On Landau's Function $g(n)$

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1. Introduction

Let S_n be the symmetric group of n letters. Landau considered the function $g(n)$ defined as the maximal order of an element of S_n; Landau observed that (cf. [9])

$$g(n) = \max \operatorname{lcm}(m_1, \ldots, m_k) \tag{1}$$

where the maximum is taken on all the partitions $n = m_1 + m_2 + \cdots + m_k$ of n and proved that, when n tends to infinity

$$\log g(n) \sim \sqrt{n \log n}. \tag{2}$$

More precise asymptotic estimates have been given in [11, 22, 25]. In [25] and [11] one also can find asymptotic estimates for the number of prime factors of $g(n)$. In [8] and [3], the largest prime factor $P^+(g(n))$ of $g(n)$ is investigated. In [10] and [12], effective upper and lower bounds of $g(n)$ are given. In [17], it is proved that $\lim_{n\to\infty} g(n+1)/g(n) = 1$. An algorithm able to calculate $g(n)$ up to 10^{15} is given in [2] (see also [26]). The sequence of distinct values of $g(n)$ is entry A002809 of [24]. A nice survey paper was written by W. Miller in 1987 (cf. [13]).

My very first mathematical paper [15] was about Landau's function, and the main result was that $g(n)$, which is obviously non decreasing, is constant on arbitrarily long intervals (cf. also [16]). I first met A. Schinzel in Paris in May 1967. He told me that he was interested in my results, but that P. Erdős would be more interested than himself. Then I wrote my first letter to Paul with a copy of my work. I received an answer dated of June 12 1967 saying "I sometimes thought about $g(n)$ but my results were very much less complete than yours". Afterwards, I met my advisor, the late Professor Pisot, who, in view of this letter, told me that my work was good for a thesis.

The main idea of my work about $g(n)$ was to use the tools introduced by S. Ramanujan to study highly composite numbers (cf. [19, 20]). P. Erdős was very well aware of this paper of Ramanujan (cf. [1, 4–6]) as well as of the symmetric group and the order of its elements, (cf. [7]) and I think that he enjoyed the connection between these two areas of mathematics. Anyway, since these first letters, we had many occasions to discuss Landau's function.

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Let us define $n_1 = 1$, $n_2 = 2$, $n_3 = 3$, $n_4 = 4$, $n_5 = 5$, $n_6 = 7$, etc. $\ldots, n_k$ (see a table of $g(n)$ in [16, p. 187]), such that

$$g(n_k) > g(n_k - 1). \tag{3}$$

The above mentioned result can be read:

$$\overline{\lim}(n_{k+1} - n_k) = +\infty. \tag{4}$$

Here, I shall prove the following result:

Theorem 1.

$$\underline{\lim}(n_{k+1} - n_k) < +\infty. \tag{5}$$

Let us set $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$, $p_k =$ the k-th prime. It is easy to deduce Theorem 1 from the twin prime conjecture (i.e. $\underline{\lim}(p_{k+1} - p_k) = 2$) or even from the weaker conjecture $\underline{\lim}(p_{k+1} - p_k) < +\infty$. (cf. Sect. 4 below.) But I shall prove Theorem 1 independently of these deep conjectures. Moreover I shall explain below why it is reasonable to conjecture that the mean value of $n_{k+1} - n_k$ is 2; in other terms one may conjecture that

$$n_k \sim 2k \tag{6}$$

and that $n_{k+1} - n_k = 2$ has infinitely many solutions. Due to a parity phenomenon, $n_{k+1} - n_k$ seems to be much more often even than odd; nevertheless, I conjecture that:

$$\underline{\lim}(n_{k+1} - n_k) = 1. \tag{7}$$

The steps of the proof of Theorem 1 are first to construct the set G of values of $g(n)$ corresponding to the so called superior highly composite numbers introduced by S. Ramanujan, and then, when $g(n) \in G$, to build the table of $g(n+d)$ when d is small. This will be done in Sects. 4 and 5. Such values of $g(n+d)$ will be linked with the number of distinct differences of the form $P - Q$ where P and Q are primes satisfying $x - x^\alpha \le Q \le x < P \le x + x^\alpha$, where x goes to infinity and $0 < \alpha < 1$. Our guess is that these differences $P - Q$ represent almost all even numbers between 0 and $2x^\alpha$, but we shall only prove in Sect. 3 that the number of these differences is of the order of magnitude of x^α, under certain strong hypothesis on x and α, and for that a result due to Selberg about the primes between x and $x + x^\alpha$ will be needed (cf. Sect. 2).

To support conjecture (6), I think that what has been done here with $g(n) \in G$ can also be done for many more values of $g(n)$, but, unfortunately, even assuming strong hypotheses, I do not see for the moment how to manage it.

I thank very much E. Fouvry who gave me the proof of Proposition 2.

1.1 Notation

p will denote a generic prime, p_k the k-th prime; P, Q, P_i, Q_j will also denote primes. As usual $\pi(x) = \sum_{p\le x} 1$ is the number of primes up to x.

$|S|$ will denote the number of elements of the set S. The sequence n_k is defined by (3).

2. About the Distribution of Primes

Proposition 1. *Let us define $\pi(x) = \sum_{p\le x} 1$, and let α be such that $\frac{1}{6} < \alpha < 1$, and $\varepsilon > 0$. When ξ goes to infinity, and $\xi' = \xi + \xi/\log\xi$, then for all x in the interval $[\xi, \xi']$ but a subset of measure $O((\xi' - \xi)/\log^3 \xi)$ we have:*

$$\left|\pi(x + x^\alpha) - \pi(x) - \frac{x^\alpha}{\log x}\right| \le \varepsilon \frac{x^\alpha}{\log x} \tag{8}$$

$$\left|\pi(x) - \pi(x - x^\alpha) - \frac{x^\alpha}{\log x}\right| \le \varepsilon \frac{x^\alpha}{\log x} \tag{9}$$

$$\left|\frac{x}{\log x} - \frac{Q^k - Q^{k-1}}{\log Q}\right| \ge \frac{\sqrt{x}}{\log^4 x} \text{ for all primes } Q, \text{ and } k \ge 2. \tag{10}$$

Proof. This proposition is an easy extension of a result of Selberg (cf. [21]) who proved that (8) holds for most x in (ξ, ξ'). In [18], I gave a first extension of Selberg's result by proving that (8) and (9) hold simultaneously for all x in (ξ, ξ') but for a subset of measure $O((\xi' - \xi)/\log^3 \xi)$. So, it suffices to prove that the measure of the set of values of x in (ξ, ξ') for which (10) does not hold is $O((\xi' - \xi)/\log^3 \xi)$.

We first count the number of primes Q such that for one k we have:

$$\frac{\xi}{\log \xi} \le \frac{Q^k - Q^{k-1}}{\log Q} \le \frac{\xi'}{\log \xi'}. \tag{11}$$

If Q satisfies (11), then $k \le \frac{\log \xi'}{\log 2}$ for ξ' large enough. Further, for k fixed, (11) implies that $Q \le (\xi')^{1/k}$, and the total number of solutions of (11) is

$$\le \sum_{k=2}^{\log \xi'/\log 2} (\xi')^{1/k} = O(\sqrt{\xi'}) = O(\sqrt{\xi}).$$

With a more careful estimation, this upper bound could be improved, but this crude result is enough for our purpose. Now, for all values of $y = \frac{Q^k - Q^{k-1}}{\log Q}$ satisfying (11), we cross out the interval $\left(y - \frac{\sqrt{\xi'}}{\log^4 \xi'}, y + \frac{\sqrt{\xi'}}{\log^4 \xi'}\right)$. We also cross out this interval whenever $y = \frac{\xi}{\log \xi}$ and $y = \frac{\xi'}{\log \xi'}$. The total sum of the lengths of the crossed out intervals is $O\left(\frac{\xi}{\log^4 \xi}\right)$, which is smaller than the length of the interval $\left(\frac{\xi}{\log\xi}, \frac{\xi'}{\log\xi'}\right)$ and if $\frac{x}{\log x}$ does not fall into one of these forbidden intervals, (10) will certainly hold. Since the derivative of the function $\varphi(x) = x/\log x$ is $\varphi'(x) = \frac{1}{\log x} - \frac{1}{\log^2 x}$ and satisfies $\varphi'(x) \sim \frac{1}{\log\xi}$ for all $x \in (\xi, \xi')$, the measure of the set of values of $x \in (\xi, \xi')$ such that $\varphi(x)$ falls into one of the above forbidden intervals is, by the mean value theorem $O\left(\frac{\xi}{\log^3\xi}\right)$, and the proof of Proposition 1 is completed. □

3. About the Differences Between Primes

Proposition 2. *Suppose that there exists $\alpha, 0 < \alpha < 1$, and x large enough such that the inequalities*

$$\pi(x + x^\alpha) - \pi(x) \geq (1 - \varepsilon)x^\alpha/\log x \tag{12}$$

$$\pi(x) - \pi(x - x^\alpha) \geq (1 - \varepsilon)x^\alpha/\log x \tag{13}$$

hold. Then the set

$$E = E(x, \alpha) = \{P - Q; P, Q \textit{ primes},\ x - x^\alpha < Q \leq x < P \leq x + x^\alpha\}$$

satisfies:

$$|E| \geq C_2 x^\alpha$$

where $C_2 = C_1\alpha^4(1 - \varepsilon)^4$ and C_1 is an absolute constant ($C_1 = 0.00164$ works).

Proof. The proof is a classical application of the sieve method that Paul Erdős enjoys very much. Let us set, for $d \leq 2x^\alpha$,

$$r(d) = |\{(P, Q); x - x^\alpha < Q \leq x < P \leq x + x^\alpha, P - Q = d\}|.$$

Clearly we have

$$|E| = \sum_{\substack{0<d\leq 2x^\alpha \\ r(d)\neq 0}} 1 \tag{14}$$

and

$$\sum_{0<d\leq 2x^\alpha} r(d) = (\pi(x + x^\alpha) - \pi(x))(\pi(x) - \pi(x - x^\alpha)) \geq (1 - \varepsilon)^2 x^{2\alpha}/\log^2 x. \tag{15}$$

Now to get an upper bound for $r(d)$, we sift the set

$$A = \{n; x - x^\alpha < n \leq x\}$$

with the primes $p \leq z$. If p divides d, we cross out the n's satisfying $n \equiv 0 \pmod p$, and if p does not divide d, the n's satisfying

$$n \equiv 0 \pmod p \quad \text{or} \quad n \equiv -d \pmod p$$

so that we set for $p \le z$:

$$w(p) = \begin{cases} 1 & \text{if } p \text{ divides } d \\ 2 & \text{if } p \text{ does not divide } d. \end{cases}$$

By applying the large sieve (cf. [14, Corollary 1]), we have

$$r(d) \le \frac{|A|}{L(z)}$$

with

$$L(z) = \sum_{n \le z} \left(1 + \frac{3}{2} n |A|^{-1} z\right)^{-1} \mu(n)^2 \left(\prod_{p|n} \frac{w(p)}{p - w(p)}\right)$$

(μ is the Möbius function), and with the choice $z = (\frac{2}{3}|A|)^{1/2}$, it is proved in [23] that

$$\frac{|A|}{L(z)} \le 16 \prod_{p \ge 3} \left(1 - \frac{1}{(p-1)^2}\right) \frac{|A|}{\log^2(|A|)} \prod_{\substack{p|d \\ p>2}} \frac{p-1}{p-2}.$$

The value of the above infinite product is $0.6602\ldots < 2/3$. We set $f(d) = \prod_{\substack{p|d \\ p>2}} \frac{p-1}{p-2}$, and we observe that $|A| \ge x^{\alpha} - 1$, so that for x large enough

$$r(d) \le \frac{32}{3\alpha^2} \frac{|A|}{\log^2 x} f(d). \tag{16}$$

Now, for the next step, we shall need an upper bound for $\sum_{n \le x} f^2(n)$. By using the convolution method and defining

$$h(n) = \sum_{a|n} \mu(a) f^2(n/a)$$

one gets $h(2) = h(2^2) = h(2^3) = \ldots = 0$ and, for $p \ge 3$, $h(p) = \frac{2p-3}{(p-2)^2}$, $h(p^2) = h(p^3) = \ldots = 0$, so that

$$\begin{aligned} \sum\nolimits_{n \le x} f^2(n) &= \sum\nolimits_{n \le x} \sum\nolimits_{a|n} h(a) = \sum\nolimits_{a \le x} h(a) \left\lfloor \frac{x}{a} \right\rfloor \\ &\le x \sum\nolimits_{a=1}^{\infty} \frac{h(a)}{a} = x \prod\nolimits_{p \ge 3} \left(1 + \frac{2p-3}{p(p-2)^2}\right) \qquad (17) \\ &= 2.63985\ldots x \le \frac{8}{3} x. \end{aligned}$$

From (15) and (16), one can deduce

$$\frac{(1-\varepsilon)^2 x^{2\alpha}}{\log^2 x} \le \sum_{\substack{0<d\le 2x^{\alpha} \\ r(d) \ne 0}} r(d) \le \frac{32}{3\alpha^2} \frac{|A|}{\log^2 x} \sum_{\substack{0<d\le 2x^{\alpha} \\ r(d) \ne 0}} f(d)$$

which implies

$$\sum_{\substack{0<d\le 2x^\alpha\\ r(d)\neq 0}} f(d) \ge \frac{3\alpha^2 x^{2\alpha}(1-\varepsilon)^2}{32|A|}.$$

By Cauchy-Schwarz's inequality, one has

$$\Bigg(\sum_{\substack{0<d\le 2x^\alpha\\ r(d)\neq 0}} 1\Bigg)\Bigg(\sum_{\substack{0<d\le 2x^\alpha\\ r(d)\neq 0}} f^2(d)\Bigg) \ge \frac{9\alpha^4 x^{4\alpha}(1-\varepsilon)^4}{1{,}024|A|^2}$$

and, by (14) and (17)

$$|E| \ge \frac{9\alpha^4 x^{4\alpha}(1-\varepsilon)^4}{1{,}024|A|^2} \Big/ \frac{8}{3}(2x^\alpha) = \frac{27}{16{,}384}\frac{x^{3\alpha}(1-\varepsilon)^4}{|A|^2}.$$

Since $|A| \le x^\alpha + 1$, and x has been supposed large enough, Proposition 2 is proved. $\square$

4. Some Properties of $g(n)$

Here, we recall some known properties of $g(n)$ which can be found for instance in [16]. Let us define the arithmetic function ℓ in the following way: ℓ is additive, and, if p is a prime and $k \ge 1$, then $\ell(p^k) = p^k$. It is not difficult to deduce from (1) (cf. [13] or [16]) that

$$g(n) = \max_{\ell(M)\le n} M. \tag{18}$$

Now the relation (cf. [16, p. 139])

$$M \in g(\mathbb{N}) \iff (M' > M \implies \ell(M') > \ell(M)) \tag{19}$$

easily follows from (18), and shows that the values of the Landau function g are the “champions” for the small values of ℓ. So the methods introduced by Ramanujan (cf. [19]) to study highly composite numbers can also be used for $g(n)$. Indeed M is highly composite, if it is a “champion” for the divisor function d, that is to say if

$$M' < M \implies d(M') < d(M).$$

Corresponding to the so-called superior highly composite numbers, one introduces the set $G : N \in G$ if there exists $\rho > 0$ such that

$$\forall M \ge 1, \quad \ell(M) - \rho \log M \ge \ell(N) - \rho \log N. \tag{20}$$

Equations (19) and (20) easily imply that $G \subset g(\mathbb{N})$. Moreover, if $\rho > 2/\log 2$, let us define $x > 4$ such that $\rho = x/\log x$ and

$$N_\rho = \prod_{p\le x} p^{\alpha_p} = \prod_p p^{\alpha_p} \tag{21}$$

with

$$\alpha_p = \begin{cases} 0 & \text{if} \quad p > x \\ 1 & \text{if} \quad \frac{p}{\log p} \le \rho < \frac{p^2-p}{\log p} \\ k \ge 2 & \text{if} \quad \frac{p^k - p^{k-1}}{\log p} \le \rho < \frac{p^{k-1}-p^k}{\log p} \end{cases}$$

then $N_\rho \in G$. With the above definition, since $x \ge 4$, it is not difficult to show that (cf. [11, (5)])

$$p^{\alpha_p} \le x \tag{22}$$

holds for $p \le x$, whence N_ρ is a divisor of the least common multiple of the integers $\le x$. Here we can prove

Proposition 3. *For every prime p, there exists n such that the largest prime factor of $g(n)$ is equal to p.*

Proof. We have $g(2) = 2, g(3) = 3$. If $p \ge 5$, let us choose $\rho = p/\log p > 2/\log 2$. N_ρ defined by (21) belongs to $G \subset g(\mathbb{N})$, and its largest prime factor is p, which proves Proposition 3. □

From Proposition 3, it is easy to deduce a proof of Theorem 1, under the twin prime conjecture. Let $P = p + 2$ be twin primes, and n such that the largest prime factor of $g(n)$ is p. The sequence n_k being defined by (3), we define k in terms of n by $n_k \le n < n_{k+1}$, so that $g(n_k) = g(n)$ has its largest prime factor equal to p. Now, from (18) and (19),

$$\ell(g(n_k)) = n_k$$

and $g(n_k + 2) > g(n_k)$ since $M = \frac{P}{p} g(n_k)$ satisfies $M > g(n_k)$ and $\ell(M) = n_k + 2$. So $n_{k+1} \le n_k + 2$, and Theorem 1 is proved under this strong hypothesis.

Let us introduce now the so-called benefit method. For a fixed $\rho > 2/\log 2$, $N = N_\rho$ is defined by (21), and for any integer M,

$$M = \prod_p p^{\beta_p},$$

one defines the benefit of M:

$$\operatorname{ben}(M) = \ell(M) - \ell(N) - \rho \log M/N. \tag{23}$$

Clearly, from (20), $\operatorname{ben}(M) \ge 0$ holds, and from the additivity of ℓ one has

$$\operatorname{ben}(M) = \sum_p \left(\ell(p^{\beta_p}) - \ell(p^{\alpha_p}) - \rho(\beta_p - \alpha_p) \log p \right). \tag{24}$$

In the above formula, let us observe that $\ell(p^\beta) = p^\beta$ if $\beta \ge 1$, but that $\ell(p^\beta) = 0 \ne p^\beta = 1$ if $\beta = 0$, and, due to the choice of α_p in (21), that, in the sum (24), all the terms are non negative: for all p and for $\beta \ge 0$, we have

$$\ell(p^\beta) - \ell(p^{\alpha_p}) - \rho(\beta - \alpha_p) \log p \ge 0. \tag{25}$$

Indeed, let us consider the set of points $(0,0)$ and $(\beta, p^\beta \log p)$ for β integer ≥ 1. For all p, the piecewise linear curve going through these points is convex, and for a given ρ, α_p is chosen so that the straight line L of slope ρ going through $\left(\alpha_p, \frac{p^{\alpha_p}}{\log p}\right)$ does not cut that curve. The left-hand side of (25), (which is $\text{ben}(Np^{\beta-\alpha_p})$) can be seen as the product of $\log p$ by the vertical distance of the point $\left(\beta, \frac{p^\beta}{\log p}\right)$ to the straight line L, and because of convexity, we shall have for all p,

$$\text{ben}(Np^t) \geq t\,\text{ben}(Np), \quad t \geq 1 \tag{26}$$

and for $p \leq x$,

$$\text{ben}(Np^{-t}) \geq t\,\text{ben}(Np^{-1}), \quad 1 \leq t \leq \alpha_p. \tag{27}$$

5. Proof of Theorem 1

First the following proposition will be proved:

Proposition 4. *Let $\alpha < 1/2$, and x large enough such that (10) holds. Let us denote the primes surrounding x by:*

$$\ldots < Q_j < \ldots < Q_2 < Q_1 \leq x < P_1 < P_2 < \ldots < P_i < \ldots.$$

Let us define $\rho = x/\log x, N = N_\rho$ by (21), $n = \ell(N)$. Then for $n \leq m \leq n + 2x^\alpha, g(m)$ can be written

$$g(m) = N\frac{P_{i_1}P_{i_2}\ldots P_{i_r}}{Q_{j_1}Q_{j_2}\ldots Q_{j_r}} \tag{28}$$

with $r \geq 0$ and $i_1 < \ldots < i_r, j_1 < \ldots < j_r, P_{i_r} \leq x + 4x^\alpha, Q_{j_r} \geq x - 4x^\alpha$.

Proof. First, from (18), one has $\ell(g(m)) \leq m$, and from (23) and (18)

$$\text{ben}(g(m)) = \ell(g(m)) - \ell(N) - \rho \log \frac{g(m)}{N} \leq m - n \leq 2x^\alpha \tag{29}$$

for $n \leq m \leq 2x^\alpha$.

Further, let $Q \leq x$ be a prime, and $k = \alpha_Q \geq 1$ the exponent of Q in the standard factorization of N. Let us suppose that for a fixed m, Q divides $g(m)$ with the exponent $\beta_Q = k + t$, $t > 0$. Then, from (24), (25), and (26), one gets

$$\text{ben}(g(m)) \geq \text{ben}(NQ^t) \geq \text{ben}(NQ) \tag{30}$$

and

$$\begin{aligned}\text{ben}(NQ) &= Q^{k+1} - Q^k - \rho \log Q \\ &= \log Q\left(\frac{Q^{k+1} - Q^k}{\log Q} - \rho\right).\end{aligned}$$

From (21), the above parenthesis is nonnegative, and from (10), one gets:

$$\operatorname{ben}(NQ) \geq \log 2 \frac{\sqrt{x}}{\log^4 x}. \tag{31}$$

For x large enough, there is a contradiction between (29), (30) and (31), and so, $\beta_Q \leq \alpha_Q$.

Similarly, let us suppose $Q \leq x$, $k = \alpha_Q \geq 2$ and $\beta_Q = k - t$, $1 \leq t \leq k$. One has, from (24), (25) and (27),

$$\operatorname{ben}(g(m)) \geq \operatorname{ben}(NQ^{-t}) \geq \operatorname{ben}(NQ^{-1})$$

and

$$\begin{aligned}\operatorname{ben}(NQ^{-1}) &= Q^{k-1} - Q^k + \rho \log Q \\ &= \log Q \left(\rho - \frac{Q^k - Q^{k-1}}{\log Q}\right) \geq \log 2 \frac{\sqrt{x}}{\log^4 x}\end{aligned}$$

which contradicts (29), and so, for such a Q, $\beta_Q = \alpha_Q$.

Now, let us suppose $Q \leq x, \alpha_Q = 1$, and $\beta_Q = 0$ for some $m, n \leq m \leq n + 2x^\alpha$. Then

$$\operatorname{ben}(g(m)) \geq \operatorname{ben}(NQ^{-1}) = -Q + \rho \log Q = y(Q)$$

by setting $y(t) = \rho \log t - t$. From the concavity of $y(t)$ for $t > 0$, for $x \geq e^2$, we get

$$\begin{aligned}y(Q) \geq y(x) + (Q - x) y'(x) &= (Q - x)\left(\frac{\rho}{x} - 1\right) \\ &= (x - Q)\left(1 - \frac{1}{\log x}\right) \geq \frac{1}{2}(x - Q)\end{aligned}$$

and so,

$$\operatorname{ben}(g(m)) \geq \frac{1}{2}(x - Q)$$

which, from (29) yields

$$x - Q \leq 4x^\alpha.$$

In conclusion, the only prime factors allowed in the denominator of $\frac{g(m)}{N}$ are the Q's, with $x - 4x^\alpha \leq Q \leq x$, and $\alpha_Q = 1$.

What about the numerator? Let $P > x$ be a prime number and suppose that P^t divides $g(m)$ with $t \geq 2$. Then, from (26) and (23),

$$\operatorname{ben}(Np^t) \geq \operatorname{ben}(Np^2) = P^2 - 2\rho \log P.$$

But the function $t \mapsto t^2 - 2\rho \log t$ is increasing for $t \geq \sqrt{\rho}$, so that,

$$\operatorname{ben}(NP^t) \geq x^2 - 2x > 2x^\alpha$$

for x large enough, which contradicts (29). The only possibility is that P divides $g(m)$ with exponent 1. In that case, from the convexity of the function $z(t) = t - \rho \log t$, inequality (26) yields

$$\begin{aligned}\operatorname{ben}(g(m)) \geq \operatorname{ben}(NP) = z(P) \geq z(x) + (P-x)z'(x) \\ = (P-x)\left(1 - \frac{1}{\log x}\right) \geq \frac{1}{2}(P-x)\end{aligned}$$

for $x \geq e^2$, which, with (29), implies

$$P - x \leq 4x^{\alpha}.$$

Up to now, we have shown that

$$g(m) = N\frac{P_{i_1} \dots P_{i_r}}{Q_{j_1} \dots Q_{j_s}}$$

with $P_{i_r} \leq x + 4x^{\alpha}, Q_{j_s} \geq x - 4x^{\alpha}$. It remains to show that $r = s$. First, since $n \leq m \leq n + 2x^{\alpha}$, and N belongs to G, we have from (18) and (19)

$$n \leq \ell(g(m)) \leq n + 2x^{\alpha}. \tag{32}$$

Further,

$$\ell(g(m)) - n = \sum_{t=1}^{r} P_{i_t} - \sum_{t=1}^{s} Q_{j_t}$$

and since $r \leq 4x^{\alpha}$, and $s \leq 4x^{\alpha}$,

$$\begin{aligned}\ell(g(m)) - n &\leq r(x + 4x^{\alpha}) - s(x - 4x^{\alpha}) \\ &\leq (r-s)x + 32x^{2\alpha}.\end{aligned}$$

From (32), $\ell(g(m)) - n \geq 0$ holds and as $\alpha < 1/2$, this implies that $r \geq s$ for x large enough. Similarly,

$$\ell(g(m)) - n \geq (r-s)x,$$

so, from (32), $(r-s)x$ must be $\leq 2x^{\alpha}$, which, for x large enough, implies $r \leq s$; finally $r = s$, and the proof of Proposition 4 is completed. $\square$

Lemma 1. *Let x be a positive real number, $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k$ be real numbers such that*

$$b_k \leq b_{k-1} \leq \dots \leq b_1 \leq x < a_1 \leq a_2 \leq \dots \leq a_k$$

and Δ be defined by $\Delta = \sum_{i=1}^{k}(a_i - b_i)$. Then the following inequalities

$$\frac{x + \Delta}{x} \leq \prod_{i=1}^{k} \frac{a_i}{b_i} \leq \exp\left(\frac{\Delta}{x}\right)$$

hold.

Proof. It is easy, and can be found in [16, p. 159]. □

Now it is time to prove Theorem 1. With the notation and hypothesis of Proposition 4, let us denote by B the set of integers M of the form

$$M = N \frac{P_{i_1} P_{i_2} \dots P_{i_r}}{Q_{j_1} Q_{j_2} \dots Q_{j_r}}$$

satisfying

$$\ell(M) - \ell(N) = \sum_{t=1}^{r} (P_{i_t} - Q_{j_t}) \leq 2x^\alpha.$$

From Proposition 4, for $n \leq m \leq 2x^\alpha$, $g(m) \in B$, and thus, from (18),

$$g(m) = \max_{\substack{\ell(M) \leq m \\ M \in B}} M. \tag{33}$$

Further, for $0 \leq d \leq 2x^\alpha$, define

$$B_d = \{M \in B : \ell(M) - \ell(N) = d\}.$$

I claim that, if $d < d'$ (which implies $d \leq d' - 2$), any element of B_d is smaller than any element of $B_{d'}$. Indeed, let $M \in B_d$, and $M' \in B_{d'}$. From Lemma 1, one has

$$\frac{M}{N} \leq \exp\left(\frac{d}{x}\right) \quad \text{and} \quad \frac{M'}{N} \geq \frac{x + d'}{x} \geq \frac{x + d + 2}{x}.$$

Since $d < 2x^\alpha < x$, and $e^t \leq \frac{1}{1-t}$ for $0 \leq t < 1$, one gets

$$\frac{M}{N} \leq \frac{1}{1 - d/x} = \frac{x}{x - d}.$$

This last quantity is smaller than $\frac{x+d+2}{x}$ if $(d+1)^2 < 2x + 1$, which is true for x large enough, because $d \leq 2x^\alpha$ and $\alpha < 1/2$.

From the preceding claim, and from (33), it follows that, if B_d is non empty, then

$$g(n + d) = \max B_d.$$

Further, since $N \in G$, we know that $n = \ell(N)$ belongs to the sequence (n_k) where g is increasing, and so, $n = n_{k_0}$. If $0 < d_1 < d_2 < \dots < d_s \leq 2x^\alpha$ denote the values of d for which B_d is non empty, then one has

$$n_{k_{0+i}} = n + d_i, 1 \leq i \leq s. \tag{34}$$

Suppose now that $\alpha < 1/2$ and x have been chosen in such a way that (12) and (13) hold. With the notation of Proposition 2, the set $E(x, \alpha)$ is certainly included in the set $\{d_1, d_2, \dots, d_s\}$, and from Proposition 2,

$$s \geq C_2 x^\alpha \tag{35}$$

which implies that for at least one i, $d_{i+1} - d_i \le \frac{2}{C_2}$, and thus

$$n_{k_0+i+1} - n_{k_0+i} \le \frac{2}{C_2}.$$

Finally, for $\frac{1}{6} < \alpha < \frac{1}{2}$, Proposition 1 allows us to choose x as wished, and thus, the proof of Theorem 1 is completed. $\square$

With ε very small, and α close to $1/2$, the values of C_1 and C_2 given in Proposition 2 yield that for infinitely many $k's$,

$$n_{k+1} - n_k \le 20{,}000.$$

To count how many such differences we get, we define

$$\gamma(n) = \mathrm{Card}\{m \le n : g(m) > g(m-1)\}.$$

Therefore, with the notation (3), we have $n_{\gamma(n)} = n$.

In [16, 162–164], it is proved that

$$n^{1-\tau/2} \ll \gamma(n) \le n - c\frac{n^{3/4}}{\sqrt{\log n}}$$

where τ is such that the sequence of consecutive primes satisfies $p_{i+1} - p_i \ll p_i^{\tau}$. Without any hypothesis, the best known τ is $> 1/2$.

Proposition 5. *We have $\gamma(n) \ge n^{3/4-\varepsilon}$ for all $\varepsilon > 0$, and n large enough.*

Proof. With the definition of $\gamma(n)$, (34) and (35) give

$$\gamma(n + 2x^{\alpha}) - \gamma(n) \ge s \gg x^{\alpha} \tag{36}$$

whenever $n = \ell(N), N = N_\rho, \rho = x/\log x$, and x satisfies Proposition 1. But, from (21), two close enough distinct values of x can yield the same N.

I now claim that, with the notation of Proposition 1, the number of primes p_i between ξ and ξ' such that there is at least one $x \in [p_i, p_{i+1})$ satisfying (8), (9) and (10) is bigger than $\frac{1}{2}(\pi(\xi') - \pi(\xi))$. Indeed, for each i for which $[p_i, p_{i+1})$ does not contain any such x, we get a measure $p_{i+1} - p_i \ge 2$, and if there are more than $\frac{1}{2}(\pi(\xi') - \pi(\xi))$ such $i's$, the total measure will be greater than $\pi(\xi') - \pi(\xi) \sim \xi/\log^2 \xi$, which contradicts Proposition 1.

From the above claim, there will be at least $\frac{1}{2}(\pi(\xi') - \pi(\xi))$ distinct N's, with $N = N_\rho, \rho = x/\log x$, and $\xi \le x \le \xi'$. Moreover, for two such distinct N, say $N' < N''$, we have from (21), $\ell(N'') - \ell(N') \ge \xi$.

Let $N^{(1)}$ and $N^{(0)}$ the biggest and the smallest of these N's, and $n^{(1)} = \ell(N^{(1)}), n^{(0)} = \ell(N^{(0)})$, then from (36),

$$\gamma(n^{(1)}) \ge \gamma(n^{(1)}) - \gamma(n^{(0)}) \ge \frac{1}{2}\left(\pi(\xi') - \pi(\xi)\right)\xi^{\alpha} \gg \frac{\xi^{1+\alpha}}{\log^2 \xi}. \tag{37}$$

But from (21) and (22), $x \sim \log N_\rho$, and from (2),

$$x \sim \log N_\rho \sim \sqrt{n \log n} \quad \text{with} \quad n = \ell(N_p)$$

so

$$\xi \sim \sqrt{n^{(1)} \log n^{(1)}}$$

and since α can be choosen in (37) as close as wished of 1/2, this completes the proof of Proposition 5. $\square$

References

1. L. Alaoglu, P. Erdős, "On highly composite and similar numbers", Trans. Amer. Math. Soc. 56, 1944, 448–469.
2. M. Deléglise, J.-L. Nicolas, P. Zimmermann, "Landau's function for one million billions", J. de Théorie des Nombres de Bordeaux, 20, 2008, 625–671.
3. M. Deléglise, J.-L. Nicolas, "Le plus grand facteur premier de la fonction de Landau", Ramanujan J., 27, 2012, 109–145.
4. P. Erdős, "On highly composited numbers", J. London Math. Soc., 19, 1944, 130–133.
5. P. Erdős, "Ramanujan and I", Number Theory, Madras 1987, Editor : K. Alladi, Lecture Notes in Mathematics n^o 1395, Springer-Verlag, 1989.
6. P. Erdős, J.-L. Nicolas, "Répartition des nombres superabondants", Bull. Soc. Math. France, 103, 1975, 65–90.
7. P. Erdős, P. Turan, "On some problems of a statistical group theory", I to VII, Zeitschr. fur Wahrschenlichkeitstheorie und verw. Gebiete, 4, 1965, 175–186; Acta Math. Hung., 18, 1967, 151–163; Acta Math. Hung., 18, 1967, 309–320; Acta Math. Hung., 19, 1968, 413–435; Periodica Math. Hung., 1, 1971, 5–13; J. Indian Math. Soc., 34, 1970, 175–192; Periodica Math. Hung., 2, 1972, 149–163.
8. J. Grantham, "The largest prime dividing the maximal order of an element of S_n", Math. Comp., 64, 1995, 407–410.
9. E. Landau, "Uber die Maximalordung der Permutation gegebenen Grades", Handbuch der Lehre von der Verteilung der Primzahlen, vol. 1, 2nd edition, Chelsea, New-York, 1953, 222–229.
10. J. P. Massias, "Majoration explicite de l'ordre maximum d'un élément du groupe symétrique", Ann. Fac. Sci. Toulouse Math., 6, 1984, 269–281.
11. J. P. Massias, J.-L. Nicolas, G. Robin, "Evaluation asymptotique de l'ordre maximum d'un élément du groupe symétrique", Acta Arithmetica, 50, 1988, 221–242.
12. J. P. Massias, J.-L. Nicolas, G. Robin, "Effective bounds for the Maximal Order of an Element in the Symmetric Group", Math. Comp., 53, 1989, 665–678.
13. W. Miller, "The Maximum Order of an Element of a Finite Symmetric Group", Amer. Math. Monthly, 94, 1987, 497–506.
14. H. L. Montgomery, R. C. Vaughan, "The large sieve", Mathematika, 20, 1973, 119–134.
15. J.-L. Nicolas, "Sur l'ordre maximum d'un élément dans le groupe S_n des permutations", Acta Arithmetica, 14, 1968, 315–332.
16. J.-L. Nicolas, "Ordre maximum d'un élément du groupe de permutations et highly composite numbers", Bull. Soc. Math. France, 97, 1969, 129–191.
17. J.-L. Nicolas, "Ordre maximal d'un élément d'un groupe de permutations", C.R. Acad. Sci. Paris, 270, 1970, 1473–1476.
18. J.-L. Nicolas, "Répartition des nombres largement composés", Acta Arithmetica, 34, 1979, 379–390.

19. S. Ramanujan, “Highly composite numbers”, Proc. London Math. Soc., Series 2, 14, 1915, 347–400; and “Collected papers”, Cambridge at the University Press, 1927, 78–128.
20. S. Ramanujan, “Highly composite numbers, annotated and with a foreword by J.-L. Nicolas and G. Robin”, Ramanujan J., 1, 1997, 119–153.
21. A. Selberg, “On the normal density of primes in small intervals and the difference between consecutive primes”, Arch. Math. Naturvid, 47, 1943, 87–105.
22. S. Shah, “An Inequality for the Arithmetical Function $g(x)$”, J. Indian Math. Soc., 3, 1939, 316–318.
23. H. Siebert, “Montgomery’s weighted sieve for dimension two”, Monatsch., Math., 82, 1976, 327–336.
24. N. J. A. Sloane, “The On-Line Encyclopedia of Integer Sequences”, http://oeis.org. Accessed 12 December 2012.
25. M. Szalay, “On the maximal order in S_n and S_n^*”, Acta Arithmetica, 37, 1980, 321–331.
26. http://math.univ-lyon1.fr/~nicolas/landaug.html.