COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
60. SETS, GRAPHS AND NUMBERS, BUDAPEST (HUNGARY), 1991

An Integral Representation for Eulerian Numbers

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1. Introduction

For $n \geq 1$, the Eulerian number $A(n,k)$ can be defined as the number of permutations of n letters with k runs up (cf. [3], t.II, p.82, and [4], p.34). It is easy to see that they verify the triangular relation

$$A(n,k) = k\,A(n-1,k) + (n-k+1)A(n-1,k-1) \tag{1}$$

and the starting values:

$$A(n,1) = A(n,n) = 1. \tag{2}$$

(1) and (2) characterize Eulerian numbers, and can be used to compute $A(n,k)$ for $n \geq 1$ and $1 \leq k \leq n$. Moreover, if we set

$$A(n,k) = 0 \qquad \text{for } n \geq 1 \text{ and } k \leq 0 \text{ or } k \geq n+1, \tag{3}$$

it is easy to see that $A(n,k)$ still verifies (1) for all $n \geq 2$ and $k \in \mathbb{Z}$.

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* Research partially supported by CNRS, SDI 5614 and PRC Math-Info.

In view of the generating function, it is good to define $A(0,k)$. In order to satisfy (1) for $n=1$, we must set $A(0,k)=0$ for $k\le 0$ and $k\ge 2$, and

$$A(0,0)+A(0,1)=1.$$

Knuth (cf. [4], p.35) sets

$$A(0,0)=0 \text{ and } A(0,1)=1$$

while Carlitz et al. (cf. [2]) and Comtet (cf. [3]) set:

$$A(0,0)=1 \text{ and } A(0,1)=0.$$

We shall choose

$$A(0,0)=A(0,1)=1/2, \tag{4}$$

and then, the generating function will be: (cf. [4])

$$\sum_{n\ge 0}\sum_{0\le k\le n} A(n,k)\frac{z^n x^k}{n!} = \left(\frac{1-x}{2}\right)\frac{e^{(x-1)z}+x}{e^{(x-1)z}-x} \tag{5}$$

Eulerian numbers also verify the symmetric relation:

$$A(n,k)=A(n,n-k+1). \tag{6}$$

The aim of this paper is to prove the following integral representation of Eulerian numbers:

Theorem 1. *For $n\ge 0$, and $k\in\mathbb{Z}$, we have*

$$A(n,k)=\frac{2(n!)}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^{n+1}\cos((n+1-2k)t)dt. \tag{7}$$

In a first step, the proof of Theorem 1 will be given. In fact the proof follows the proof of Lemma 2 of §III of [5], where (7) was proved in the particular case $n=2p-1$, $k=p$. It is also possible to define $A(n,k)$ by (7). Then, classical formulas (1), (2), (6) can be easily deduced.

In §3, formula (7) will be extended to all k real and an interpretation of function $k\mapsto A(n,k)$ will be given in terms of Fourier analysis.

In [5] the maximum

$$M_n=\max_k A(n,k)$$

was extensively studied. In §4 another proof of some results of [5] concerning M_n is given or suggested using Theorem 1, and the different behaviours of M_{2n} and M_{2n+1} is explained.

In §5, by applying the Laplace method to the right hand side of (7), we give another proof of Sirazdinov expansion, and we explicitly give one more term.

Finally, an other integral representation of $A(n,k)$ is given, specially simple when $n=2k-1$ or $n=2k$.

I am pleased to thank very much L. Lesieur for several ideas and fruitful discussions, H. Cohen who pointed out to me the integrals

$$\int_0^\infty \frac{\sin^n x}{x^m}dx,$$

L. Comtet for giving me the reference [1], V. Glaymann for kindly translating the papers [6] and [7] from Russian, and my grand-daughter Alexandra Seidel, I was baby-sitting when I proved Theorem 1.

2. Proof of Theorem 1.

Let us define the Eulerian polynomial:

$$A_n(\lambda)=\sum_{k=0}^n A(n,k)\lambda^k. \tag{8}$$

The generating function is: (cf. [2])

$$-\frac{1}{2}+\frac{1}{1-\lambda e^{-z}}=\sum_{n=0}^\infty -\frac{A_n(\lambda)}{(\lambda-1)^{n+1}}\frac{z^n}{n!} \tag{9}$$

with $A_0(\lambda)$ defined by (4) and (8).

Further, we have for $n\ge 2$, and $\lambda\in\mathbb{C}$, $\lambda\ne 0,1$

$$\frac{A_n(\lambda)}{(\lambda-1)^{n+1}n!}=\sum_{q\in\mathbb{Z}}\frac{1}{(\log|\lambda|+i\arg\lambda+2iq\pi)^{n+1}} \tag{10}$$

with $-\pi < \arg\lambda \leq \pi$.

In fact, (10) is lemma III of [5]. It is easily obtained by complex integration. Next we have from (8) by Cauchy's formula

$$A(n,k) = \frac{1}{2i\pi}\int_C A_n(\lambda)\lambda^{-k-1}\,d\lambda$$

where C denotes the circle of center 0 and radius 1. Each term of the infinite sum is integrable along C, normally for the family, when $|\lambda| = 1$ so that we can transpose the signs $\int$ and $\sum$. Hence

$$\frac{A(n,k)}{n!} = \frac{1}{2i\pi}\sum_{q\in\mathbb{Z}}\int_C \frac{(\lambda-1)^{n+1}\lambda^{-k-1}\,d\lambda}{(i(\arg\lambda + 2q\pi))^{n+1}}.$$

Next we put $\lambda = \exp(2it)$, $-\pi/2 < t \leq \pi/2$, so that

$$\frac{A(n,k)}{n!} = \frac{1}{\pi}\sum_{q\in\mathbb{Z}}\int_{-\pi/2}^{\pi/2}\frac{(e^{2it}-1)^{n+1}e^{-2itk}}{(2i(t+q\pi))^{n+1}}dt$$

$$= \frac{1}{\pi}\sum_{q\in\mathbb{Z}}\int_{\frac{-\pi}{2}+q\pi}^{\frac{\pi}{2}+q\pi}\frac{(e^{2iu}-1)^{n+1}e^{-2iuk}\,du}{(2iu)^{n+1}}$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{\sin u}{u}\right)^{n+1} e^{iu(n+1-2k)}\,du.$$

Taking the real part of the integral yields (7).

A second proof of Theorem 1 will now be given. For all $n \geq 0$, and $k \in \mathbb{Z}$, let us define $A(n,k)$ by (7). Let us observe first that the symmetric relation (6) is immediate. We shall prove (1), (2), (4), so that $A(n,k)$ will thus coincide with Eulerian numbers.

From the Fresnel integral

$$\int_0^\infty \frac{\sin t}{t}dt = \frac{2}{\pi},$$

it is easy to deduce that

$$A(0,0) = A(0,1) = 1/2 \text{ and } A(0,k) = 0 \text{ for } k \neq 0, 1. \tag{11}$$

Now, for $n \geq 2$, integrating by parts the right hand side of (7) and setting

$$u = (\sin t)^{n+1}\cos((n+1-2k)t) \qquad dv = t^{-n-1}\,dt \tag{12}$$

yields (1) for all $k \in \mathbb{Z}$ after some trigonometric calculation.

Finally by (11) and (1), it is easy to see that

$$A(1,1) = 1 \quad\text{and}\quad A(1,k) = 0 \text{ for } k \leq 0$$

and by induction on n, to prove by (1) that

$$A(n,1) = 1 \quad\text{and}\quad A(n,k) = 0 \text{ for } k \leq 0$$

which, with (6), yields (2). ■

3. Interpolation of Eulerian Numbers

Let $n \geq 0$ be an integer and x a real number. We set

$$A(n,x) = \frac{2n!}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^{n+1}\cos((n+1-2x))t\,dt. \tag{13}$$

For $n = 0$, the right hand side of (13) can be calculated from the Fresnel integral, and this gives

$$A(0,x) \begin{array}{llll} = 0 & \text{for } x < 0 & \text{and} & x > 1 \\ = 1/2 & \text{for } x = 0 & \text{and} & x = 1 \\ = 1 & \text{for } 0 < x < 1. & & \end{array}$$

It can be seen from (13) that for $n \geq 1$, $A(n,x)$ is of class C^{n-1}.

It will be convenient to introduce the function f such that for all x, $f(x) = A(0,x)$.

As usual, the convolution $f_1 * f_2$ of two real functions is defined by

$$f_1 * f_2(x) = \int_{-\infty}^{+\infty} f_1(t)f_2(x-t)dt,$$

and the Fourier transform is

$$\mathcal{F}(f)(y) = \int_{-\infty}^{+\infty} f(t)e^{-iyt}\,dt.$$

It is easy to see that

$$g(y) = \mathcal{F}(f)(y) = \int_0^1 e^{-iyt}\,dt = \frac{2}{y}\sin\left(\frac{y}{2}\right)e^{-iy/2}.$$

We now define the sequence f_n by $f_1 = f$ and $f_n = f_{n-1} * f$. From the classical properties of convolution, we know that f_{n+1} is piecewise polynomial, and that $f_{n+1}(x)$ vanishes for $x \leq 0$ and $x \geq n+1$. The Fourier transform of f_{n+1} is g_{n+1}:

$$g_{n+1}(y) = \mathcal{F}(f_{n+1})(y) = \left(\frac{\sin(y/2)}{y/2}\right)^{n+1} e^{-i(n+1)y/2},$$

and by the inversion formula

$$f_{n+1}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} g_{n+1}(y)e^{iyx}\,dy = \frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{\sin t}{t}\right)^{n+1} e^{-(n+1)it+2itx}\,dt.$$

Hence,

$$f_{n+1}(x) = \frac{1}{n!}A(n,x). \tag{14}$$

From (14), and the properties of f_{n+1}, it follows that $A(n,x)$ vanishes for $x \leq 0$ and $x \geq n+1$, and is piecewise polynomial.

Furthermore, as it has been pointed out to me by L. Lesieur, it is possible to give an explicit form of these polynomials:

Proposition. *Let $0 \leq k \leq n$ be two integers. The restriction of $A(n,x)$ to the interval $[k,k+1]$ is a polynomial $P_{n,k}$ of degree n which can be written:*

$$P_{n,k}(x) = \sum_{0\leq j\leq k}(-1)^j\binom{n+1}{j}(x-j)^n. \tag{15}$$

Proof. First we give a triangular recurrence relation for $A(n,x)$:

$$A(n+1,x) = (n-x+2)A(n,x-1) + xA(n,x). \tag{16}$$

It is obtained from (13) by an integration by part similar to (12). Then we prove (15) by induction on n. We have:

$$P_{0,0}(x) = 1$$

$$P_{1,0}(x) = x\ ; \qquad P_{1,1}(x) = x - 2(x-1) = 2 - x.$$

Further, assuming that (15) holds for n and all $k \leq n$, we have to show that:

$$P_{n+1,k}(x) = (n-x+2)P_{n,k-1}(x-1) + xP_{n,k}(x)$$

verifies (15). This is easily done by using the usual properties of the binomial coefficients. ■

From a probabilistic point of view, f can be interpreted as the density of probability of a random real variable X taking the value 1 with equal probability in the interval $[0,1]$. So f_{n+1} is the density of probability of a random variable Y which is the sum of $(n+1)$ independent random variables equal to X.

Such a point of view has already been given by Sackov (cf. for instance [6], formula (3.4) p.44).

4. Estimations of the maxima

Theorem 2. *For all integer n, let us set:*

$$m_n = \max_f f_n(x) = f_n(n/2).$$

Then the sequence m_n is not increasing.

Proof. From (14) and (13), it follows

$$m_n = \frac{1}{(n-1)!}A(n-1,n/2) = \frac{2}{\pi}\int_0^\infty\left(\frac{\sin t}{t}\right)^n dt. \tag{17}$$

Then, from the definition of f_n and the properties of the convolution, it is easy to see by induction that $f_n(x)$ is increasing for $0 \leq x \leq n/2$ and decreasing for $n/2 \leq x \leq n$. Moreover

$$f_n(x) = \int_{-\infty}^{+\infty} f_{n-1}(t)f(x-t)\,dt = \int_{x-1}^{x} f_{n-1}(t)\,dt \leq m_{n-1} \tag{18}$$

so that

$$m_n \leq m_{n-1}.$$

In [5], it was proved that m_{2p} is a decreasing sequence. From the integral value of m_{2p} given by (17), this is quite obvious, since $|\sin(t)/t| \leq 1$.

It was also proved in [15] that

$$\mu_{2p+1} = \frac{A(2p,p)}{(2p)!} = \frac{2}{\pi}\int_0^\infty \left(\frac{sint}{t}\right)^{2p+1} \cos t\, dt \tag{19}$$

is decreasing in p, and that for all p,

$$m_{2p+5} \le \mu_{2p+1} \le m_{2p+3}. \tag{20}$$

To prove that μ_{2p+1} is decreasing with its integral representation (19), we may observe that

$$\mu'_{2p+1} = \frac{2}{\pi}\int_0^{\pi/2} \left(\frac{\sin t}{t}\right)^{2p+1} \cos t\, dt$$

is obviously decreasing, and that μ_{2p+1} and μ'_{2p+1} are very close. This can be put in form, but calculations are a bit technical.

Similarly, to get (20), we observe that for t small enough, we have from Mac Laurin's formula:

$$(\frac{\sin t}{t})^4 \le \cos t \le \left(\frac{\sin t}{t}\right)^2. \tag{21}$$

It can be proved that (21) holds for $0 \le t \le 1$, and, as above, the proof of (20) can be completed with some technical estimations.

The lag between indices of μ and m in (20) can be explained: let us define for all n,

$$\mu_n = \max_{x\in\mathbb{Z}} f_n(x)$$

so that, if n is even,

$$\mu_n = m_n = \frac{1}{(n-1)!}A(n-1, n/2)$$

but if n is odd, $n = 2p+1$, then $\mu_n < m_n$, and μ_n is given by (19).

Similarly, from (14) and (18), it follows

$$A(n,x) = n! f_{n+1}(x) = n! \int_{x-1}^{x} f_n(t)dt = n\int_{x-1}^{x} A(n-1,t)dt.$$

Since we know that $A(n-1,x)$ is increasing for $x \le n/2$ and decreasing for $x > n/2$, we deduce that

$$\text{for } x \le n/2, \qquad A(n,x) \le n\,A(n-1,x)$$

and for

$$x \ge n/2+1 \qquad A(n,x) \ge n\,A(n-1,x).$$

Therefore, for x fixed the sequence $A(n,x)/n!$ is unimodal that is to say increasing for $n \le n_0$ and decreasing for $n \ge n_0$.

For $x \le n/2$, using (1) we obtain

$$A(n-1,x-1) \le \frac{n-x}{n+1-x}A(n-1,x)$$

but this inequality is weaker than Theorem 3 of [5].

5. Sirazdinov's asymptotic expansion

Theorem 3. *Let x be a fixed real number, and n an integer. Let us define*

$$I(n,x) = \int_0^\infty \left(\frac{\sin t}{t}\right)^n \cos\left(xt\sqrt{\frac{n}{3}}\right) dt.$$

When n goes to infinity, we have the following asymptotic expansion

$$I(n,x) = \sqrt{\frac{6}{\pi n}}e^{-x^2/2}\Big(1 - \frac{1}{20n}(x^4-6x^2+3) + \frac{1}{n^2}\Big(\frac{x^8}{800} - \frac{107}{4200}x^6 + \frac{67}{560}x^4 - \frac{27}{280}x^2 - \frac{13}{1120}\Big)\Big) + O\left(\frac{(\log n)^{11}}{n^{7/2}}\right). \tag{22}$$

Remark. *When $k = \frac{n+1}{2} + x\sqrt{\frac{n+1}{12}}$, then, from(7),*

$$A(n,k) = \frac{2}{\pi}(n!)I(n+1,x)$$

and the two first terms of (22) coincide with the asymptotic expansion given by Sirazdinov (cf. [7]).

Lemma 1. *Let us set*

$$U_{2k}(x) = \int_0^\infty e^{-v^2/2}v^{2k}\cos vx\, dv.$$

We have

$$U_0(x) = \sqrt{\frac{\pi}{2}}\exp(-x^2/2) \tag{23}$$

and

$$U_{2k}(x) = (-1)^k \frac{d^{2k}}{dx^{2k}}(U_0(x)). \tag{24}$$

Proof. If we set

$$W_{2k}(x) = \int_0^\infty e^{-v^2/2} v^{2k} e^{-ivx}\, dv,$$

we have

$$U_{2k}(x) = \mathrm{Re}(W_{2k}(x)),$$

and

$$W_{2k}(x) = (-i)^{2k} \frac{d^{2k}}{dx^{2k}} W_0(x).$$

Moreover $W_0(x)$ is half of the Fourier transform of the function $v \mapsto \exp(-v^2/2)$, which can be found in the tables. ■

Proof of Theorem 3. It will follow the Laplace method. The case $x = 0$ has been treated in [5]. The derivative of $\log \sin t$ is $\mathrm{cotg}\, t$, and so, for $|t| < \pi$, we have:

$$\log \frac{\sin t}{t} = -\frac{t^2}{6} - \frac{t^4}{180} - \frac{t^6}{2835} - \cdots - \frac{2^{2k} B_{2k}}{2^k (2k)!} t^{2k} - \cdots \tag{25}$$

where B_{2k} are positive Bernoulli numbers.

From (25) it follows that for $|t| < \pi$

$$\log \frac{\sin t}{t} \le -\frac{t^2}{6} \tag{26}$$

holds. Now,

$$\begin{aligned}\int_{\frac{\log n}{\sqrt{n}}}^{+\infty} \left(\frac{\sin t}{t}\right)^n \cos\left(xt\sqrt{\frac{n}{2}}\right) dt &\le \left|\int_{\frac{\log n}{\sqrt{n}}}^{2}\right| + \left|\int_2^{+\infty}\right| \\ &\le \int_{\frac{\log n}{\sqrt{n}}}^{2} \exp\left(-\frac{nt^2}{6}\right) dt + \int_2^\infty t^{-n}\, dt \\ &\le 2\exp\left(-\frac{\log^2 n}{6}\right) + \frac{1}{(n-1)2^{n-1}} = O(n^{-B})\end{aligned}$$

where B is a positive number as large as we want.

+Hence,

$$\begin{aligned} I(n,x) &= \\ &= \int_0^{\frac{\log n}{\sqrt{n}}} \exp\left(n \log\left(\frac{\sin t}{t}\right)\right) \cos\left(xt\sqrt{\frac{n}{3}}\right) dt + O(n^{-B}) \\ &= \int_0^{\frac{\log n}{\sqrt{n}}} \exp\left(-n\left(\frac{t^2}{6} + \frac{t^4}{180} + \frac{t^6}{2835} + O(t^8)\right)\right) \cos\left(xt\sqrt{\frac{n}{3}}\right) dt + O(n^{-B}) \\ &= \int_0^{\frac{\log n}{\sqrt{n}}} \exp\left(-n\left(\frac{t^2}{6} + \frac{t^4}{180} + \frac{t^6}{2835}\right) + O\left(\frac{(\log n)^8}{n^3}\right)\right) \cos\left(xt\sqrt{\frac{n}{3}}\right) dt \\ &\quad + O(n^{-B}) \\ &= J(n,x) + O\left(\frac{(\log n)^9}{n^{7/2}}\right) + O(n^{-B}) \end{aligned}$$

with

$$J(n,x) = \int_0^{\frac{\log n}{\sqrt{n}}} \exp\left(-n\left(\frac{t^2}{6} + \frac{t^4}{180} + \frac{t^6}{2835}\right)\right) \cos\left(xt\sqrt{\frac{n}{3}}\right) dt.$$

By setting $u = \sqrt{\frac{n}{3}}t$, we get

$$\begin{aligned} J(n,x) &= \sqrt{\frac{3}{n}} \int_0^{\frac{\log n}{\sqrt{3}}} \exp\left(-\frac{u^2}{2} - \frac{u^4}{20n} - \frac{u^6}{105n^2}\right) \cos(ux)\, du \\ &= \sqrt{\frac{3}{n}} \int_0^{\frac{\log n}{\sqrt{3}}} \exp\left(-\frac{u^2}{2}\right) \exp(-v) \cos(ux)\, du, \end{aligned}$$

with

$$v = \frac{u^4}{20n} + \frac{u^6}{105n^2}.$$

When n goes to infinity, we have for $0 \le u \le \frac{\log n}{\sqrt{3}}$,

$$\exp(-v) = 1 - \frac{u^4}{20n} - \frac{u^6}{105n^2} + \frac{u^8}{800n^2} + O\left(\frac{(\log n)^{10}}{n^3}\right)$$

and hence

$$J(n,x)=\sqrt{\frac{3}{n}}\int_0^{\frac{\log n}{\sqrt{3}}}\exp\Big(-\frac{u^2}{2}\Big)\Big(1-\frac{u^4}{20n}-\frac{u^6}{105n^2}+\frac{u^8}{800n^2}\Big)\cos(ux)du + O\Big(\frac{(\log n)^{11}}{n^{7/2}}\Big)$$
$$=\sqrt{\frac{3}{n}}\int_0^{\infty}\exp\Big(-\frac{u^2}{2}\Big)\Big(1-\frac{u^4}{20n}-\frac{u^6}{105n^2}+\frac{u^8}{800n^2}\Big)\cos(ux)du + O\Big(\int_{\frac{\log n}{\sqrt{3}}}^{+\infty}\exp\Big(-\frac{u^2}{2}\Big)du\Big)+O\Big(\frac{(\log n)^{11}}{n^{7/2}}\Big).$$

Now, by a classical estimation (cf., for instance [5], III 2.3), the first error term is included in the second, and, with the notation of Lemma 1

$$J(n,x)=\sqrt{\frac{3}{n}}\Big(U_0(x)-\frac{U_4(x)}{20n}-\frac{1}{n^2}\Big(\frac{U_6(x)}{105}-\frac{U_8(x)}{800}\Big)\Big)+O\Big(\frac{(\log n)^{11}}{n^{7/2}}\Big)$$

and, by Lemma 1, the proof of Theorem 3 is completed. ∎

6. Another Integral Representation

It has been known for a long time (cf., for instance [1], t.II, p.203), that for $2\le m\le n$, and n even,

$$\int_0^\infty \frac{\sin^n x}{x^m}dx=\frac{n!}{(m-1)!}\int_0^\infty\frac{z^{m-2}dz}{\prod_{1\le r\le n/2}(z^2+4r^2)}. \tag{27}$$

A similar relation exists also when n is odd. The proof of (27) follows from

$$\frac{1}{x^m}=\frac{1}{(m-1)!}\int_0^\infty e^{-zx}z^{m-1}dz, \tag{28}$$

from the Laplace transform of $\sin^n x$:

$$\int_0^\infty e^{-zx}\sin^n x dx=\frac{n!}{z\prod_{1\le r\le n/2}(z^2+4r^2)}, \tag{29}$$

and from Fubini's theorem.

The same method shows that the integral (7) can be expressed as the integral between 0 and infinity of a rational function. Unfortunately the numerator of this fraction does not look simple.

Theorem 4. *For $n\ge 2$, and $0\le k\le (n+1)/2$, we have*

$$A(n,k)=\frac{2}{\pi}(n+1)!\int_0^\infty\frac{z^{n-1}Q_{n-2k+1,k}(z)dz}{\prod_{1\le r\le n+1-k}(z^2+4r^2)}, \tag{30}$$

where $Q_{d,k}(z)$ is a monic polynomial of degree d in z, satisfying

$$Q_{0,k}(z)=1\,;\qquad Q_{1,k}(z)=z\,; \tag{31}$$

and

$$(d+k)Q_{d,k}(z)=z(d+2k+1)Q_{d-1,k+1}(z)-(k+1)(z^2+4(d+k)^2)Q_{d-2,k+1}(z). \tag{32}$$

Proof. Let us set

$$J(n,k)=J(n,k)(z)=\int_0^\infty e^{-zx}\sin^{n+1}x\cos((n+1-2k)x)dx. \tag{33}$$

Integrating by parts with

$$u=\sin^{n+1}x\cos((n+1-2k)x)\qquad dv=e^{-zx}dx$$

in a similar way like (12) gives

$$zJ(n,k)=(n+1-k)J(n-1,k-1)+kJ(n-1,k). \tag{34}$$

The symmetric relation

$$J(n,k)=J(n,n-k+1) \tag{35}$$

also holds. We first prove:

$$J(n,k)=\frac{(n+1)!Q_{n-2k+1,k}(z)}{z\prod_{1\le r\le n+1-k}(z^2+4r^2)} \tag{36}$$

where Q is defined by (31) and (32).

The proof of (36) is by induction on $d=n-2k+1$, for all $k\ge 0$. First, for $d=0$, $n=2k-1$, and from (29), since $n+1$ is even

$$J(n,k)=J(2k-1,k)=\int_0^\infty e^{-zx}\sin^{n+1}(x)dx$$
$$=\frac{(n+1)!}{z\prod_{1\le r\le (n+1)/2}(z^2+4r^2)}. \tag{37}$$

Further, for $d = 1$, it follows from (34) and (35) that

$$J(n,k) = J(2k,k) = \frac{z}{2k+2} J(2k+1, k+1). \tag{38}$$

If we substitute $k+1$ to k in (37), (38) yields

$$J(n,k) = J(2k,k) = \frac{z}{2k+2} \frac{(2k+2)!}{z\prod_{1\le r\le k+1}(z^2+4r^2)}. \tag{39}$$

So, by (31), (37) and (39) prove (36) for $d = 0$ and $d = 1$. Now, we suppose that (36) holds for $n - 2k + 1 \le d - 1$. From (34) we deduce

$$(d+k)J(d+2k-1,k) = zJ(d+2k,k+1) - (k+1)J(d+2k-1,k+1). \tag{40}$$

But the above right hand side can be expressed by (36) under the induction hypothesis, yielding

$$\frac{z(d+2k+1)!Q_{d-1,k+1}}{z\prod_{1\le r\le d+k}(z^2+4r^2)} - (k+1)\frac{(d+2k)!Q_{d-2,k+1}}{z\prod_{1\le r\le d+k-1}(z^2+4r^2)}$$
$$= \frac{(d+2k)!(z(d+2k+1)Q_{d-1,k+1} - (k+1)(z^2+4(d+k)^2)Q_{d-2,k+1})}{z\prod_{1\le r\le d+k}(z^2+4r^2)}$$

which completes the proof of (36).

To prove (30), we first deduce from (7) and (28)

$$A(n,k) = \frac{2}{\pi}\int_0^\infty \int_0^\infty e^{-zt} z^n (\sin t)^{n+1} \cos((n+1-2k)t)dzdt. \tag{41}$$

By Fubini's theorem, (41) gives

$$A(n,k) = \frac{2}{\pi}\int_0^\infty z^n J(n,k)(z)dz$$

which, with (36) yields (30).

From (31) and (32), it follows by induction that $Q_{d,k}$ is monic, and of degree d. It follows in the same way that the coefficients of $Q(d,k)$ are polynomials in k with integral coefficients divided by $(k+2)(k+3)\dots(k+d)$. From the first values:

$$Q_{2,k} = z^2 - 4(k+1)(k+2)$$
$$Q_{3,k} = z^3 - 4(3k^2+12k+11)$$
$$Q_{4,k} = z^4 - 4(6k^2+30k+35) + 16(k+1)(k+2)(k+3)(k+4)$$

it can be seen that there is a simplification, and that the coefficients of $Q_{d,k}$ belong to $\mathbb{Z}[k]$. This fact has been checked up to $d = 20$, by the computer algebra system MAPLE.

As a possible hint, we may observe that substituting complex exponentials to trigonometric functions in (33) gives:

$$J(n,k) = \frac{1}{2^{n+2} i^{n+1}} \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} \left(\frac{-1}{z - 2i(n+1-j-k)} + \frac{-1}{z+2i(j-k)} \right).$$

It is possible to prove by induction that the coefficient of degree $d-2$ of $Q_{d,k}$ is

$$\frac{-d(d-1)}{6}(12k^2 + 12(d+1)k + (d+1)(3d+2))$$

and the constant coefficient of $Q_{2,k}$ is

$$(-1)^d 2^{2d}(k+1)(k+2)\dots(k+2d). \quad \blacksquare$$

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