```
> read "hdenHRAust.mpl":
> const();
```

$$\text{"x0=", } 10000000019$$
$$\text{"n0=", } 2220822442581729257$$
$$\text{"L0=", } 42.244409270801490066$$
$$\text{"lambda\_0=", } 3.7434720200960200353$$
$$\text{"nu0=", } 0.08861461397410129538$$
$$\text{"c=", } 0.04611764442150902382 \tag{1}$$

```
> lemma2_1();
```

"lis=", $[[4, 2], [8, 2], [9, 3], [16, 2], [25, 5], [27, 3], [32, 2], [49, 7], [64, 2], [81, 3], [121, 11]]$

"lisf=", $\left[2, 2\sqrt{2} - \ln(2), 3 - 2\ln(2), 4 - \ln(12), 5 - \ln(24), 3\sqrt{3} - \ln(120), 4\sqrt{2} \right.$
$\left. - \ln(360), 7 - \ln(720), 8 - \ln(5040), 9 - \ln(10080), 11 - \ln(30240) \right]$

"lisf=", $[2., 2.1352, 1.6137, 1.5151, 1.8219, 0.4088, -0.2293, 0.4207, -0.5252, -0.2183, 0.683]$

$$\text{"maxi=", } 2\sqrt{2} - \ln(2), \text{ "=", } 2.1352 \tag{2}$$

```
> liVal();
```

"Values of li = ", $[2., 7.3890560989306502272], [1.0451637801174927848,$
$4.9542343560018901634]$

$$\text{"values of li\textasciicircum(-1) = ", } [0., 1.], [1.4513692348833810503, 1.9690474892247508501] \tag{3}$$

```
> formula2_7(3);
```

$$\text{" N=", } 1, \text{ " f= ", } \mathrm{Li}(t^s), \text{ " f ' = ", } \frac{t^{-1+s}}{\ln(t)}$$

$$\text{" N=", } 2, \text{ " f= ", } s\,\mathrm{Li}(t^s) - \frac{t^s}{\ln(t)}, \text{ " f ' = ", } \frac{t^{-1+s}}{\ln(t)^2}$$

$$\text{" N=", } 3, \text{ " f= ", } \frac{1}{2}\,s^2\,\mathrm{Li}(t^s) - \frac{1}{2}\,\frac{s\,t^s}{\ln(t)} - \frac{1}{2}\,\frac{t^s}{\ln(t)^2}, \text{ " f ' = ", } \frac{t^{-1+s}}{\ln(t)^3} \tag{4}$$

```
> lemma2_2();
```

$$\text{"f=", } \mathrm{Li}(t) - \frac{t}{\ln(t)}, \text{ "f ' =", } \frac{1}{\ln(t)^2}, \text{ "t0=", } 3.8464677170468563268$$

$$\text{"f=", } t - \mathrm{Li}(t), \text{ "f ' =", } 1 - \frac{1}{\ln(t)}, \text{ "min=", } 0.82316401210310843799$$

$$\text{"f=", } \mathrm{Li}(t) - \frac{1.49\,t}{\ln(t)}, \text{ "f ' =", } -\frac{0.49\,\ln(t) - 1.49}{\ln(t)^2}, \text{ "max=", } -0.041069019144568969$$

$$\text{"f=", } \mathrm{Li}(t) - \frac{t}{\ln(t)} - \frac{1.101\,t}{\ln(t)^2}, \text{ "f ' =", } -\frac{0.101\,\ln(t) - 2.202}{\ln(t)^3}$$

$$\text{"t1=", } 2.9408961889482060139\ 10^9, \text{ "f(10\textasciicircum10)=", } -5015.157607445647 \tag{5}$$

```
> lemma2_3();
```

$$\text{"f=", } \mathrm{Li}(t) - \frac{t}{\ln(t)} - \frac{t}{\ln(t)^2} - \frac{2\,t}{\ln(t)^3} - \frac{u\,t}{\ln(t)^4}, \text{ "f ' =", } -\frac{u\,\ln(t) - 6\,\ln(t) - 4\,u}{\ln(t)^5}$$

$$\text{"u=6", "t0=", } 76.541674567754891281$$

$$\text{"u=7", "exp(28)=", } 1.4462570642914751737\ 10^{12}, \text{ "f(4.96\ 10\textasciicircum12)=", } -259.076627445622$$

$$(6)$$

"u=40/3", "exp(80/11)=", 1440.4736653116977523, "f(exp(80/11))=",
     -0.0033583591209519827

(6)

```
> lemma2_4();
```

$$\text{"f=", } \mathrm{Li}(t\,(\ln(t) + \ln(\ln(t)))) - t, \text{ "f(3.28)=", } 0.0073000362527432097$$

$$\text{"f=", } \mathrm{Li}(t\,\ln(t)) - t, \text{ "f(41)=", } -0.048032352402360906$$

$$\text{"num=", } (L-1)\,(\ln(L) - 2) - 1, \text{ "subs(L=log(4678),num)=", } 0.000175865786473378$$

$$\text{"f=", } t\,(\ln(t) + \ln(\ln(t)) - 1), \text{ "t0=", } 3.1973744820798380856$$

$$\text{"y(12218)=", } 0.001065641590597$$

(7)

```
> lemma2_5();
```

$$\text{"The derivative of li}^\wedge\text{(-1) (t) is log u with u=li}^\wedge\text{(-1) (t)"}$$

$$\text{"df1/dt=", } \frac{1}{2}\,\frac{\ln(u)}{\sqrt{u}}, \text{ " d2f1/dt2=", } -\frac{1}{4}\,\frac{(-2+\ln(u))\,\ln(u)}{u^{3/2}}$$

$$\text{"d2f2/dt2=", } -\frac{1}{16}\,\frac{3\,\ln(t)^2 + 2\,\ln(t) + 3}{(t\,\ln(t))^{7/4}}, \text{ " B=", } \frac{1}{4}\,\frac{3 + \dfrac{8}{L} + \dfrac{19}{L\,(L-2)}}{u^{1/4}\left(1 - \dfrac{\ln(L)}{L}\right)^{7/4}}$$

"Li(103)=", 30.775491300599445293, "subs(u=103,L=log(103),B)=",
     0.99588426164105960428

(8)

```
> pirdex(x);
```

$$\text{"i=", } 1, \text{ "f[i]=", } \mathrm{Li}(t^s), \text{ "f[i]=", } \frac{t^{-1+s}}{\ln(t)}$$

$$\text{"i=", } 2, \text{ "f[i]=", } s\,\mathrm{Li}(t^s) - \frac{t^s}{\ln(t)}, \text{ "f[i]=", } \frac{t^{-1+s}}{\ln(t)^2}$$

$$\text{"i=", } 3, \text{ "f[i]=", } \frac{1}{2}\,s^2\,\mathrm{Li}(t^s) - \frac{1}{2}\,\frac{s\,t^s}{\ln(t)} - \frac{1}{2}\,\frac{t^s}{\ln(t)^2}, \text{ "f[i]=", } \frac{t^{-1+s}}{\ln(t)^3}$$

$$\text{"i=", } 4, \text{ "f[i]=", } \frac{1}{6}\,s^3\,\mathrm{Li}(t^s) - \frac{1}{6}\,\frac{s^2\,t^s}{\ln(t)} - \frac{1}{6}\,\frac{s\,t^s}{\ln(t)^2} - \frac{1}{3}\,\frac{t^s}{\ln(t)^3}, \text{ "f[i]=", } \frac{t^{-1+s}}{\ln(t)^4}$$

$$\text{"i=", } 5, \text{ "f[i]=", } \frac{1}{24}\,s^4\,\mathrm{Li}(t^s) - \frac{1}{24}\,\frac{s^3\,t^s}{\ln(t)} - \frac{1}{24}\,\frac{s^2\,t^s}{\ln(t)^2} - \frac{1}{12}\,\frac{s\,t^s}{\ln(t)^3} - \frac{1}{4}\,\frac{t^s}{\ln(t)^4},$$

$$\text{"f[i]=", } \frac{t^{-1+s}}{\ln(t)^5}$$

$$\text{"################## upper bound ##################"}$$

$$\text{"eq. (2.26) f=", } \frac{1}{24}\,(3\,\alpha\,s^4 - 4\,\alpha\,s^3 + 24)\,\mathrm{Li}(t^s) - \frac{1}{24}\,\frac{(3\,\alpha\,s^3 - 4\,\alpha\,s^2 + 24)\,t^s}{L}$$

$$-\frac{1}{24}\,\frac{\alpha\,s\,(3\,s-4)\,t^s}{L^2} - \frac{1}{12}\,\frac{\alpha\,(3\,s-4)\,t^s}{L^3} + \frac{1}{4}\,\frac{\alpha\,t^s}{L^4}$$

$$\text{"eq. (2.27)", } \frac{x^s}{s\,L} + \frac{x^s}{s^2\,L^2} + \frac{2\,x^s}{s^3\,L^3} + \frac{1}{24}\,\frac{x^s\,(51\,\alpha\,s^4 - 28\,\alpha\,s^3 + 168)}{s^4\,L^4}$$

"eq. (2.19)", $\dfrac{x^{r+1}}{(r+1)\,L} + \dfrac{x^{r+1}}{(r+1)^2\,L^2} + \dfrac{2\,x^{r+1}}{(r+1)^3\,L^3}$

$\qquad + \dfrac{1}{24}\ \dfrac{x^{r+1}\left(51\,\alpha\,r^4 + 176\,\alpha\,r^3 + 222\,\alpha\,r^2 + 120\,\alpha\,r + 23\,\alpha + 168\right)}{(r+1)^4\,L^4}$

"eq. (2.20) C0=", $pix1 - \dfrac{x1^r\,thx1}{\ln(x1)} + \dfrac{1}{24}\ \left(-3\,\alpha\,r^4 - 8\,\alpha\,r^3 - 6\,\alpha\,r^2 + \alpha - 24\right)\mathrm{Li}\!\left(x1^{r+1}\right)$

$\qquad + \dfrac{1}{24}\ \dfrac{\left(3\,\alpha\,r^3 + 5\,\alpha\,r^2 + \alpha\,r - \alpha + 24\right)x1^{r+1}}{L1} + \dfrac{1}{24}\ \dfrac{\alpha\left(3\,r^2 + 2\,r - 1\right)x1^{r+1}}{L1^2}$

$\qquad + \dfrac{1}{12}\ \dfrac{\alpha\left(3\,r - 1\right)x1^{r+1}}{L1^3} - \dfrac{1}{4}\ \dfrac{\alpha\,x1^{r+1}}{L1^4}$

"############## lower bound ######################"

"eq. (2.28) fh=", $\dfrac{1}{24}\ \left(-3\,\alpha\,s^4 + 4\,\alpha\,s^3 + 24\right)\mathrm{Li}\!\left(t^s\right) - \dfrac{1}{24}\ \dfrac{\left(-3\,\alpha\,s^3 + 4\,\alpha\,s^2 + 24\right)t^s}{L}$

$\qquad + \dfrac{1}{24}\ \dfrac{\alpha\,s\left(3\,s - 4\right)t^s}{L^2} + \dfrac{1}{12}\ \dfrac{\alpha\left(3\,s - 4\right)t^s}{L^3} - \dfrac{1}{4}\ \dfrac{\alpha\,t^s}{L^4}$

"eq. (2.23) C0h=", $pix1 - \dfrac{x1^r\,thx1}{\ln(x1)} + \dfrac{1}{24}\ \left(3\,\alpha\,r^4 + 8\,\alpha\,r^3 + 6\,\alpha\,r^2 - \alpha - 24\right)\mathrm{Li}\!\left(x1^{r+1}\right)$

$\qquad + \dfrac{1}{24}\ \dfrac{\left(-3\,\alpha\,r^3 - 5\,\alpha\,r^2 - \alpha\,r + \alpha + 24\right)x1^{r+1}}{L1} - \dfrac{1}{24}\ \dfrac{\alpha\left(3\,r^2 + 2\,r - 1\right)x1^{r+1}}{L1^2}$

$\qquad - \dfrac{1}{12}\ \dfrac{\alpha\left(3\,r - 1\right)x1^{r+1}}{L1^3} + \dfrac{1}{4}\ \dfrac{\alpha\,x1^{r+1}}{L1^4}$

" ################# lower bound for r smaller than r0 #############"

"eq. (2.21) in s=", $\dfrac{x^s}{s\,L} + \dfrac{x^s}{s^2\,L^2} + \dfrac{2\,x^s}{s^3\,L^3} - \dfrac{x^s\left(2\,\alpha\,s^4 - \alpha\,s^3 - 6\right)}{s^4\,L^4}$

"eq. (2.21) in r", $\dfrac{x^{r+1}}{(r+1)\,L} + \dfrac{x^{r+1}}{(r+1)^2\,L^2} + \dfrac{2\,x^{r+1}}{(r+1)^3\,L^3}$

$\qquad - \dfrac{x^{r+1}\left(2\,\alpha\,r^4 + 7\,\alpha\,r^3 + 9\,\alpha\,r^2 + 5\,\alpha\,r + \alpha - 6\right)}{(r+1)^4\,L^4}$

" ############# lower bound for r greater than r0 ################"

"eq. (2.22) in s", $\dfrac{x^s}{s\,L} + \dfrac{x^s}{s^2\,L^2} + \dfrac{2\,x^s}{s^3\,L^3} - \dfrac{1}{24}\ \dfrac{x^s\left(51\,\alpha\,s^4 - 28\,\alpha\,s^3 - 168\right)}{s^4\,L^4}$

"eq. (2.22) in r", $\dfrac{x^{r+1}}{(r+1)\,L} + \dfrac{x^{r+1}}{(r+1)^2\,L^2} + \dfrac{2\,x^{r+1}}{(r+1)^3\,L^3}$

$\qquad - \dfrac{1}{24}\ \dfrac{x^{r+1}\left(51\,\alpha\,r^4 + 176\,\alpha\,r^3 + 222\,\alpha\,r^2 + 120\,\alpha\,r + 23\,\alpha - 168\right)}{(r+1)^4\,L^4}$

" ############### Corollary 2.7: r=1 ####################"

$$\text{"eq. (2.30) with alpha=1",} \quad \frac{1}{2}\frac{x^2}{L} + \frac{1}{4}\frac{x^2}{L^2} + \frac{1}{4}\frac{x^2}{L^3} + \frac{95}{48}\frac{x^2}{L^4}$$

$$\text{"eq. (2.30) with alpha=0.5",} \quad \frac{1}{2}\frac{x^2}{L} + \frac{1}{4}\frac{x^2}{L^2} + \frac{1}{4}\frac{x^2}{L^3} + \frac{29}{24}\frac{x^2}{L^4}$$

$$\text{"eq. (2.30) with alpha=0.15",} \quad \frac{1}{2}\frac{x^2}{L} + \frac{1}{4}\frac{x^2}{L^2} + \frac{1}{4}\frac{x^2}{L^3} + \frac{107}{160}\frac{x^2}{L^4}$$

$$\text{"eq. (2.31) with alpha=1",} \quad \frac{1}{2}\frac{x^2}{L} + \frac{1}{4}\frac{x^2}{L^2} + \frac{1}{4}\frac{x^2}{L^3} - \frac{9}{8}\frac{x^2}{L^4}$$

$$\text{"eq. (2.31) with alpha=0.5",} \quad \frac{1}{2}\frac{x^2}{L} + \frac{1}{4}\frac{x^2}{L^2} + \frac{1}{4}\frac{x^2}{L^3} - \frac{3}{8}\frac{x^2}{L^4}$$

$$\text{"eq. (2.31) with alpha=0.15",} \quad \frac{1}{2}\frac{x^2}{L} + \frac{1}{4}\frac{x^2}{L^2} + \frac{1}{4}\frac{x^2}{L^3} + \frac{3}{20}\frac{x^2}{L^4}$$

$$\text{"f(x1)=with alpha=1",} \quad -2.1454187882181860682\ 10^{14}$$

$$\text{"f(x2)= with alpha=0.5",} \quad -1.4011940545985513337\ 10^{16}$$

$$\text{"f(x3)= with alpha=0.15",} \quad -7.4854212583894368509\ 10^{18}$$

$$\text{"C0= with alpha=1",} \quad -6.23669119269738\ 10^{9}$$

$$\text{"C0= with alpha=0.5",} \quad -1.84046284412225\ 10^{11}$$

$$\text{"C0= with alpha=0.15",} \quad -1.58654448223331\ 10^{13}$$

$$\text{"C0h= with alpha=1",} \quad 7.0273359600975794\ 10^{10}$$

$$\text{"C0h= with alpha=0.5",} \quad 1.5896391695668743\ 10^{12}$$

$$\text{"C0h= with alpha=0.15",} \quad 1.655065313609409\ 10^{14}$$

$$\text{"r0(1)=",}\ 1.1445015218697486108,\ \text{"r0(1/2)=",}\ 1.4377710581235880920,\ \text{"r0(0.15)=",} \tag{9}$$
$$2.1086611695777568369$$

```
> lemma2_8();
```

$$\text{"\#\#\#\#\#\#\#\#\#\#\#\#\#\#\# upper bound \#"}$$

$$\text{"A=",}\ L\left(1 + \frac{1}{2}\frac{\lambda}{L}\right)^2\left(L + \lambda + \frac{\lambda}{L} + 1\right) - \left(L + \lambda + \frac{\lambda}{L}\right)^2$$

$$\text{"A ordered in lambda =",}\ \left(\frac{1}{4L} + \frac{1}{4L^2}\right)\lambda^3 + \left(\frac{1}{4} - \frac{3}{4L} - \frac{1}{L^2}\right)\lambda^2 + L$$

$$\text{"\#\#\#\#\#\#\#\#\#\#\#\#\#\#\# lower bound \#"}$$

$$\text{"f(n)=",}\ \sqrt{nL}\left(1 + \frac{b\lambda}{L}\right)$$

$$\text{"1+1/logx0+107/40/logx0^2=",}\ 1.0484748110809695873$$

$$\text{"anum=",}\ 1.049,\ \text{"bnum=",}\ 0.365$$

$$\text{"2b/(1+b*la0/L0)=",}\ 0.70712840269136439096,\ \text{"c0num=",}\ 0.7$$

$$\text{"A1=",}\ L\left(1 + \frac{b\lambda}{L}\right)^2\left(L + \lambda + \frac{c0\lambda}{L} + a\right) - \left(L + \lambda + \frac{c0\lambda}{L}\right)^2$$

"A2=expand(A1/(lambda*L))", $-\dfrac{\lambda\,c0^2}{L^3} + \left(-\dfrac{1}{L} + \dfrac{(2\,b-2)\,\lambda}{L^2} + \dfrac{\lambda^2\,b^2}{L^3}\right)c0 + \dfrac{a}{\lambda} + 2\,b - 1$

$+ \dfrac{(b^2+2\,b-1)\,\lambda + 2\,b\,a}{L} + \dfrac{a\,b^2\,\lambda + b^2\,\lambda^2}{L^2}$

"c0(2-2*b-b^2*lambda0/L0)=", 0.88073602263731024843, "dnum=", 0.88

"B=", $\dfrac{a}{\lambda} + 2\,b - 1 + \dfrac{-c0 + (b^2+2\,b-1)\,\lambda + 2\,b\,a}{L} + \dfrac{b^2\,\lambda^2 + (a\,b^2-d)\,\lambda}{L^2}$

"B=", $\dfrac{a}{\lambda} + 2\,b - 1 - \dfrac{c1}{L} + \dfrac{(b^2+2\,b-1)\,\lambda + 2\,b\,a - c2}{L} + \dfrac{b^2\,\lambda^2 + (a\,b^2-d)\,\lambda - c3\,L}{L^2}$

"a/lambda0-c1/L0=", 0.26980556136906525897

"********** case 1, lambda0 ! = lambda ! =4.3, c2=0.26, c3=0"

"(b^2+2b-1)*lambda0+2ab-c2=", −0.00624338554863314033

"4.3*b^2*+(ab^2-d)=", −0.167379475

"********** case 2, lambda O 4.3, c2=0.18, c3=0.08"

"4.3*(b^2+2b-1)+2ab-c2=", −0.0023625

"4b^2/e^2-c3=", −0.007879827563209096492

(10)

```
> deltak(10,30,4);
```

"m=", 2, "gamma=", −0.0096903631928723184845303860 3521, "delta=",
0.18754623284036522459720338 4605, "sum 1/rho^m =",
0.04615431729580460275710799 0385

"m=", 3, "gamma=", 0.00205383442030334586616004654275, "delta=",
0.05168863203319289380200822 30828, "sum 1/rho^m =",
0.00011115823145210592276266 82372

"m=", 4, "gamma=", 0.002325370065467300057468170 17753, "delta=",
0.01475165882545374406458023 68139, "sum 1/rho^m =",
−0.00007362722126168951832677 13039

"m=", 5, "gamma=", 0.000793323817301062701753334877444, "delta=",
0.00452447788849537874124611609894, "sum 1/rho^m =",

−7.15093355762607735801088 94 10^{-7}

"m=", 6, "gamma=", −0.000238769345430199609872421841908, "delta=",
0.00144679520452518314021698042219, "sum 1/rho^m =",

2.81436416938766261606717 81 10^{-7}

"m=", 7, "gamma=", −0.000527289567057751046074097505479, "delta=",
0.00047154407818540505033952 0328858, "sum 1/rho^m =",

4.57419114970477211116114 2 10^{-9}

"m=", 8, "gamma=", −0.00035212335380303950960205 2165001, "delta=",
0.00015518029416423025374796 8308257, "sum 1/rho^m =",

−1.26886811095076071900825 7 10^{-9}

"m=", 9, "gamma=", −0.000034394774418088048177914 6237982, "delta=",
0.00005134521211814414337677 14358418, "sum 1/rho^m =",

$$-2.82743715505588708958418 \ 10^{-11}$$

"m=", 10, "gamma=", 0.000205332814909064794683722289237, "delta=",

0.0000170413570471106410320277049357, "sum 1/rho^m =",

$$5.9977148471518746050643 \ 10^{-12}$$

(11)

```
> calculcn(10,4);
```

"n=", 0, "c_n=", 1, "=", 1., "c_n/2^n=", 1.

"n=", 1, "c_n=", 1, "=", 1., "c_n/2^n=", 0.5000000

"n=", 2, "c_n=", 2, "=", 2., "c_n/2^n=", 0.5000000

"n=", 3, "c_n=", 3, "=", 3., "c_n/2^n=", 0.3750000

"n=", 4, "c_n=", $\dfrac{11}{2}$, "=", 5.500000, "c_n/2^n=", 0.3437500

"n=", 5, "c_n=", $\dfrac{19}{2}$, "=", 9.500000, "c_n/2^n=", 0.2968750

"n=", 6, "c_n=", $\dfrac{35}{2}$, "=", 17.50000, "c_n/2^n=", 0.2734375

"n=", 7, "c_n=", 32, "=", 32., "c_n/2^n=", 0.2500000

"n=", 8, "c_n=", $\dfrac{479}{8}$, "=", 59.87500, "c_n/2^n=", 0.2338867

"n=", 9, "c_n=", $\dfrac{899}{8}$, "=", 112.3750, "c_n/2^n=", 0.2194824

"n=", 10, "c_n=", 213, "=", 213., "c_n/2^n=", 0.2080078

(12)

```
> somrho(40,40,-2);
```

"n=", 0, 0.04615431729580460275710799037900773035300, "a=", 0.04615432

"n=", 1, 0.0462654755272567086798706586179188200360, "a=", 0.0001111582

"n=", 2, 0.0461182210847333296432171160039306789032, "a=", -0.0001472544

"n=", 3, 0.0461160758046660418200097127221912816254G, "a=", -0.000002145280

"n=", 4, 0.0461176237049592050344854965821390030425, "a=", 0.000001547900

"n=", 5, 0.0461176671597751272297836057037935042529Z, "a=", $4.345482 \ 10^{-8}$

"n=", 6, 0.0461176449545831855914710229842109732761Z, "a=", $-2.220519 \ 10^{-8}$

"n=", 7, 0.0461176440498032959735871543972039722974З, "a=", $-9.047799 \ 10^{-10}$

"n=", 8, 0.0461176444089164724468056457663640463403H, "a=", $3.591132 \ 10^{-10}$

"n=", 9, 0.046117644427915587390218535951681680684ZO, "a=", $1.899911 \ 10^{-11}$

"n=", 10, 0.04611764442179436116312131756237723086908, "a=", $-6.121226 \ 10^{-12}$

"n=", 11, 0.046117644421395757273032484394864930678G, "a=", $-3.986039 \ 10^{-13}$

"n=", 12, 0.04611764442150272773412605170438237686815, "a=", $1.069705 \ 10^{-13}$

"n=", 13, 0.04611764442151105218752489619187533428099, "a=", $8.324453 \ 10^{-15}$

"n=", 14, 0.04611764442150916013398881391628191160451, "a=", $-1.892054 \ 10^{-15}$

"n=", 15, 0.0461176444215089872761215559780541789304, "a=", $-1.728579 \ 10^{-16}$

"n=", 16, 0.046117644421509020918596416365577765091343, "a=", $3.364247\ 10^{-17}$

"n=", 17, 0.046117644421509024487266834755701253234, "a=", $3.568670\ 10^{-18}$

"n=", 18, 0.046117644421509023888417775674146882975, "a=", $-5.988247\ 10^{-19}$

"n=", 19, 0.046117644421509023815171356739334086234 97, "a=", $-7.327042\ 10^{-20}$

"n=", 20, 0.046117644421509023825808880137706445749 18, "a=", $1.063752\ 10^{-20}$

"n=", 21, 0.046117644421509023827305511231552535230 39, "a=", $1.496631\ 10^{-21}$

"n=", 22, 0.046117644421509023827117416967852132180 47, "a=", $-1.880943\ 10^{-22}$

"n=", 23, 0.046117644421509023827086992960326220021 12, "a=", $-3.042401\ 10^{-23}$

"n=", 24, 0.046117644421509023827090294992472017526 90, "a=", $3.302032\ 10^{-24}$

"n=", 25, 0.046117644421509023827090910685324726896 85, "a=", $6.156929\ 10^{-25}$

"n=", 26, 0.046117644421509023827090853300311172758 44, "a=", $-5.738501\ 10^{-26}$

"n=", 27, 0.046117644421509023827090840893386831525 73, "a=", $-1.240692\ 10^{-26}$

"n=", 28, 0.046117644421509023827090841877033769691 12, "a=", $9.836469\ 10^{-28}$

"n=", 29, 0.046117644421509023827090842126032970705 49, "a=", $2.489992\ 10^{-28}$

"n=", 30, 0.046117644421509023827090842109215980412 36, "a=", $-1.681699\ 10^{-29}$

"n=", 31, 0.046117644421509023827090842104307335298 79, "a=", $-4.908645\ 10^{-30}$

"n=", 32, 0.046117644421509023827090842103735106108 42, "a=", $-5.722292\ 10^{-31}$

"n=", 33, 0.046117644421509023827090842103944155508 61, "a=", $2.090494\ 10^{-31}$

"n=", 34, 0.046117644421509023827090842099532924861 38, "a=", $-4.411231\ 10^{-30}$

"n=", 35, 0.046117644421509023827090842096812073221 51, "a=", $-2.720852\ 10^{-30}$

"n=", 36, 0.046117644421509023827090842084882349127 14, "a=", $-1.192972\ 10^{-29}$

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"n=", 39, 0.046117644421509023827090842004925169012 41, "a=", $-9.533853\ 10^{-31}$

"n=", 40, 0.046117644421509023827090841789286868788 71, "a=", $-2.156383\ 10^{-28}$

0.046117644421509023827090841789286868788 71

(13)

`> lemma5_5();`

"(1+7.5*10^(-7))*1+1.101/log(x0))=", 1.0478166083154692983

"k0=", 3, "S=", 33.219280951614744053

"k0=", 4, "S=", 5.6634236243991674240

"k0=", 5, "S=", 2.4526975733267236630

"k0=", 6, "S=", 1.6670967531186623334

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"k0=", 14, "S=", 1.0892500059711039555
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"k0=", 30, "S=", 1.0662274695852914633
"k0=", 31, "S=", 1.0664312214354453824
"k0=", 32, "S=", 1.0666799822470031602
"k0=", 33, "S=", 1.0669675345510592661

(14)

> **lemma5_6();**

$$"a=", 0, "delta\_a=", -2.8853900817779268147$$
$$"a=", 1, "delta\_a=", -3.1439210137763880403$$
$$"a=", \frac{1}{2}, "delta\_a=", -3.0736770371330802349$$
$$"a=", \frac{1}{3}, "delta\_a=", -3.0254995261715077034$$

(15)

> **lemma5_7();**

"Estimate of the constant in formula (5.13)"

$$"2^\wedge(3/2)*(2-1/\log(2))/\log(2)-\mathrm{li}(2^\wedge(5/2))=", -1.7534079067235937292$$

(16)

> **prop5_9();**

$$"(1+0.365\ \mathrm{nu})^\wedge 3(1+\mathrm{nu})(1-0.37\ \mathrm{nu})^\wedge 2-(1+1.018\ \mathrm{nu})^\wedge 2(1-0.3405)^\wedge 2=", 0.31552675\ v^2$$
$$+ 0.09873042000\ v^3 - 0.19864710364100\ v^4 + 0.0253884884125\ v^5$$
$$+ 0.0066570534125\ v^6$$

$$"\text{roots of } P=", -7.7889904047479667286, -0.99893233674347343633, 0., 0.,$$
$$2.1807015063260369535, 2.7934485583790277948$$

ν
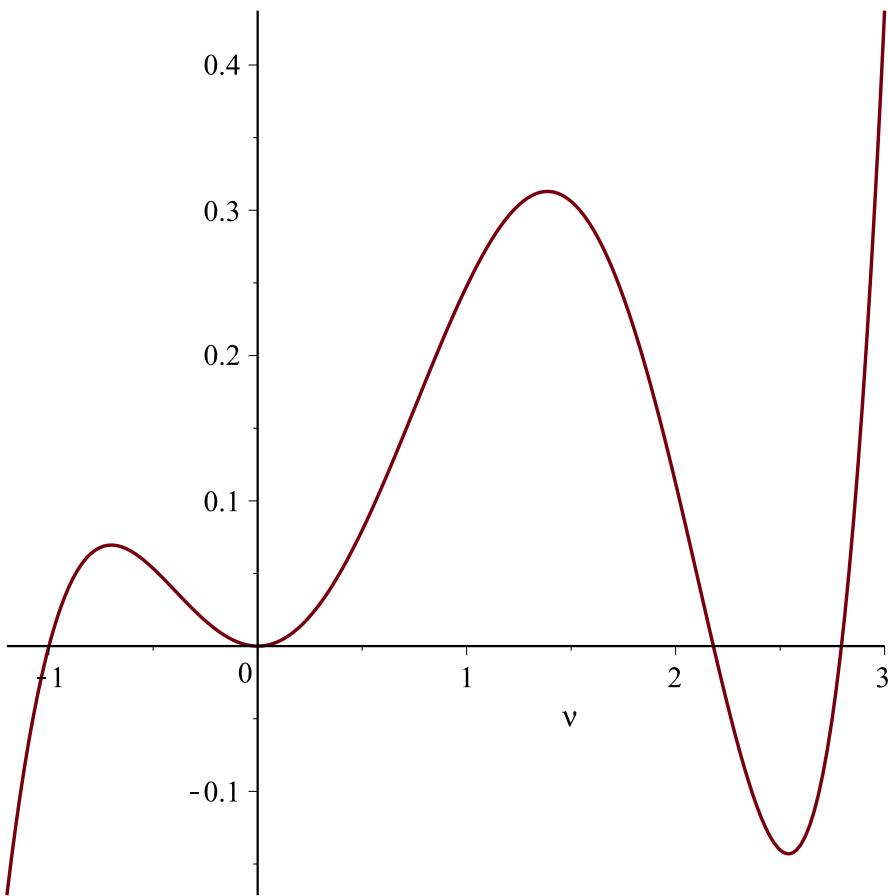
```
> bninterval(2,100);
```

"b_n is defined in (1.12) and mu_n by b_n=2/3-cC mu_n*(log log n)/log n"

"n=", 2, "h=", 2, "b=", 0.9102345781, "mu=", −0.5478516991

"n=", 3, "h=", 3, "b=", 0.6773389156, "mu=", 0.6633866652

"n=", 4, "h=", 3, "b=", 0.8274642410, "mu=", 0.8781852858

"n=", 5, "h=", 6, "b=", 0.5601001710, "mu=", −0.2044373617

"n=", 6, "h=", 6, "b=", 0.7235439661, "mu=", 0.3164313671

"n=", 7, "h=", 10, "b=", 0.6045006053, "mu=", −0.04690908657

"n=", 8, "h=", 15, "b=", 0.5499714651, "mu=", −0.2004671916

"n=", 9, "h=", 15, "b=", 0.6911304069, "mu=", 0.1970072873

"n=", 10, "h=", 30, "b=", 0.5033518867, "mu=", −0.3235562102

"n=", 11, "h=", 30, "b=", 0.6321697244, "mu=", 0.03186085209

"n=", 12, "h=", 42, "b=", 0.6068880741, "mu=", −0.03729386095

"n=", 13, "h=", 42, "b=", 0.7209821344, "mu=", 0.2734847150

"n=", 14, "h=", 70, "b=", 0.6200877879, "mu=", −0.001254324481

"n=", 15, "h=", 105, "b=", 0.5638793724, "mu=", −0.1540451776

"n=", 16, "h=", 105, "b=", 0.6654222494, "mu=", 0.1220016355

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"n=", 18, "h=", 210, "b=", 0.5934117778, "mu=", −0.07390034284
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"n=", 22, "h=", 330, "b=", 0.7723356258, "mu=", 0.4157513018
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"n=", 26, "h=", 1155, "b=", 0.6496744177, "mu=", 0.08034026266
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"n=", 29, "h=", 2310, "b=", 0.6342396573, "mu=", 0.03797054505
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"n=", 40, "h=", 15015, "b=", 0.7252292566, "mu=", 0.2958293407
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"n=", 42, "h=", 30030, "b=", 0.6332787004, "mu=", 0.03608698600
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"n=", 50, "h=", 51870, "b=", 0.8689379917, "mu=", 0.7123639622
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"n=", 72, "h=", 1939938, "b=", 0.8210466970, "mu=", 0.5900613093
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"n=", 77, "h=", 9699690, "b=", 0.6135250315, "mu=", -0.02077333289
"n=", 78, "h=", 9699690, "b=", 0.6474562091, "mu=", 0.07965307544
"n=", 79, "h=", 9699690, "b=", 0.6809483656, "mu=", 0.1789673279
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"n=", 83, "h=", 13123110, "b=", 0.7416794542, "mu=", 0.3602289715
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"n=", 100, "h=", 223092870, "b=", 0.6232867607, "mu=", 0.008255578559   **(17)**

```
> read "superhden.m":
```

20   **(18)**

```
> testOK(2,305926023);
```

"comp=", 10000, [160693319, 160730664], "temps=", 543.066

"Procedure ok(n), n=", 157933211

"Procedure ok(n), n=", 157933210

"Procedure ok(n), n=", 157933210

"Procedure ok(n), n=", 157933209

"n=", 157933209, "does not satisfy inequality (iv) of Thm 1.1"

"comp=", 11395, "compOK=", 4   (19)

```
> ti:=time():testppi(100);"temps=",time()-ti;
```

"sigk=", 305844740, "sigkp1=", 305926023, "muk=", 0.7700919520

"sigk=", 185307, "sigkp1=", 186914, "bptauk=", 1.044395558

"min of b_{sigma_k} (for sigma_k >= 100) =", 0.6232867609, "for sigma_k=", 100

"max of b_{sigma_k} (for sigma_k >= 100) =", 0.8844763739, "for sigma_k=", 31117

"temps=", 194.472   (20)

1   (21)

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>
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