CLOSE FREQUENCY RESOLUTION BY MAXIMUM ENTROPY SPECTRAL ESTIMATORS

J. L. LACOUME, MOHAMED GHARBI, CLAUDINE LATOMBE and JEAN LOUIS NICOLAS

Reprinted from IEEE Transactions on Acoustics, Speech, and Signal Procerssing, Vol. ASSP - 32, No. 5, October 1984

Close Frequency Resolution by Maximum Entropy Spectral Estimators

J. L. LACOUME, MOHAMED GHARBI, CLAUDINE LATOMBE, AND JEAN LOUIS NICOLAS

Abstract—We study the properties of the maximum entropy spectral estimator in pure frequencies estimation. We model this situation by a sum of pure frequencies added to white noise. We study the effect of the signal-to-noise ratio, of the autoregressive filter order, and of the pure frequencies amplitude ratio on the resolving power. Neglecting the uncertainties in the autoregressive filter coefficient estimates, we show that the poles of the autoregressive filter transfer function can be classified into two families: one gives the pure frequencies and the other the white noise. By a first-order development we can specify the position of the pure frequencies associated poles. This allows us to give analytical results on the bias of the pure frequency estimation and on the resolving power. These theoretical results are confirmed and illustrated by computer simulations.

I. Introduction

The maximum entropy spectral estimation is a very useful tool in the analysis of pure frequencies corrupted by additive white noise. Several studies have used this technique in order to obtain spectral estimation in time-dependent signal analysis [3], [4] or in space-dependent signal analysis (array processing) [6]-[8]. In these papers the resolving power of the maximum entropy spectral estimator was stated "experimentally." Recently, papers giving a theoretical analysis of this problem have appeared. Lang and McClellan [2] give the estimator variance and bias issuing from the uncertainties in the autoregressive (AR) filter coefficients, and Herring [1] studies the line splitting with two complex sinusoidal signals when the Burg algorithm is used.

In this paper, we present a theoretical approach to the properties of the maximum entropy spectral estimator of one or two pure frequencies corrupted by additive white noise with the assumption that the AR filter coefficient estimates are based on exact correlation samples. After a presentation of the model, we will show, using results presented in [5], that in the case of one frequency, the poles of the AR filter can be classified into two families: one pole representing the pure frequency, and the others the white noise. This allows us to propose a detection procedure of the pure frequency and to relate the "apparent bandwidth" of the pure frequency to the signal-to-noise ratio. In the two-frequency case, we generalize the preceding results and give a theoretical study of the poles associated with the pure frequency locations in the complex z plane. This allows us to give theoretical results on the bias and on the resolution power of the ME spectral estimator as a function of the signal-to-noise ratio, AR filter order, and frequency separation.

Manuscript received January 10, 1983; revised March 26, 1984.

J. L. Lacoume and C. Latombe are with CEPHAG (INPG), 38402 Saint-Martin-d'Heres, France, and the National Polytechnic Institute of Grenoble, Grenoble, France.

M. Gharbi was with CEPHAG (INPG), 38402 Saint-Martin-d'Heres, France, and the National Polytechnic Institute of Grenoble, Grenoble, France. He is now with the Commissariat aux Energies Nouvelles, Algiers, Algeria.

J. L. Nicolas is with the Département de Mathématiques, University of Limoges, 87100 Limoges, France.

II. Model

Let us state the Yule-Walker equations in vector notation:

$c_i^+ = (z_i^* \cdots z_i^{*M})$ with $z_i = e^{j\varphi_i}$, the M-dimensional vector associated with a pure frequency

$a^+ = (a_1^* \cdots a_M^*)$ is the M-dimensional vector associated with the AR filter coefficients

$+$ denotes complex transpose.

With these notations, the exact correlation matrix $(M \times M)$ of the sum of L statistically independant pure frequencies and white noise is

$$\mathbf{C} = \sum_{i=1}^{L} p_i c_i \cdot c_i^+ + p_n \mathbf{I}$$

p_i power of the ith pure frequency
p_n power of the white noise
$\mathbf{I}$ identity matrix

and the Yule-Walker equations are

$$\sum_{i=1}^{L} (p_i c_i \cdot c_i^+ + p_n \mathbf{I}) \quad a = \sum_{i=1}^{L} p_i c_i. \tag{1a}$$

Developing (1a)

$$\sum_{i=1}^{L} p_i (c_i^+ \cdot a)\, c_i + p_n a = \sum_{i=1}^{L} p_i c_i \tag{1b}$$

shows that the AR filter coefficients can be written

$$a = \sum_{j=1}^{L} \alpha_j c_j \tag{2}$$

and the α_j are given [substitute (2) into (1)] by

$$\sum_{j=1}^{L} \alpha_j c_i^+ \cdot c_j + \rho_i \alpha_i = 1; \qquad i = 1, \cdots, L \tag{3}$$

where $\rho_i = p_n/p_i$ is the noise-to-signal ratio for the ith component.

The maximum entropy (ME) spectral estimator is

$$\gamma_{\text{MEN}}(z) = \frac{1}{|G(z)|^2}$$

where $G(z)$ is the transfer function of the AR filter

$$G(z) = 1 - (z^+ \cdot a) = 1 - \sum_{j=1}^{L} \alpha_j (z^+ \cdot c_j) \tag{4}$$

and $z^+ = (z^{-1}, z^{-2}, \cdots, z^{-M})$.

From this formulation we easily obtain some general results.

1) For $z^+ = c_i^+ (z = e^{j\varphi_i})$: $G(z) = \rho_i \alpha_i$ and so when $\rho_i = 0$ (no white noise), L poles of the AR filter transfer function give exactly the L pure frequencies.

2) In the general case (3) gives

$$\left(1 + \frac{\rho_i}{M}\right) \alpha_i + \sum_{j \neq i} \alpha_j \cdot \frac{(c_i^+ \cdot c_j)}{M} = \frac{1}{M}$$

the second term of this equation (given before in [4]) "couples" the different frequencies.

If all the pure frequencies are well separated, the terms $(c_i^+ \cdot c_j)/M$ are small and (3) can be written

$$\left(1 + \frac{\rho_i}{M}\right) \alpha_i = \frac{1}{M} \qquad i = 1, \cdots, L.$$

In this case, as quoted before by [4], the "L frequencies" problem is equivalent to L "one frequency" problems.

III. Analysis of One Frequency

For a single frequency, the AR filter transfer function is given by

$$(M + \rho_1) z^M G(z) = (M + \rho_1) z^M - \sum_{1}^{M} z_1^n z^{M-n}. \tag{5}$$

It is shown in [5] that, when $\rho_1 \to 0$, the roots of this equation (poles of the ME spectral estimator) can be classified in two families and that:

1) One root is near $z = e^{j\varphi_1}$. Specifically, this root is inside the unit circle. Its angular position is φ_1 (no bias in the frequency estimate due to this root alone) and it is at a distance

$$\epsilon = \frac{2\rho_1}{M(M+1)} \tag{6}$$

from the unit circle.

2) The $(M - 1)$ other roots are regularly distributed inside the unit circle in the ring

$$(2M)^{-1/M} < |z| < 1 - \frac{1}{M}. \tag{7}$$

These roots represent the white noise (Fig. 1). The limits given in (7) allow us to propose the following "pure frequency detection" criterion: the number of pure frequencies is the number of poles satisfying

$$|\ |\ \ \ 1 \qquad (8)$$

A very similar criterion (for the detection of sinusoids in white noise by AR processing) was given recently by [9].

The pole associated to the pure frequency is, in the presence of noise, inside the unit circle. In the estimated PSD, the pure frequency is not represented by an infinitely sharp Dirac function but by a function peaked at

$$\omega_1 = \varphi_1$$

and the extension in ω of the peak depends on the distance of the poles from the unit circle. So, as in spectral analysis using the Fourier transforms of the signal windowed by a finite duration function, a pure frequency has an "apparent bandwidth" characteristic of the method of analysis. There exist different definitions of this bandwidth and we have chosen the "white noise equivalent bandwidth" [10] given by

$$B_{Wn} = \frac{1}{2} \frac{\left[\int_{-\pi}^{+\pi} \gamma(\omega)\, d\omega\right]^2}{\int_{-\pi}^{+\pi} \gamma^2(\omega)\, d\omega}. \tag{9}$$

In the analysis of a pure frequency, with a signal-to-noise ratio of

$$\sigma = \frac{1}{\rho}$$

with an autoregressive model of order M, the "white noise equivalent bandwidth" of a pure frequency corrupted by white noise is [5]

$$B_{Wn} = \frac{6}{\pi M(M+1)\sigma}. \tag{10}$$

This result is interesting in two respects. First, this bandwidth depends on the autoregressive model order and on the signal-to-noise ratio, and second, this characterization can be used in order to compare the analysis by autoregressive method and by the Fourier transform.

For a Fourier transform on kM samples of the signal, the "white noise equivalent bandwidth" is

$$B_{\text{WNF}} = \frac{1}{2\pi kM}.$$

So, if

$$k > \frac{(M+1)\sigma}{12}$$

the PSD obtained by the Fourier transform is more peaked than the autoregressive one, and if

$$k < \frac{(M+1)\sigma}{12}$$

the autoregressive estimate is more peaked than the Fourier transform.

IV. Analysis of Two Close Frequencies–Resolving Power

Let us consider two pure frequencies with

$$z_1 = 1$$

for the locations in the complex z plane of the poles associated with the two pure frequencies (it is always possible to locate one of the frequencies at $z_1 = 1$ by a rotation in the complex z plane), and with the amplitudes p_1 and p_2 giving the noise-to-signal ratios ρ_1 and ρ_2.

Then a straightforward calculation using (3) and (4) gives the AR filter transfer function ($G(z)$) by

$$F(z) = \alpha_0 z^M G(z)$$

$$= \alpha_0 z^M - \alpha_1 \sum_1^M z^{M-n} - \alpha_2 \sum_1^M z_2^n z^{M-n} \tag{11}$$

with

$$\alpha_0 = (M + \rho_1)(M + \rho_2) - |A|^2$$

$$\alpha_1 = M + \rho_2 - A$$

$$\alpha_2 = M + \rho_1 - A^*$$

$$A = \sum_{n=1}^{M} z_2^n.$$

The polynomial $F(z)$ can be written

$$F(z) = F_0(z) + F_{\rho_1\rho_2}(z)$$

with

$$F_0(z) = F(z) \quad \text{for} \quad \rho_1 = \rho_2 = 0$$

$F_{\rho_1\rho_2}(z)$ = the remaining part of $F(z)$.

Furthermore, when ρ_1, ρ_2, and φ are small, in a domain Δ in the neighborhood of 1 and z_2, we can write

$$F(z) \sim (z - 1)(z - z_2) B_{M-2}(z).$$

When φ tends to 0, it can be shown (Appendix B) that the roots of $B_{m-2}(z)$ are inside the unit circle and outside the sector defined by

$$\arg z = \pm \frac{\pi}{2(M-2)}.$$

The domain Δ being included in this sector, $F_0(z)$, can be approximated by

$$\hat{F}_0(z) \sim (z - z_2)(z - 1)\frac{F_0''(1)}{2}$$

with $F_0''(1)$ the second derivative of $F_0(z)$.

With the hypothesis ρ_1, ρ_2 small, $F_{\rho_1\rho_2}(z)$ can be approximated by

$$\hat{F}_{\rho_1\rho_2}(z) = M(\rho_1 + \rho_2) z^M - \rho_1 \sum_1^M z^{M-n} - \rho_2 \sum_1^M z_2^n z^{M-N}$$

and, with $z_2 = e^{j\varphi}$, $\varphi \sim 0$, a first-order development in φ of $F_0''(z)$ and $F_{\rho_1\rho_2}(z)$ gives

$$F_0''(1) \sim \frac{M^2(M+1)^2(M-1)(M+2)}{72}\varphi^2$$

$$\hat{F}_{\rho_1\rho_2}(z) \sim \frac{M(M+1)(\rho_1+\rho_2)}{2}\left[z - \frac{\rho_2 + z_2\rho_1}{\rho_1+\rho_2}\right].$$

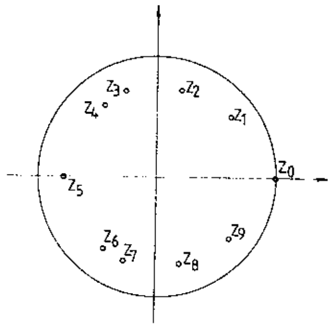

Fig. 1. Pole locations in the z complex plane for the limiting case $\rho \to 0$ and an autoregressive filter order 10.

Finally,

$$\hat{F}(z) = \frac{F_0''(1)}{2}\hat{f}(z)$$

with

$$\hat{f}(z) \sim z^2 - (1 + z_2 - \Gamma)z + z_2 - \Gamma\frac{\rho_2 + \rho_1 z_2}{\rho_1 + \rho_2}$$

$$\Gamma = \frac{72(\rho_1+\rho_2)}{M(M+1)(M-1)(M+2)\varphi^2}. \tag{12}$$

We will study two cases:

- two frequencies with equal amplitude
- two frequencies with different amplitude.

A. *Two Frequencies with Equal Amplitude*

For equal amplitude, $\rho_1 = \rho_2 = \rho$, the noise-to-signal ratio for each frequency.

In this case (12) gives

$$\hat{f}(z) = z^2 - (1 + z_2 - \Gamma)z + z_2 - \Gamma\frac{1+z_2}{2}$$

$$\Gamma = \frac{144\rho}{M(M+1)(M-1)(M+2)\varphi^2}. \tag{13}$$

We are interested in the roots of (13) (pole of the PSD estimated) which represent the "apparent frequencies" seen by this spectral analysis.

These roots are

$$z_{a,b} = \frac{1 + z_2 - \Gamma}{2} \pm \frac{\sqrt{\Delta}}{2}$$

$$\Delta = \Gamma^2 - \varphi^2. \tag{14}$$

For $0 \leqslant \Gamma < \varphi$ the two roots for (14) are located symmetrically on the semicircle shown in Fig. 2.

For $\Gamma \to 0$ the two roots tends to 1 and z_2 and (as stated before) the true frequencies are obtained.

When ρ (and Γ) increases, the two roots go inside the unit circle, so the PSD estimate is less "peaked." The arguments of the roots are no longer 0 and φ, introducing a bias on the estimated frequencies.

When $\Gamma = \varphi$, the two roots collapse and then one frequency is seen by the analyzer.

When $\Gamma > \varphi$, the two roots move along the radius of argument $\varphi/2$. One root goes inside the unit circle and will be

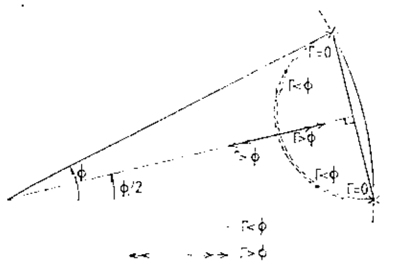

Fig. 2. Location of the autoregressive filter poles versus Γ (two frequencies with equal amplitude).

interpreted as a "noise root;" the other root goes toward the unit circle and will be interpreted as a "pure frequency root." In this situation, the two frequencies will be "seen" by the analyzer as "one frequency" with a value equal to the mean of the two initial values. It must be noted that the root closer to the unit circle stays inside the unit circle [it is seen from (14)].

These results merit comment. First of all, one can say that the two frequencies are separated as long as the two roots have a different argument (this is in some way an artificial definition, and other kinds of definitions can be used). With this definition the resolution power of the AR analysis is given by $\Gamma < \varphi$, for resolution which gives

$$\frac{144\rho}{M(M+1)(M-1)(M+2)\varphi^3} < 1. \tag{15}$$

This relation shows us the effect of the noise-to-signal ratio (ρ) and of the AR filter order (M) on the resolution power. The noise-to-signal ratio must be lower than a threshold value in order to separate the two freqencies. This result gives a justification of the "hybrid methods" [3] in which the resolution power is increased by subtraction of noise.

The regular approach to the resolving power can be obtained through the plot of the PSD in "normalized frequency." Letting

$$z = e^{i\omega}$$

we obtain

$$\gamma(\omega) = \frac{1}{|G(e^{i\omega})|^2}$$

which is plotted for $[-1/\pi, +1/\pi]$.

Using the previous results one can see (Appendix A) that the maxima of the PSD are located at

$$\omega_{a,b} = \frac{\varphi}{2}\left[1 - \left(1 - 2\frac{\Gamma^2}{\varphi^2}\right)^{1/2}\right] \tag{16}$$

for $\Gamma < \varphi/\sqrt{2}$ and there is only one maximum, located at

$$\omega_{a,b} = \frac{\varphi}{2}$$

for $\Gamma \geqslant \varphi/\sqrt{2}$. We see that with this representation, the two frequencies are separated for $\Gamma < \varphi/\sqrt{2}$ and not, as previously, for $\Gamma < \varphi$. In the range of value of Γ between $\varphi/\sqrt{2}$ and φ the two frequencies are represented by two poles with different angular position, but give rise to only one maximum in the PSD plot. This effect is shown in Fig. 3.

(a)

(b)

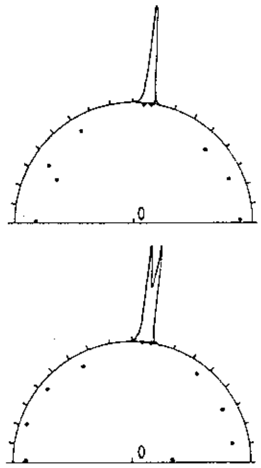

Fig. 3. PSD (plotted around the unit circle) and pole filter presentation. (a) $\rho = 0.1$, $\varphi = 0.02$ rad, $M = 16$. The pure frequencies are separated by the pole presentation but not separated by the PSD presentation. (b) $\rho = 0.1$, $\varphi = 0.02$ rad, $M = 18$. The pure frequencies are separated by the pole and PSD presentations.

The bias in the frequency estimate (given by the maxima of the PSD) is

$$\Delta\varphi = \frac{\varphi}{2}\left(1 - 2\frac{\Gamma^2}{\varphi^2}\right)^{1/2}.$$

So, the estimation of the frequencies by the pole locations is better than the direct examination of the estimated PSD.

Another "technical reason" is favorable at the poles. The plot of the PSD is sampled and, in order to get a good definition of the maxima of the PSD, it is necessary tc oversample the spectrum. However, a "sampling bias," which can be made negligible, will always remain in this presentation.

B. *Two Frequencies with Different Amplitudes*

The two roots of $G(z)$ given by (12) are now

$$z_{a,b} = \frac{1 + z_2 - \Gamma}{2} \pm \frac{\sqrt{\Delta}}{2}$$

with

$$\Delta = \Gamma^2 - \varphi^2 + 2j\Gamma\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}. \tag{17}$$

With $\rho_2 = k\rho_1$ ($k > 1$: the amplitude of the $z = 1$ pure frequency is greater than the amplitude of the $z = z_2$ one), the root locus of $G(z)$ in the z complex plane, for Γ increasing from 0, is given in Fig. 4.

As previously for $\Gamma \to 0$, the two poles tend to 1 and z_2. For increasing values of Γ the two poles go inside the unit circle. Pole 1, associated to the frequency of greater amplitude, follows a curve situated in the semicircle obtained with two frequencies of equal amplitude. Pole 2, associated to the frequency of lower amplitude, follows a curve situated outside the same semicircle. When Γ becomes greater than φ, pole 1 goes toward the unit circle and pole 2 toward the center. The

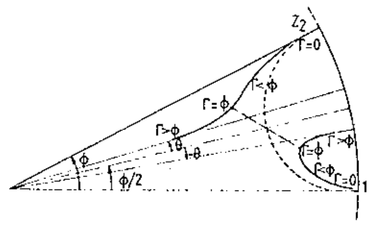

Fig. 4. Location of the autoregressive filter poles versus for two frequencies with different amplitude ($\theta = (\varphi/2)\,(\rho_2 - \rho_1)/(\rho_2 + \rho_1)$).

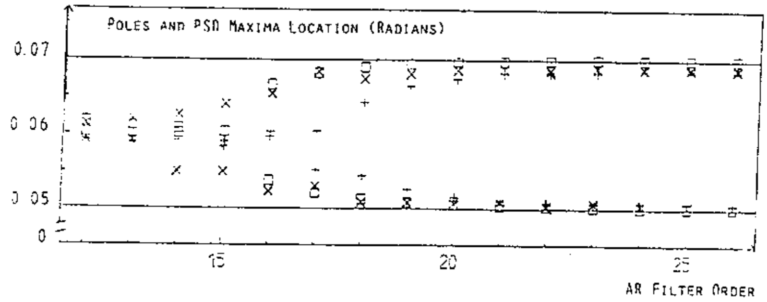

Fig. 5. PSD maxima (+) poles (experimental: X; theory: □) frequency separation versus AR filter order. Noise-to-signal ratio: $\rho = 0.1$. Frequency separation: $\varphi = 0.02$ rad.

limiting values of the angular positions of the two poles ($\Gamma \to \infty$) are

$$\theta_{1,2} = \frac{\varphi}{2} \pm \frac{\varphi}{2}\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}.$$

In the case of two frequencies of different amplitudes, it is shown in Appendix A that the PSD presents two different maxima for

$$\left(\frac{\Gamma}{\varphi}\right)^2 \leqslant \frac{1}{2}\left[1 - \left(3\sqrt{3}\,\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)^{2/3}\left(\frac{\Gamma}{\varphi}\right)^{4/3}\right]. \quad (18)$$

When the two frequencies have different amplitudes, the resolution power is "lowered" by the second term in (18).

V. Experimental Illustration

A simulation was made using two frequencies of equal amplitude corrupted by additive white noise with

$\varphi = 0.02$ (rad) as frequency separation

$\rho = 0.1$ noise-to-signal ratio.

The AR filter coefficients were estimated by the Burg method using 256 samples of the signal studied.

In Fig. 5, we present the evolution of

- the maxima of the PSD (+)
- the pole location (X)
- the theoretical pole location [using (13) (□)]
- versus filter order obtained by averaging over 20 independent realizations.

In Fig. 6, we show the evolution of the distance between the unit circle (ϵ) and the AR filter poles versus AR filter order (M) in the same conditions ($\varphi = 0.02$ rad, $\rho = 0.1$). In this situation, the two roots collapse for $M = 15$. One can see that for $M > 15$, the two poles (X, +) are at the same distance from the unit circle. For $M < 15$, one pole goes inside the unit circle and the other one towards the unit circle. The experimental values follow the theoretical curves except near the "collapsing point," where it seems that the two curves tend to separate themselves. We think that this minor discrepancy is due to the approximations in the pole location calculation and that in this region a more precise calculation is needed.

VI. Conclusion

We have presented a theoretical calculation of the bias and the "resonance amplitude" of the maximum entropy spectral estimate of pure frequencies corrupted by additive white noise. This study allows us in the one-frequency case to obtain a "separation criterion" of the poles of the AR filter associated with the white noise and the pure frequency, and to calculate the amplitude of the ME spectral estimator peak. In the two-frequency case the theoretical calculation gives the value of the frequency estimation bias and relates the resolving power of the method to the noise-to-signal ratio, AR filter order, and frequency separation.

Appendix A

Position of the Maxima of the PSD

The modulus of $\hat{f}(z)$ is

$$|\hat{f}(z)| = \left|(z-1)(z-z_2) + \Gamma\left(z - \frac{\rho_2 + z_2\rho_1}{\rho_1 + \rho_2}\right)\right|.$$

We are near the point $z = 1$ so we can let

$$z = e^{i\omega} \sim 1 + i\omega$$

$$z_2 = e^{i\varphi} \sim 1 + i\varphi$$

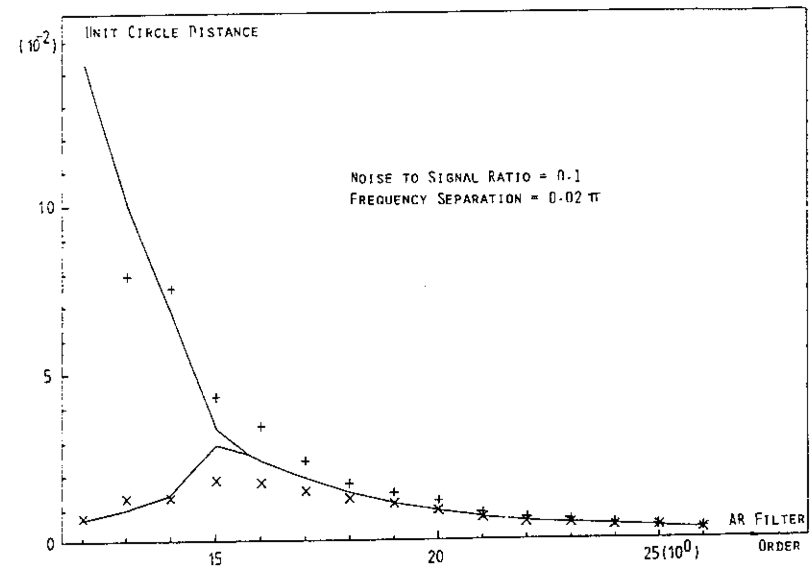

Fig. 6. Evolution of the unit circle distance of the two pole (experimental: ×, +) results and theoretical curve. Noise-to-signal ratio: $\rho = 0.1$. Frequency separation: $\varphi = 0.02$ rad.

which gives

$$|\hat{f}(\omega)| \simeq \left| -\omega(\omega - \varphi) + i\Gamma \left(\omega - \varphi \frac{\rho_1}{\rho_1 + \rho_2}\right) \right|$$

and with $\omega = \beta + (\varphi/2)$

$$|\hat{f}(\beta)|^2 = \beta^4 - \beta^2 \left(\frac{\varphi^2}{2} - \Gamma^2\right) + \Gamma^2 \varphi \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \beta + \frac{\varphi^2}{16} + \frac{\Gamma^2 \varphi^2}{4} \left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)^2.$$

The extrema of the PSD (proportional to $1/|\hat{f}(\beta)|^2$) are the roots of

$$\frac{d|\hat{f}(\beta)|^2}{d\beta} = 4\beta^3 - 2\beta\left(\frac{\varphi^2}{2} - \Gamma^2\right) + \Gamma^2 \varphi \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}. \qquad (19)$$

If the two frequencies have equal amplitude, there are three extrema located at

$$\beta = 0, \quad \beta = \pm \left(\frac{\varphi^2}{4} - \frac{\Gamma^2}{2}\right)^{1/2}$$

if $\Gamma < \varphi/\sqrt{2}$, and one extrema, located at

$$\beta = 0$$

if $\Gamma \geqslant \varphi/\sqrt{2}$. If the two frequencies have different amplitudes, (19) has three roots if and only if

$$\frac{\Gamma^2}{\varphi^2} \leqslant \frac{1}{2}\left[1 - \left(3\sqrt{3}\,\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)^{2/3} \left(\frac{\Gamma}{\varphi}\right)^{4/3}\right].$$

Appendix B

Localization of the Roots of $F_0(z)$ when z_2 Is Close to 1

Letting $\rho_1 = \rho_2 = 0$ and multiplying (11) by $(z - 1)(z - z_2)$ gives

$$\begin{aligned}(z - 1)(z - z_2) F_0(z) &= \alpha_0 z^{m+2} - (\alpha_0 + \alpha_0 z_2 + \alpha_1 + \alpha_2 z_2) z^{m+1} \\ &+ (\alpha_0 z_2 + \alpha_0 z_2 + \alpha_2 z_2) z^m \\ &+ (\alpha_1 + \alpha_2 z_2^{m+1}) z - \alpha_1 z_2 - \alpha_2 z_2^{m+1}. \end{aligned} \qquad (20)$$

When $z_2 \to 1$ the roots of (20) tends to the root of the polynomial

$$A_{m+2}(z) = z^{m+2} - 2\frac{m+2}{m} z^{m+1} + \frac{(m+1)(m+2)}{m(m-1)} z^m - 2\frac{(m+2)}{m(m-1)} z + \frac{2}{m}.$$

In order to get this expression, let us define

$$\alpha_0 = f(z_2).$$

The first derivative of $f(z_2)$ for $z_2 = 1$ is

$$f'(1) = 0$$

and the second derivative is

$$f''(1) = \frac{m^2(m+1)(m-1)}{6}.$$

So, when z_2 tends to 1,

$$\alpha_2 \sim \frac{(z_2 - 1)^2}{2} \frac{m^2(m+1)(m-1)}{6}.$$

The same calculations give the other coefficients of (20). $z = 1$ is a four-order root of $A_{m+2}(z)$, so

$$A_{m+2}(z) = (z - 1)^4 B_{m-2}(z)$$

and

$$B_{m-2}(z) = \frac{1}{(z-1)^4} A_{m+2}(z)$$

$$= \left(\sum_{k=0}^{\infty} \frac{(k+1)(k+2)(k+3)}{6} z^k \right) A_{m+2}(z)$$

which gives (by identification)

$$B_{m-2}(z) = \frac{1}{m(m-1)} \sum_{k=0}^{m-2} (k+1)(k+2)(m-1-k) z^k.$$

A polynomial with all coefficients > 0.

The roots of $B_{m-2}(z)$ are all inside the unit circle: Let us define

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

$$\bar{P}(z) = a_n^* + a_{n-1}^* z + \cdots + a_0^* z^n = z^n P^*\left(\frac{1}{z}\right)$$

$*$ = complex conjugate

and

$$TP(z) = a_0^* P(z) - a_n \bar{P}(z).$$

The polynomial $TP(z)$ is of degree $n-1$ and here

$$TB_{m-2} = -\frac{(m+2)}{m} B_{m-3}^*.$$

At that place we use the "Rouche theorem" [11]: if two complex functions f and g are holomorph in the unit circle ($|z| < 1$) and verify

$$|g(z)| < |f(z)| \quad \text{for} \quad |z| = 1$$

then the two equations

$$f(z) = 0$$

$$f(z) + g(z) = 0$$

have the same number of roots in the unit circle ($|z| < 1$).

Let us apply this theorem to

$$TB_{m-2} = \frac{2}{m} B_{m-2} - \bar{B}_{m-2} = -\frac{m+2}{m} \bar{B}_{m-3}.$$

For $|z| = 1$, $|B_{m-2}(z)| = |\bar{B}_{m-2}(z)|$ and so, if $m > 2$, $|2/m \cdot B_{m-2}(z)| < |\bar{B}_{m-2}(z)|$.

The hypothesis of the Rouche theorem is fulfilled, and so $\bar{B}_{m-2}$ and $\bar{B}_{m-3}$ have the same number of roots in the unit circle.

By iteration we can so prove that $\bar{B}_{m-2}$ and $\bar{B}_0$ have the same number of roots in the unit circle. The last ($\bar{B}_0$) has no roots in the unit circle, so $\bar{B}_{m-2}$ has no roots in the unit circle *and B_{m-2} has all its roots in the unit circle*.

The roots of B_{m-2} are outside the sector $|\arg z| < \pi/m$: This follows from the fact that all the coefficients of B_{m-2} are positive or null.

References

[1] R. W. Herring, "The cause of the line splitting in Burg entropy spectral analysis," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, Dec. 1980.

[2] S. W. Lang and J. H. McClellan, "Frequency estimation with maximum entropy spectral estimators," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, Dec. 1980.

[3] O. L. Frost, "Power spectrum estimation," in *Proc. NATO Adv. Study Inst. Signal Processing with Emphasis on Underwater Acoustics*, Portovenere (La Spezia), Italy, 1976, pp. 125–162.

[4] E. H. Satorius and J. R. Zeidler, "Maximum entropy spectral analysis of multiple sinusoids in noise," *Geophysics*, vol. 43, pp. 1111–1118, Oct. 1978.

[5] J. L. Lacoume, C. Hanna, and J. L. Nicolas, "Calibration de l'analyse spectrale par la méthode du modèle autorégressif," *Ann. Télécommun.*, vol. 36, Nov.-Dec. 1981.

[6] H. P. Bucker, "High resolution cross-sensor beamforming for a uniform array," *J. Acoust. Soc. Amer.*, vol. 63, p. 420, Feb. 1978.

[7] D. P. Skinner, S. M. Hedlicka, and A. D. Matthews, "Maximum entropy array processing," *J. Acoust. Soc. Amer.*, vol. 66, pp. 488–493, Aug. 1979.

[8] W. F. Gabriel, "Spectral analysis and adaptive array superresolution techniques," *Proc. IEEE*, vol. 68, pp. 654–666, June 1980.

[9] S. M. Kay, "Robust detection by autoregressive spectrum analysis," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 256–269, Apr. 1982.

[10] G. M. Jenkins and D. J. Watts, *Spectral Analysis and Its Applications*. San Francisco, CA: Holden-Day, 1968, p. 256.

[11] M. Marden, *Geometry of Polynomials* (Amer. Math. Soc. Math. Surveys, no. 3), 2nd ed., 1966.

J. L. Lacoume was born in Mirepoix, France, in 1940. He received the Licentiate of Physics, 3rd Cycle Doctorate, and Sc.D. degrees from the Ecole Normale Supérieure, Paris, France, in 1962, 1965, and 1969, respectively.

From 1964 to 1965 he was an Assistant and from 1965 to 1970 a Lecturer at the University of Orsay, Orsay, France. He is currently a Professor at the National Polytechnic Institute of Grenoble, Grenoble, France, and Director of the Random Phenomena and Geophysics Study Center (CEPHAG), Saint-Martin-d'Hères, France. He has published over 40 papers on geophysics and signal processing and has lectured widely. He is currently interested in spectral and cross-spectral analysis and applications to geophysics and underwater acoustics and array processing.

Prof. Lacoume is a member of the International Scientific Radio Union (URSI), UGGI the French Society of Physics, the European Association for Signal Processing, and the Association of Former Pupils of the Ecole Normale Supérieure. He has received the Hughes Price Award from the Academy of Sciences.

Mohamed Gharbi was born in Algiers, Algeria, on May 11, 1957. He received the Maitrise d'Electronique from the University of Grenoble, Grenoble, France, in 1979, and the Doctorat de 3ème Cycle from the National Polytechnic Institute of Grenoble, Grenoble, in 1983.

He is currently an Assistant Researcher at the Commissariat aux Energies Nouvelles, Algiers, Algeria. He has lectured at the University of Grenoble and published several papers on spectral analysis. His present research interests are digital image processing and teledetection.

Claudine Latombe was born in Evreux, France, on June 18, 1945. She received the B.S. degree in mathematics from the Ecole Normale Supérieure de l'Enseignement Technique, France, in 1967, and the Third Cycle degree in computer science and the Ph.D. degree in electrical engineering from Grenoble University, Grenoble, France, in 1974 and 1982, respectively.

Since 1969, she has been a teacher at the National Polytechnic Institute of Grenoble, Grenoble.

Jean Louis Nicolas was born in Orleans, France, on January 17, 1942. He received the Master's degree in mathematics, agrégation, and Ph.D. degree from the Ecole Normale Supérieure, Paris, France, in 1962, 1963, and 1968, respectively.

Between 1964 and 1977 he was successively Assistant at the University of Paris–Sorbonne, Professor at the University of Sherbrooke, Sherbrooke, P.Q., Canada, and Lecturer at the University of Paris–Orsay. Since 1973 he has been Professor at the University of Limoges, Limoges, France, where he directs a research team on computers in number theory. He has published over 30 research papers on number theory, many concerning optimization problems in integers. He has also published several survey papers on the use of computers in arithmetic, factorization methods, and primality tests. Since 1982 he has been the Dean of the Faculty of Sciences of the University of Limoges.

Dr. Nicolas is a member of the French Mathematics Society (of which he was the Treasurer from 1976 to 1978), of the American Mathematics Society, and of the Association of Former Pupils of the Ecole Normale Supérieure.