_{2p+1}}{(2p+1)!} < 1.03 \times 0.5 < 0.55.$$

However, we see in the table of Appendix 1 that the property $M_{2p+1}/(2p+1)! \leqslant 0.55$ is also true for all values of p from 2 to 19. Thus (iv) holds. When $p = 2$ $(n = 5)$ we have the equality $M_{2p+1}/(2p+1)! = 0.55$. The only exceptional values of p are $p = 0$ $(n = 1)$ and $p = 1$ $(n = 3)$.

(v) The probability distribution associated with the line n is given by $A(n, k)/n!$. The mean is $m = (n+1)/2$ and the variance is $v = \sigma^2 = (n+1)/12$ (see [5, p. 51]). When $n \to +\infty$, these authors prove, by means of the central limit theorem of the probability theory, that this distribution is equivalent to the normal Gaussian distribution

$$\frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-(x-m)^2/2\sigma^2},$$

and that the maximum $M_n/n!$ is equivalent to

$$\frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{\sqrt{(2\pi/12)(n+1)}} = \sqrt{\frac{6}{\pi(n+1)}}.$$

(see also [1, 2, 13]). Consequently, $M_n/n!$ tends to 0 when $n \to +\infty$, and this is illustrated in the table of Appendix 1. On the other hand, the series $M_n/n!$ is divergent, as could already be seen before with (ii): $M_n/n! > 1/n$. □

2.3. *The Sequences $M_{2p}/(2p)!$ and $M_{2p+1}/(2p+1)!$ are decreasing*

Let us consider the inequality:

$$\frac{M_{2p+2}}{(2p+2)!} < \frac{M_{2p}}{(2p)!}. \tag{1}$$

We have (cf. Section 2, Theorem 2(ii)):

$$M_{2p} = A(2p, p) = A(2p, p+1), \qquad M_{2p+1} = A(2p+1, p+1) = (2p+2)M_{2p}.$$

Let us apply the triangular recurrence relation to:

$$M_{2p+2} = A(2p+2, p+1) = (p+2)A(2p+1, p) + (p+1)A(2p+1, p+1),$$

$$A(2p+1, p) = (p+2)A(2p, p-1) + pA(2p, p).$$

We obtain:

$$M_{2p+2}=(p+2)^2A(2p,p-1)+[p(p+2)+2(p+1)^2]M_{2p},$$

$$\frac{M_{2p+2}}{(2p+2)!}=\frac{M_{2p}}{(2p)!}\left[\frac{p(p+2)+2(p+1)^2}{(2p+1)(2p+2)}+\frac{(p+2)^2}{(2p+1)(2p+2)}\frac{A(2p,p-1)}{A(2p,p)}\right].$$

Hence the inequality (1) is equivalent to $[\ ]<1$, or:

$$A(2p,p-1)<\frac{2(p+1)(2p+1)-p(p+2)-2(p+1)^2}{(p+2)^2}A(2p,p).$$

The numerator of the fraction is equal to p^2. This proves the following property:

Property 1. The inequality (1) is equivalent to:

$$A(2p,p-1)<\frac{p^2}{(p+2)^2}A(2p,p). \tag{2}$$

Now we are going to prove (2) as being a consequence of a more general inequality, as follows.

Theorem 3. *We have, for all $n\geqslant 2k-1$, $k\geqslant 2$:*

$$A(n,k-1)<\left(\frac{n-k}{n-k+2}\right)^{n-2k+2}A(n,k) \tag{3}$$

and this inequality implies that sequences $M_{2p}/(2p)!$ and $M_{2p+1}/(2p+1)!$ are decreasing.

Setting $n=2p$, $k=p$ in (3) gives the inequality (2) of Property 1. This proves that the sequence $M_{2p}/(2p)!$ is decreasing.

The decreasing property of $M_{2p+1}/(2p+1)!$ follows immediately because:

$$\frac{M_{2p+1}}{(2p+1)!}=\frac{(2p+2)M_{2p}}{(2p+1)!}=\frac{2p+2}{2p+1}\frac{M_{2p}}{(2p)!}$$

and the right member is the product of two positive decreasing functions.

Remark. When $n=2p+1$, $k=p+1$, inequaltiy (3) gives:

$$A(2p+1)<\frac{p}{p+2}A(2p+1,p+1). \tag{4}$$

But we can see, as we saw in Property 1, that the decreasing property of $M_{2p+1}/(2p+1)!$ is equivalent to:

$$A(2p+1,p)<\left(\frac{p+1}{p+2}\right)^2A(2p+1,p+1). \tag{5}$$

We observe that $(4)\Rightarrow(5)$ because

$$\frac{p}{p+2}<\left(\frac{p+1}{p+2}\right)^2.$$

So the decreasing property of $M_{2p}/(2p)!$ ensures that of $M_{2p+1}/(2p+1)!$, but the converse is not true.

Proof of Theorem 3. Let us put $k=p+1$, $n=2p+h$, $p\geqslant 1$, $h\geqslant 1$. Inequality (3) reads:

$$A(n,p)<\left(\frac{p+h-1}{p+h+1}\right)^h A(n,p+1), \qquad n=2p+h, \quad p\geqslant 1, \quad h\geqslant 1. \tag{3)'$$

We shall prove it by induction on p (the columns) and, for each column p, by induction on h (the lines).

$p=1$.

$$A(n,1)<\left(\frac{n-2}{n}\right)^{n-2} A(n,2), \qquad n\geqslant 3.$$

Let us take the exact values (formula (4) of Section 1):

$$A(n,1)=1, \qquad A(n,2)=2^n-(n+1).$$

The inequality is:

$$\left(\frac{n}{n-2}\right)^{n-2}<2^n-(n+1) \qquad \text{or} \qquad \left(1+\frac{2}{n-2}\right)^{n-2}<2^n-(n+1).$$

But, if $n\geqslant 4$, $n-2\geqslant 2$, $1+2/(n-2)\leqslant 1+1=2$. So, it is enough that:

$$2^{n-2}<2^n-(n+1) \qquad \text{or} \qquad 3\cdot 2^{n-2}>n+1.$$

But $2^{n-2}=(1+1)^{n-2}>1+n-2=n-1$, and we have $3(n-1)>n+1$ when $n>2$. This is the case since we have assumed $n\geqslant 4$. The inequality is also true when $n=3$ $(1<\frac{1}{3}\times 4)$.

Induction from column p to column $p+1$. Replacing p by $p+1$ in (3)′, we have to prove:

$$A(n,p+1)<\left(\frac{p+h}{p+h+2}\right)^h A(n,p+2), \qquad n=2(p+1)+h \tag{3)''$$

for all $h\geqslant 1$, knowing that (3)′ is true.

We do this by induction on h.

$h=1$.

$$A(n,p+1)<\frac{p+1}{p+3}A(n,p+2), \qquad n=2p+3.$$

Let us apply the triangular recurrence relation to compute (by Theorem 2(ii) of Section 2):

$$A(2p+3,p+1)=(p+3)A(2p+2,p)+(p+1)A(2p+2,p+1)$$
$$A(2p+3,p+2)=(2p+4)A(2p+2,p+1).$$

So we have to prove:

$$(p+3)^2A(2p+2,p)+(p+1)(p+3)A(2p+2,p+1)$$
$$<(p+1)(2p+4)A(2p+2,p+1)$$

or

$$(p+3)^2A(2p+2,p)<[2(p+1)(p+2)-(p+1)(p+3)]A(2p+2,p+1):$$

that is to say,

$$(p+3)^2A(2p+2,p)<(p+1)^2A(2p+2,p+1)$$

and therefore

$$A(2p+2,p)<\left(\frac{p+1}{p+3}\right)^2 A(2p+2,p+1).$$

This results from (3)′ when $h=2$.

Induction from h to h + 1. Replacing h by $h+1$ in (3)″, we have to prove:

$$A(n+1,p+1)<\left(\frac{p+h+1}{p+h+3}\right)^{h+1} A(n+1,p+2) \qquad (3)'''$$

assuming (3)″ for fixed h and (3)′ for every h.

The triangular recurrence relation allows us to go back to the preceding lines and columns.

In effect we have:

$$A(n+1,p+1)=(p+h+3)A(n,p)+(p+1)A(n,p+1),$$

$$A(n+1,p+2)=(p+h+2)A(n,p+1)+(p+2)A(n,p+2).$$

Then, (3)‴ reads:

$$(p+h+3)A(n,p)+(p+1)A(n,p+1)$$
$$<\left(\frac{p+h+1}{p+h+3}\right)^{h+1}[(p+h+2)A(n,p+1)+(p+2)A(n,p+2)].$$

But $n=2p+h+2$, and we can apply (3)′ with $h'=h+2$:

$$A(n,p)<\left(\frac{p+h+1}{p+h+3}\right)^{h+2} A(n,p+1).$$

In order to verify (3)‴ it is enough to have:

$$[(p+h+1)^{h+2}-(p+h+2)(p+h+1)^{h+1}+(p+1)(p+h+3)^{h+1}]A(n,p+1)$$
$$<(p+2)(p+h+1)^{h+1}A(n,p+2):$$

that is,

$$[(p+1)(p+h+3)^{h+1}-(p+h+1)^{h+1}]A(n,p+1)$$
$$<(p+2)(p+h+1)^{h+1}A(n,p+2).$$

Now we can apply (3)″ to $A(n,p+1)$, so that it will be enough to verify:

$$(p+h)^h[(p+1)(p+h+3)^{h+1}-(p+h+1)^{h+1}]<(p+2)(p+h+1)^{h+1}(p+h+2)^h$$

or

$$(p+1)(p+h)^h(p+h+3)^{h+1}<(p+h+1)^{h+1}[(p+2)(p+h+2)^h+(p+h)^h].$$

Now we have only to check this inequality for all integers $p\geqslant 1$, $h\geqslant 1$. It will be proved true, because of the following lemma.

LEMMA 1. *Let x, y be real positive numbers. Then:*

$$(y+1)(x+y)^x(x+y+3)^{x+1}<(x+y+1)^{x+1}[(y+2)(x+y+2)^x+(x+y)^x].$$

The details of the proof are given in [10]. It requires an intensive use of the variation of certain real functions, with some help from computer algebra. Elementary proofs of this lemma can be found in [15]. □

2.4. *Other Applications of the Inequality*

$$A(n, k-1) < \left(\frac{n-k}{n-k+2}\right)^{n-2k+2} A(n, k), \qquad n \geqslant 2k-1, \quad k \geqslant 2. \tag{3}$$

In addition to the decreasing property of $M_{2p}/(2p)!$ and $M_{2p+1}/(2p+1)!$, this inequality allows us to prove some supplementary properties of the table of the values of $A(n, k)/n!$ below.

n \ k	1	2	3	4	5	6	7	8
1	1							
2	0.5	0.5						
3	0.16667	0.66667	0.16667					
4	0.04167	0.45833	0.45833	0.04167				
5	0.00833	0.21667	0.55	0.21667	0.00833			
6	0.00139	0.07917	0.41944	0.41944	0.07917	0.00139		
7	0.00020	0.02381	0.23631	0.47937	0.23631	0.02381	0.00020	
8	0.00002	0.00613	0.10647	0.38738	0.38738	0.10647	0.00613	0.00002
9	0.28×10^{-5}	0.00138	0.04026	0.24315	0.43042	0.24315	0.04026	0.00138
10	0.28×10^{-6}	0.00028	0.01318	0.12544	0.36110	0.36110	0.12544	0.01318
11	0.25×10^{-7}	0.00005	0.00382	0.05520	0.24396	0.39393	0.24396	0.05520
12	0.21×10^{-8}	0.85×10^{-5}	0.00100	0.02127	0.13845	0.33927	0.33927	0.13845

Let us study the variation of the function $A(n, k)/n!$ of n for fixed k; that is, in the column k.

THEOREM 4. *$A(n, k)/n!$ increases from $1/k!$ to $A(2k+1, k)/(2k-1)!$ in the interval $[k, 2k-1]$ and decreases afterwards from $A(2k-1, k)/(2k-1)!$ to 0 in the interval $[2k-1, +\infty[$.*

n	k		$2k-1$		$+\infty$
$\dfrac{A(n,k)}{n!}$	$\dfrac{1}{k!}$	$\nearrow$	$\dfrac{A(2k-1,k)}{(2k+1)!}$	$\searrow$	0

PROOF. (1)

$$\underline{k \leqslant n < 2k-1} \Rightarrow \frac{A(n, k)}{n!} < \frac{A(n+1, k)}{(n+1)!}.$$

Let us apply the triangular recurrence relation. Then we have to prove:

$$(n+1)A(n, k) < (n-k+2)A(n, k-1) + kA(n, k)$$

or

$$(n-k+1)A(n, k) < (n-k+2)A(n, k-1).$$

But we have $(n-k+1) < (n-k+2)$ and $A(n, k) \leqslant A(n, k-1)$ in the interval concerned, because

$$A(n, k) = A(n, k'), \qquad k' = n + 1 - k, \qquad A(n, k-1) = A(n, k'+1),$$

and we know that $A(n, k') \leqslant A(n, k'+1)$ if $n \geqslant 2k'$, which is the case here since $n \geqslant 2k'$ reads $n \geqslant 2(n+1-k)$ or $n \leqslant 2k-2$.

(2) $$\underline{n \geqslant 2k-1} \Rightarrow \frac{A(n, k)}{n!} > \frac{A(n+1, k)}{(n+1)!}.$$

This time we shall verify that:

$$\frac{n-k+1}{n-k+2} A(n, k) > A(n, k-1).$$

We have $A(n, k) > A(n, k-1)$, but this is not sufficient. We need inequality (3). It is enough to prove:

$$\frac{n-k+1}{n-k+2} > \left(\frac{n-k}{n-k+2}\right)^{n-2k+2} \quad \text{or} \quad (n-k+1)(n-k+2)^{n-2k+1} > (n-k)^{n-k+2}$$

and this is obvious. □

Moreover, $\lim_{n\to+\infty} A(n, k)/n! = 0$ for fixed k. This is a well known fact; for instance, with expression (4) of Section 1 for $A(n, k)$. We even know more; the *series* $u_n = A(n, k)/n!$ is convergent: $u_n \sim k^n/n!$.

A Few Words About the Series $u_n = A(n, k)/n!$. Let us recall the calculation of the sum S_k. The generating vertical function

$$S_k(t) = \sum_{n=k}^{\infty} \frac{A(n, k)}{n!} t^n$$

can be obtained by means of the double series expansion [3, t. 1, p. 64]:

$$1 + \sum_{1 \leqslant k \leqslant n} A(n, t) \frac{t^n}{n!} = \frac{1-u}{1 - ue^{t(1-u)}} = (1-u)(1 + ue^t e^{-tu} + \cdots + u^k e^{kt} e^{-ktu} + \cdots).$$

Taking the coefficients of u^k in both members, we have

$$S_k(t) = \sum_{0 \leqslant j \leqslant k-1} (-1)^j \frac{e^{(k-j)t}(k-j)^{j-1} t^{j-1}}{j!} (j + (k-j)t),$$

so that

$$S_k = S_k(1) = \sum_{0 \leqslant j \leqslant k-1} (-1)^j e^{k-j} \frac{(k-j)^{j-1}}{j!} k,$$

the first term ($j = 0$) being e^k.

Examples.

$k=1$	$S_1 = e - 1$	$= 1.71828..$
$k=2$	$S_2 = e^2 - 2e$	$= 1.95249..$
$k=3$	$S_3 = e^3 - 3e^2 + \frac{3}{2}e$	$= 1.99579..$

$$k=4 \qquad S_4 = e^4 - 4e^3 + 4e^2 - \frac{4}{3!}e \qquad = 2.00003..$$

$$k=5 \qquad S_5 = e^5 - 5e^4 + \frac{5.3}{2!}e^3 - \frac{5.2^2}{3!}e^2 + \frac{5}{4!}e \qquad = 2.00005..$$

$$k=6 \qquad S_6 = e^6 - 6e^5 + \frac{6.4}{2!}e^4 - \frac{6.3^2}{3!}e^3 + \frac{6.2^3}{4!}e^2 - \frac{6}{5!}e \qquad = 2.00000..$$

$$k=7 \qquad S_7 = e^7 - 7e^6 + \frac{7.5}{2!}e^5 - \frac{7.4^2}{3!}e^4 + \frac{7.3^3}{4!}e^3 - \frac{7.2^4}{5!}e^2 + \frac{7e}{6!} = 1.99999..$$

We see that the sums S_k very quickly reach the neighbourhood of 2; but they are never equal to 2 since e is a transcendent number. Nevertheless:

PROPOSITION 1 [8, p. 42, formula (29)]. *When $k \to +\infty$, S_k tends to the limit $S = 2$. A weaker form of inequality* (3) *is*

$$A(n, k-1) < \frac{n-k}{3n-5k+4} A(n, k), \qquad n \geq 2k-1, \quad k \geq 2. \tag{3_w}$$

PROOF. In inequality (3) we have:

$$\left(\frac{n-k}{n-k+2}\right)^{n-2k+2} = \frac{1}{(1+2/(n-k))^{n-2k+2}} < \frac{1}{1+2(n-2k+2)/(n-k)}.$$

$(3)_w$ is simpler than (3), but not strong enough to prove the inequality (2) of property 1 in Section 2.3, which is the decreasing property of $M_{2p}/(2p)!$.

Going back to the series $u_n = A(k, n)/n!$, we have, with $(3)_w$,

$$\frac{u_{n+1}}{u_n} < \frac{k}{n+1} + \frac{(n-k+2)(n-k)}{(n+1)(3n-5k+4)}.$$

If $n \to +\infty$, the right member has the limit $\frac{1}{3}$ and, once more, this proves the convergence of the series. □

3. ASYMPTOTIC EXPANSION OF $M_{2p-1}/(2p-1)!$ AND CONSEQUENCES

3.1. *Series expansion and first bounds*

The eulerian polynomial

$$A_n(\lambda) = \sum_{k=1}^{n} A(n, k)\lambda^k$$

can be defined with the expansion of the analytical function $z \mapsto 1/(1-\lambda e^{-z})$, $\lambda \in \mathbb{C}$, $\lambda \neq 1$:

$$\frac{1}{1-\lambda e^{-z}} = \sum_{n=0}^{\infty} - \frac{A_n(\lambda)}{(\lambda-1)^{n+1}} \frac{z^n}{n!}.$$

LEMMA 1. *For every $n \geq 2$, we have*:

$$\frac{A_n(\lambda)}{(\lambda-1)^{n+1} n!} = \sum_{q \in \mathbb{Z}} \frac{1}{(\log|\lambda| + i \arg \lambda + 2iq\pi)^{n+1}}$$

with $-\pi < \arg \lambda \leq \pi$.

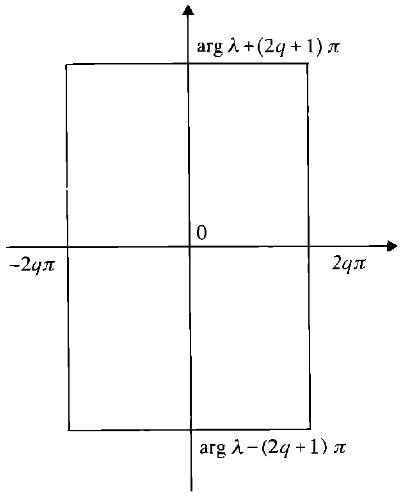

FIGURE 1.

PROOF. Compute the integral $\int [(1-\lambda e^{-z})z^{n+1}]^{-1}\,dz$ along the sides of the rectangle (see Figure 1)

$$x = \pm 2q\pi, \qquad y = \arg\lambda \pm (2q+1)\pi.$$

Apply the theorem of residues and make q tend to $+\infty$. □

LEMMA 2. *We have*:

$$\frac{M_{2p-1}}{(2p-1)!} = \frac{A(2p-1,p)}{(2p-1)!} = \frac{2}{\pi}\int_0^{\pi/2}\left(\frac{\sin t}{t}\right)^{2p} dt + \frac{2}{\pi}\sum_{q=1}^{\infty}\int_{-\pi/2}^{+\pi/2}\left(\frac{\sin t}{t+q\pi}\right)^{2p} dt$$
$$= \frac{2}{\pi}\int_0^{\infty}\left(\frac{\sin t}{t}\right)^{2p} dt.$$

PROOF. Cauchy's formula gives the coefficient $A(2p-1,p)$ of λ^p in the polynomial $A_{2p-1}(\lambda)$:

$$A(2p-1,p) = \frac{1}{2i\pi}\int_C A_{2p-1}(\lambda)\lambda^{-p-1}\,d\lambda,$$

where C denotes the circle with centre O and radius 1. Then we apply Lemma 1. Each term of the infinite sum is integrable along C, normally for the family when $|\lambda| = 1$, so that we can transpose the signs $\int$ and $\sum$. Hence:

$$\frac{M_{2p-1}}{(2p-1)!} = \frac{1}{2i\pi}\sum_{q\in\mathbb{Z}}\int_C \frac{(\lambda-1)^{2p}\lambda^{-p-1}\,d\lambda}{(i\arg\lambda + 2iq\pi)^{2p}}.$$

Next we put $\lambda = e^{2it}$, $-\pi/2 < t \leq \pi/2$. It remains to be seen that, if $q = 0$, the function $\sin t/t$ is even, and that the terms q and $-q$ give an equal contribution. □

COROLLARY 3. *Let us write* $J_p = \int_0^{\pi/2}(\sin t/t)^p\,dt$. *Then*:

$$\frac{2}{\pi}J_{2p} \leq \frac{M_{2p-1}}{(2p-1)!} \leq \frac{2}{\pi}J_{2p} + \left(\frac{2}{\pi}\right)^{2p}.$$

PROOF. The lower bound is an immediate consequence of Lemma 2. Next we have:

$$\frac{2}{\pi}\int_{\pi/2}^{\infty}\left(\frac{\sin t}{t}\right)^{2p} dt \leqslant \frac{2}{\pi}\int_{\pi/2}^{\infty} t^{-2p}\, dt = \frac{1}{2p-1}\left(\frac{2}{\pi}\right)^{2p} \leqslant \left(\frac{2}{\pi}\right)^{2p}.$$

We could have found a better upper bound, but this one is enough for our purpose. □

3.2. *Some Preliminary Inequalities*

3.2.1. Let a and b be two real positive numbers. Then the function $x \mapsto x^a e^{-bx}$ is a decreasing one if $x \geqslant a/b$.

3.2.2. Let p be a natural integer. Then:

$$K_{2p} = \int_0^{+\infty} x^{2p} e^{-x^2/2}\, dx = \sqrt{\frac{\pi}{2}}(1\cdot 3\cdot 5\cdots(2p-1)) = \sqrt{\frac{\pi}{2}}\frac{(2p)!}{2^p p!}.$$

3.2.3. When $x > 0$, we have:

$$\int_x^{+\infty} e^{-t^2/2}\, dt \leqslant \frac{1}{x} e^{-x^2/2}.$$

3.2.4. When $x \geqslant 0$, we have:

$$1 - x + \frac{x^2}{2} - \frac{x^3}{6} \leqslant e^{-x} \leqslant 1 - x + \frac{x^2}{2}.$$

3.2.5. If $0 \leqslant x \leqslant 1$, we have:

$$(1-x)^{\frac{1}{2}} \leqslant \left(1 - \frac{x^2}{8}\right)\left(1 - \frac{x}{2}\right).$$

3.2.6. If $0 \leqslant x \leqslant \frac{1}{2}$, we have:

$$(1+x)^{-\frac{1}{2}} \geqslant \left(1 + \frac{2x^2}{100}\right)\Big/\left(1 + \frac{x}{2}\right).$$

PROOF. 3.2.1 follows immediately by derivation. One easily obtains 3.2.2 and 3.2.3 using integration by parts. 3.2.4 results from the MacLaurin's formula.

3.2.5 is true for $x = 1$; if $0 \leqslant x < 1$ the third derivative of the function $x \mapsto \sqrt{1-x}$ is negative, and by the MacLaurin's formula we have:

$$(1-x)^{\frac{1}{2}} \leqslant 1 - \frac{x}{2} - \frac{x^2}{8} \leqslant \left(1 - \frac{x}{2}\right)\left(1 - \frac{x^2}{8}\right).$$

As for 3.2.6, we notice that the fourth derivative of the function $x \mapsto (1+x)^{-\frac{1}{2}}$ is positive if $x \geqslant 0$ and hence

$$(1+x)^{-\frac{1}{2}} \geqslant 1 - \frac{x}{2} + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3$$

and, if $x \leqslant \frac{1}{2}$,

$$\left(1 + \frac{x}{2}\right)(1+x)^{-\frac{1}{2}} \geqslant 1 + \frac{x^2}{8} - \frac{x^3}{8} - \frac{5x^4}{32} \geqslant 1 + \frac{x^2}{8}\left(1 - \frac{1}{2} - \frac{5}{16}\right)$$

$$\geqslant 1 + \frac{3}{128}x^2 \geqslant 1 + \frac{2}{100}x^2.$$

□

3.2.7. For every $t \in [0, \pi/2]$ we have:

$$-\frac{t^2}{6}-\frac{t^4}{180}-\frac{t^6}{2835} \geqslant \log\frac{\sin t}{t} \geqslant -\frac{t^2}{6}-\frac{t^4}{180}-\frac{t^6}{2835}-\frac{t^8}{37800}-ct^{10}$$

with $c = 1/364500$.

Proof. The derivative of $\log \sin t$ is $\operatorname{cotg} t$, the expansion of which in Laurent's series is known:

$$\log\frac{\sin t}{t} = \sum_{k=1}^{\infty} \frac{-2^{2k}B_{2k}}{2k(2k)!} t^{2k}, \qquad |t| < \pi,$$

where the B_{2k} are the positive Bernouilli's numbers. The left inequality follows immediately. Next we define $c(t)$ by:

$$\log\frac{\sin t}{t} = -\frac{t^2}{6}-\frac{t^4}{180}-\frac{t^6}{2835}-\frac{t^8}{37800}-c(t)t^{10}.$$

Thus we have, if $0 \geqslant t < \pi$,

$$c(t) = \sum_{k \geqslant 5} \frac{2^{2k}B_{2k}}{2^k(2k)!} t^{2k-10};$$

$c(t)$ is an increasing function of t; and

$$c\left(\frac{\pi}{2}\right) = 2.705 \cdots 10^{-6} \leqslant \frac{1}{364500}. \qquad \square$$

3.3. *The Inequality*

$$\frac{M_{2p-1}}{(2p-1)!} < \sqrt{\frac{3}{\pi p}}\left(1-\frac{3}{40p}\right).$$

Now we are able to obtain a precise upper bound for $M_{2p-1}/(2p-1)!$. According to Corollary 3 of Section 3.1, we can find it with an upper bound of J_{2p}, where:

$$J_p = \int_0^{\pi/2} \left(\frac{\sin t}{t}\right)^p dt = \int_0^{\pi/2} \exp\left(p \log\frac{\sin t}{t}\right) dt.$$

Upper Bound of J_p. By 3.2.7, we have:

$$J_p \leqslant \int_0^{\pi/2} \exp\left(-\frac{pt^2}{6}-\frac{pt^4}{180}-\frac{pt^6}{2835}\right) dt.$$

By putting

$$t = \sqrt{\frac{3}{p}}\,u, \qquad v = \frac{u^4}{20p}+\frac{u^6}{105p^2}$$

we have

$$J_p \leqslant \sqrt{\frac{3}{p}} \int_0^{\pi/2\sqrt{p/3}} \exp\left(-\frac{u^2}{2}-v\right) du < \sqrt{\frac{3}{p}} \int_0^{+\infty} \exp\left(-\frac{u^2}{2}\right) \exp(-v)\, du.$$

By 3.2.4, we have:

$$\begin{aligned} J_p &< \sqrt{\frac{3}{p}} \int_0^{+\infty} \exp\left(-\frac{u^2}{2}\right)\left(1-v+\frac{v^2}{2}\right) du \\ &= \sqrt{\frac{3}{p}} \int_0^{+\infty} \exp\left(-\frac{u^2}{2}\right)\left(1-\frac{u^4}{20p}-\frac{u^6}{105p^2}+\frac{u^8}{800p^2}+\frac{u^{10}}{2100p^3}+\frac{u^{12}}{22050p^4}\right) du. \end{aligned}$$

Hence, applying 3.2.2,

$$J_p < \sqrt{\frac{3}{p}}\left(K_0 - \frac{K_4}{20p} - \frac{K_6}{105p^2} + \frac{K_8}{800p^2} + \frac{K_{10}}{2100p^3} + \frac{K_{12}}{22050p^4}\right).$$

This inequality is valid for every $p \geqslant 1$. When $p \geqslant 100$ we have

$$J_p < \sqrt{\frac{3\pi}{2p}}\left(1 - \frac{3}{20p} - \frac{1}{p^2}\left(\frac{13}{1120} - \frac{9}{2000} - \frac{33}{7\times 10^5}\right)\right)$$

and hence

$$J_p < \sqrt{\frac{3\pi}{2p}}\left(1 - \frac{3}{20p} - \frac{0.10}{p^2}\right), \qquad p \geqslant 100. \tag{1}$$

This inequality will give an upper bound for $M_{2p}/(2p-1)!$ according to the following lemma.

LEMMA 1. *For every $p \geqslant 3$, we have*:

$$\frac{M_{2p-1}}{(2p-1)!} = \frac{A(2p-1,p)}{(2p-1)!} < \sqrt{\frac{3}{\pi p}}\left(1 - \frac{3}{40p}\right).$$

PROOF. By Corollary 3 of Section 3.1, and the upper bound of J_p given by (1) we have, for $p \geqslant 50$,

$$\frac{M_{2p-1}}{(2p-1)!} < \sqrt{\frac{3}{\pi p}}\left(1 - \frac{3}{40p} - \frac{0.10}{p^2}\right) + \left(\frac{2}{\pi}\right)^{2p}$$
$$= \sqrt{\frac{3}{\pi p}}\left(1 - \frac{3}{40p}\right) - p^{-\frac{5}{2}}\left\{\sqrt{\frac{3}{\pi}}\frac{0.10}{4} - p^{\frac{5}{2}}\exp\left(-p\log\frac{\pi^2}{4}\right)\right\}.$$

If $p \geqslant 50$, $\{\ \}$ is, because of 3.2.1, greater than its value for $p = 50$, which is positive. This proves the lemma for $p \geqslant 50$. The computing of $A(2p-1,p)/(2p-1)!$ for $p \leqslant 50$ using a computer completes the proof. □

3.4. *The Inequality*

$$\frac{M_{2p-1}}{(2p-1)!} > \sqrt{\frac{3}{\pi p}}\left(1 - \frac{3}{40p} - \frac{13}{4480p^2}\right).$$

Applying 3.2.7 with $c = 1/364500$, we obtain:

$$J_p \geqslant \int_0^{\pi/2} \exp\left(-\frac{pt^2}{6} - \frac{pt^4}{180} - \frac{pt^6}{2835} - \frac{pt^8}{37800} - cpt^{10}\right) dt.$$

We put:

$$t = \sqrt{\frac{3}{p}}\,u, \qquad v = \frac{u^4}{20p} + \frac{u^6}{105p^2} + \frac{3u^8}{1400p^3} + \frac{u^{10}}{1500p^4},$$

then

$$J_p \geqslant \sqrt{\frac{3}{p}}\int_0^{\pi/2\sqrt{p/3}} \exp\left(-\frac{u^2}{2}\right)\exp(-v)\,du$$
$$\geqslant \sqrt{\frac{3}{p}}\int_0^{+\infty} \exp\left(-\frac{u^2}{2}\right)\exp(-v)\,du - \sqrt{\frac{3}{p}}\int_{\pi/2\sqrt{p/3}}^{+\infty} \exp\left(-\frac{u^2}{2}\right) du$$
$$> \sqrt{\frac{3}{p}}\int_0^{+\infty} \exp\left(-\frac{u^2}{2}\right)\left(1 - v + \frac{v^2}{2} - \frac{v^3}{6}\right) du - \frac{6}{\pi p}\exp\left(-\frac{\pi^2 p}{24}\right)$$

using 3.2.4 and 3.2.3. The calculation of the above integral can be made with the help of 3.2.2 and MAPLE. It gives:

$$\sqrt{\frac{\pi}{2}}\Big(1-\frac{3}{20}\frac{1}{p}-\frac{13}{1120}\frac{1}{p^2}+\frac{27}{3200}\frac{1}{p^3}-\frac{183}{280}\frac{1}{p^4}-\frac{619047}{224000}\frac{1}{p^5}$$
$$-\frac{20115953}{392000}\frac{1}{p^6}-\frac{932679891}{3136000}\frac{1}{p^7}-\frac{217411623}{140000}\frac{1}{p^8}-\frac{186700048821}{22400000}\frac{1}{p^9}$$
$$-\frac{1845296739}{64000}\frac{1}{p^{10}}-\frac{16263470223}{160000}\frac{1}{p^{11}}-\frac{122277202047}{400000}\frac{1}{p^{12}}\Big)$$
$$=\sqrt{\frac{\pi}{2}}\sum_{j=0}^{12}a_jp^{-j}\geqslant\sqrt{\frac{\pi}{2}}\Big(\Big(\sum_{j=0}^{2}a_jp^{-j}\Big)+p^{-3}\sum_{j=3}^{12}a_j100^{3-j}\Big)\qquad\text{for } p\geqslant 100$$

because all the coefficients $a_4,\ldots,a_{12}$ are negative. Thus we obtain, for $p\geqslant 100$,

$$J_p\geqslant\sqrt{\frac{3\pi}{2p}}\Big(1-\frac{3}{20p}-\frac{13}{1120p^2}+\frac{0.0015}{p^3}\Big)-\frac{6}{\pi p}\exp\Big(-\frac{\pi^2p}{24}\Big)$$
$$=\sqrt{\frac{3\pi}{2p}}\Big(1-\frac{3}{20p}-\frac{13}{1120p^2}\Big)+p^{-\frac{7}{2}}\Big\{0.0015\sqrt{\frac{3\pi}{2}}-\frac{6p^{\frac{5}{2}}}{\pi}\exp\Big(-\frac{\pi^2p}{24}\Big)\Big\}.$$

But, with 3.2.1, { } is, for $p\geqslant 100$, less than its value for $p=100$, which is positive. Then we have, for $p\geqslant 100$,

$$J_p\geqslant\sqrt{\frac{3\pi}{2p}}\Big(1-\frac{3}{20p}-\frac{13}{1120p^2}\Big).\tag{1}$$

We shall deduce from this inequality a lower bound for $M_{2p-1}/(2p-1)!$.

LEMMA 2. *We have, for every* $p\geqslant 1$,

$$\frac{M_{2p-1}}{(2p-1)!}=\frac{A(2p-1,p)}{(2p-1)!}>\sqrt{\frac{3}{\pi p}}\Big(1-\frac{3}{40p}-\frac{13}{4480p^2}\Big).$$

PROOF. When $p\geqslant 50$, this lemma is a direct consequence of Corollary 3 of Section 3.1, and the inequality (1) above. If $1\leqslant p\leqslant 49$, it could be verified using a computer. □

In order to understand the meaning of the bounds given by Lemma 1 of Section 3.3 and Lemma 2 of Section 3.4, let us recall the asymptotic expression of $A(n,k)/n!$ given by Siraždinov's formula [13]:

$$\frac{A(n,k)}{n!}=\sqrt{\frac{6}{\pi(n+1)}}\,e^{-x^2/2}-\frac{x^4-6x^2+3}{20(n+1)^{\frac{3}{2}}}\sqrt{\frac{6}{\pi}}\,e^{-x^2/2}+O\Big(\frac{1}{n^{\frac{5}{2}}}\Big)$$

where $1\leqslant k\leqslant n$ and

$$x=\frac{n-2k+1}{2\sqrt{(n+1)/12}}.$$

If $n=2p-1$, $k=p$, we have $x=0$ and the formula reduces to:

$$\frac{M_{2p-1}}{(2p-1)!}=\frac{A(2p-1,p)}{(2p-1)!}=\sqrt{\frac{3}{\pi p}}\Big(1-\frac{3}{40p}\Big)+O\Big(\frac{1}{p^{\frac{5}{2}}}\Big).$$

Our Lemma 1 of Section 3.3 and Lemma 2 of Section 3.4 therefore give information on the remainder $R_p = O(1/p^{\frac{5}{2}})$ in this formula, that is:

$$-\frac{13}{4480p^2}\sqrt{\frac{3}{\pi p}} < R_p < 0.$$

This precision is necessary in order to obtain the comparison properties between the numbers $M_n/n!$, which was the main object of this work.

3.5. *Comparison Properties Between the Numbers* $f(n) = M_n/n!$

Let us denote $f(n) = M_n/n! = \max_{1\leqslant k\leqslant n} A(n, k)/n!$. By Theorem 2(ii) of Section 2.2, we have $M_{2p+1} = (2p+2)M_{2p}$. Hence:

THEOREM 1. $f(2p) = [(2p+1)/(2p+2)]f(2p+1) < f(2p+1)$.

This result permits the comparison of $f(n)$, n even, with $f(n)$, n odd. The following inequalities will permit the comparison of the $f(n)$ between themselves when n is odd.

THEOREM 2. *We have, for all* $p \geqslant 1$,

$$\frac{2p}{2p+1} < \frac{f(2p+1)}{f(2p-1)} < \frac{2p+1}{2p+2}.$$

PROOF. (a) $[f(2p+1)]/[f(2p-1)] < (2p+1)/(2p+2)$. We first verify this inequality when $p=1$ and $p=2$. Let us now assume $p \geqslant 3$. It is convenient to set $k = p+1$ and $x = 1/k$. Thus we have $k \geqslant 4$ and $x \leqslant \frac{1}{4}$.

By Lemma 1 of Section 3.3, we may write:

$$f(2p+1) < \sqrt{\frac{3}{\pi k}}\left(1 - \frac{3}{40}x\right) \tag{1}$$

and, by Lemma 2 of Section 3.4,

$$f(2p-1) > \sqrt{\frac{3}{\pi(k-1)}}\left(1 - \frac{3}{40(k-1)} - \frac{13}{4480(k-1)^2}\right).$$

But, when $k \geqslant 4$, we have:

$$\frac{1}{k-1} < \frac{1}{k}\left(1 + \frac{1}{k} + \frac{1}{k^2}\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right)\right) = x\left(1 + x + \frac{4}{3}x^2\right)$$

and

$$k/(k-1) \leqslant \tfrac{4}{3} < \sqrt{2},$$

so that

$$f(2p-1) > \sqrt{\frac{3}{\pi k}}(1-x)^{-\frac{1}{2}}\left[1 - \frac{3}{40}x\left(1 - x + \frac{4}{3}x^2\right) - \frac{13}{2240}x^2\right]. \tag{2}$$

From these inequalities (1), (2) and from 3.2.5, we deduce that:

$$\frac{f(2p+1)}{f(2p-1)} < \left(1 - \frac{x}{2}\right)\left\{\frac{\left(1 - \frac{3}{40}x\right)\left(1 - \frac{x^2}{8}\right)}{\left(1 - \frac{3}{40}x - \frac{181}{2240}x^2 - \frac{x^3}{10}\right)}\right\}.$$

But we have $\{\ \} \leqslant 1$ if $0 \leqslant x \leqslant 99/245 = 0.404 \cdots$. Finally we obtain:

$$\frac{f(2p+1)}{f(2p-1)} < 1 - \frac{x}{2} = \frac{2p+1}{2p+2}.$$

(b) $2p/(2p+1) < [f(2p+1)]/[f(2p-1)]$. By Lemma 2 of Section 3.4, we have:

$$f(2p+1) > \sqrt{\frac{3}{\pi(p+1)}}\left(1 - \frac{3}{40(p+1)} - \frac{13}{4480(p+1)^2}\right)$$
$$> \sqrt{\frac{3}{\pi p}}\left(1+\frac{1}{p}\right)^{-\frac{1}{2}}\left(1 - \frac{3}{40p} - \frac{13}{4480p^2}\right).$$

Using 3.2.6 we have, when $p \geqslant 2$:

$$\left(1+\frac{1}{2p}\right)f(2p+1) > \sqrt{\frac{3}{\pi p}}\left\{\left(1+\frac{2}{100p^2}\right)\left(1-\frac{3}{40p}-\frac{13}{4480p^2}\right)\right\}.$$

The quantity between $\{\ \}$ is:

$$1 - \frac{3}{40p} - \frac{13}{4480p^2} + \frac{2}{100p^2} - \frac{6}{4000p^3} - \frac{26}{448000p^4}$$
$$\geqslant 1 - \frac{3}{40p} + \frac{1}{p^2}\left(\frac{2}{100} - \frac{6}{4000} - \frac{13}{4480} - \frac{26}{448000}\right)$$
$$> 1 - \frac{3}{40p}.$$

By Lemma 1 of Section 3.3, this result proves part (b) of Theorem 2 when $p \geqslant 2$. The theorem is also valid for $p = 1$. □

We are now able to state the following theorem showing the comparison relations between the different numbers $M_n/n! = f(n)$.

THEOREM 3. (i) *The sequence* $f(2p+1)$ *is decreasing.*
(ii) *The sequence* $f(2p)$ *is decreasing.*
(iii) *We have, for every* $p \geqslant 1$,

$$f(2p+5) < f(2p) < f(2p+3).$$

PROOF. (i) and (ii) have already been proved in Theorem 3 of Section 2.3, but we shall prove them again here as consequences of the analytical results of Section 3 of this paper.

(i) This is obvious from Theorem 2(a).

(ii) Using Theorem 1 and Theorem 2(a), we may write:

$$f(2p) = \frac{2p+1}{2p+2}f(2p+1), \qquad f(2p+2) = \frac{2p+3}{2p+4}f(2p+3),$$

$$\frac{f(2p)}{f(2p+2)} = \frac{(2p+1)(2p+4)f(2p+1)}{(2p+2)(2p+3)f(2p+3)} > \frac{(2p+1)(2p+4)(2p+4)}{(2p+2)(2p+3)(2p+3)}$$
$$= \frac{4p^3+18p^2+24p+8}{4p^3+16p^2+21p+9} > 1$$

for every $p \geqslant 1$.

(iii)

$$f(2p)=\frac{2p+1}{2p+2}f(2p+1)<\frac{(2p+1)(2p+3)}{(2p+2)(2p+2)}f(2p+3)$$
$$=\frac{4p^2+8p+3}{4p^2+8p+4}f(2p+3)$$

Hence, $f(2p)<f(2p+3)$ for every $p\geqslant 1$. What remains now is the study of:

$$f(2p+5)<f(2p+3)\frac{2p+5}{2p+6}<f(2p+1)\frac{2p+3}{2p+4}\frac{2p+5}{2p+6}$$
$$=f(2p)\frac{(2p+2)(2p+3)(2p+5)}{(2p+1)(2p+4)(2p+6)}.$$

The last coefficient c is equal to:

$$c=\frac{4p^3+20p^2+36p+15}{4p^3+22p^2+34p+12}.$$

We have $c<1$ if $2p^2-2p-3>0$ or $2p(p-1)>3$, which is true when $p\geqslant 2$. The case $p=1$ is also true. □

Let us remark that (ii) could easily be proved as a consequence of (i) and (iii), but the given proof for (i) and (ii) only needs part 2(a) of Theorem 2, while (iii) also uses part 2(b) of this theorem.

REMARK 1. By using the Laplace method, it can be shown that J_p has an asymptotic expansion of any order k:

$$J_p=\sqrt{\frac{3\pi}{2p}}\left(\sum_{n=0}^{k}a_np^{-n}+O(p^{-k-1})\right),$$

and MAPLE gives $a_0=1$, $a_1=-3/20$, $a_2=-13/1120$, $a_3=27/3200$, $a_4=52791/3942400$, etc. B. Salvy (cf. [12]) has proved:

$$a_k=\frac{2}{\sqrt{\pi}}\frac{(2k+1)^{-\frac{1}{2}}\Gamma(k+1/2)}{[\log^2(-\cos\alpha_1)+\pi^2]^{k/2}}(\sin(2k\theta_1)+O(1/k)),$$

where $\alpha_1=4.4934\cdots$ is a root of $\tan x=x$, and

$$\theta_1=\tfrac{1}{2}\arctan\left(\frac{\pi}{\log(-\cos\alpha_1)}\right)=-0.5592\cdots.$$

It would then be impossible to improve the results of Lemma 1 of Section 3.3 and Lemma 2 of Section 3.4 with similar but longer computing.

REMARK 2. We can easily prove, as we did in Property 1 of Section 2.3, that the recent Theorem 2 of Section 3.5 is equivalent to:

$$\frac{p(p-1)}{(p+1)^2}A(2p-1,p)<A(2p-1,p-1)<\frac{p(2p^2-1)}{2(p+1)^3}A(2p-1,p).$$

This last bound is better than the one obtained from inequality (3) of Section 2.3:

$$A(2p-1,p-1)<\frac{p-1}{p+1}A(2p-1,p).$$

But inequality (3) of Section 2.3 has the advantage of providing an upper bound for $A(n, k-1)/A(n, k)$ that is valid for every k such that $n \geqslant 2k-1$, $k \geqslant 2$. It would also be interesting to find such a lower bound, including the one resulting from the case $n = 2p-1$, $k = p$. This problem is currently open.

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Appendix 1

n	$s(n)$	$s(n)/n!$
1	1	1.0000
2	1	.50000
3	4	.66667
4	11	.45833
5	66	.55000
6	302	.41944
7	2416	.47937
8	15619	.38738
9	156190	.43042
10	1310354	.36110
11	15724248	.39393
12	162512286	.33927
13	2275172004	.36537
14	27971176092	.32085
15	447538817472	.34224
16	6382798925475	.30506
17	114890380658550	.32301
18	1865385657780650	.29136
19	37307713155613000	.30669
20	679562217794156938	.27932
21	14950368791471452636	.29262
22	301958232385734088196	.26865
23	7246997577257618116704	.28033
24	160755658074834738495566	.25910
25	4179647109945703200884716	.26946
26	101019988341178648636047412	.25049
27	2828559673553002161809327536	.25977
28	73990373947612503295166622044	.24268
29	2219711218428375098854998661320	.25105
30	62481596875767023932367207962680	.23555
31	1999411100024544765835750654805760	.24315
32	60261990727996752483262854173443875	.22902
33	2048907684751889584430937041897091750	.23596
34	65835846167447988443323906979298454170	.22300
35	2370090462028127583959660651254744350120	.22937
36	80879977062516354460442890520154715103250	.21742
37	3073439128375621469496829839765879173923500	.22330
38	111009720815037710423740263008973631965904500	.21225
39	4440388832601508416949610520358945278636180000	.21769
40	169238447880147569395192525660609383274639835610	.20742
41	7108014810966197914598086077745594097534873095620	.21248
42	285090468862200438356418365594138405888960374563420	.20291
43	12543980629936819287682408086142089859114256480790480	.20763
44	528147173325184215023781757578506036590710931288450020	.19868
45	24294769972958473891093960848611277683172702839268700920	.20310
46	1071383436266089365034069030820446336325215649859423413640	.19470
47	51426404940772289521635313479381424143610351193252323854720	.19885
48	2370520507059560311291333595101302393901665190651705237350030	.19096
49	118526025352978015564566679755065119695083259532585261867501500	.19485
50	5700123907773416224716099708737159306764363732140229880240069124	.18742