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On Large Values of the Divisor Function

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Jean-Louis Nicolas and Andràs Sárközy dedicate this paper to the memory of Paul Erdős

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Abstract. Let $d(n)$ denote the divisor function, and let $D(X)$ denote the maximal value of $d(n)$ for $n \le X$. For $0 < z \le 1$, both lower and upper bounds are given for the number of integers n with $n \le X$, $zD(X) \le d(n)$.

Key words: division function, highly composite numbers, maximal order

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1. Introduction

Throughout this paper, we shall use the following notations: **N** denotes the set of the positive integers, $\pi(x)$ denotes the number of the prime numbers not exceeding x, and p_i denotes the ith prime number. The number of the positive divisors of $n \in \mathbf{N}$ is denoted by $d(n)$, and we write

$$D(X) = \max_{n \le X} d(n).$$

Following Ramanujan we say that a number $n \in \mathbf{N}$ is highly composite, briefly h.c., if $d(m) < d(n)$ for all $m \in \mathbf{N}$, $m < n$. For information about h.c. numbers, see [13, 15] and the survey paper [11].

The sequence of h.c. numbers will be denoted by $n_1, n_2, \ldots : n_1 = 1, n_2 = 2, n_3 = 4, n_4 = 6, n_5 = 12, \ldots$ (for a table of h.c. numbers, see [13, Section 7, or 17]. For $X > 1$, let $n_k = n_{k(X)}$ denote the greatest h.c. number not exceeding X, so that

$$D(X) = d\big(n_{k(X)}\big).$$

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It is known (cf. [13, 8]) that n_k is of the form $n_k = p_1^{r_1} p_2^{r_2} \cdots p_\ell^{r_\ell}$, where $r_1 \geq r_2 \geq \cdots \geq r_\ell$,

$$\ell = (1+o(1))\frac{\log X}{\log\log X}, \tag{1}$$

$$r_i = (1+o(1))\frac{\log p_\ell}{\log 2 \log p_i} \quad \left(\text{for } X \to \infty \text{ and } \frac{\log p_i}{\log p_\ell} \to 0\right) \tag{2}$$

and, if m is the greatest integer such that $r_m \geq 2$,

$$p_m = p_\ell^{\theta} + O\left(p_\ell^{\tau_o \theta}\right) \tag{3}$$

where

$$\theta = \frac{\log(3/2)}{\log 2} = 0.585\ldots \tag{4}$$

and τ_0 is a constant <1 which will be given later in (8).

For $0 < z \leq 1$, $X > 1$, let $S(X, z)$ denote the set of the integers n with $n \leq X$, $d(n) \geq zD(X)$. In this paper, our goal is to study the function $F(X, z) = \text{Card}(S(X, z))$.

In Section 4, we will study $F(X, 1)$, further we will prove (Corollary 1) that for some $c > 0$ and infinitely many X's with $X \to +\infty$, we have $F(X, z) = 1$ for all z and X satisfying

$$1 - \frac{1}{(\log X)^c} < z \leq 1.$$

Thus, to have a non trivial lower bound for $F(X, z)$ for *all* X, one needs an assumption of the type $z < 1 - f(X)$, cf. (6).

In Section 2, we shall give lower bounds for $F(X, z)$. Under a strong, but classical, assumption on the distribution of primes, the lower bound given in Theorem 1 is similar to the upper bound given in Section 3. The proofs of the lower bounds will be given in Section 5: in the first step we construct an integer $\hat{n} \in S(X, z)$ such that $d(\hat{n})$ is as close to $zD(X)$ as possible. This will be done by using diophantine approximation of θ (defined by (4)), following the ideas of [2, 8]. Further, we observe that slightly changing large prime factors of $\hat{n}$ will yield many numbers n not much greater than $\hat{n}$, and so belonging to $S(X, z)$. The proof of the upper bound will be given in Section 7. It will use the superior h.c. numbers, introduced by Ramanujan (cf. [13]). Such a number N_ε maximizes $d(n)/n^\varepsilon$. The problem of finding h.c. numbers is in fact an optimization problem

$$\max_{n \leq x} d(n)$$

and, in this optimization problem, the parameter ε plays the role of a Lagrange multiplier. The properties of the superior h.c. numbers that we shall need will be given in Section 6.

In [10, p. 411], it was asked whether there exists a positive constant c such that, for n_j large enough,

$$\frac{d(n_{j+1})}{d(n_j)} \le 1 + \frac{1}{(\log n_j)^c}.$$

In Section 8, we shall answer this question positively, while in Section 4 we shall prove that for infinitely many n_j, one has $d(n_{j+1})/d(n_j) \ge 1 + (\log n_j)^{-0.71}$.

We are pleased to thank J. Rivat for communicating us reference [1].

2. Lower bounds

We will show that

Theorem 1. *Assume that τ is a positive number less than 1 and such that*

$$\pi(x) - \pi(x-y) > A\frac{y}{\log x} \quad \textit{for } x^\tau < y < x \tag{5}$$

for some $A > 0$ and x large enough. Then for all $\varepsilon > 0$, there is a number $X_0 = X_0(\varepsilon)$ such that, if $X > X_0(\varepsilon)$ and

$$\exp(-(\log X)^\lambda) < z < 1 - \log X)^{-\lambda_1} \tag{6}$$

where λ is any fixed positive real number <1 and λ_1 a positive real number ≤ 0.03, then we have:

$$F(X,z) > \exp\left((1-\varepsilon)\min\{2(A\log 2\log X\log(1/z))^{1/2}, 2(\log X)^{1-\tau}\log\log X\log(1/z)\}\right). \tag{7}$$

Note that (5) is known to be true with

$$\tau = \tau_0 = 0.535 \quad \text{and} \quad A = 1/20 \tag{8}$$

(cf. [1]) so that we have

$$F(X,z) > \exp((1-\varepsilon)2(\log X)^{0.465}\log\log X\log(1/z))$$

for all z satisfying (6), and assuming the Riemann hypothesis, (5) holds for all $\tau > 1/2$ so that

$$F(X,z) > \exp((\log X)^{1/2-\varepsilon}\log(1/z))$$

for all $\varepsilon > 0$, X large enough and z satisfying (6). Moreover, if (5) holds with some $\tau < 1/2$ and $A > 1 - \varepsilon/2$ (as it is very probable), then for a fixed z we have

$$F(X,z) > \exp((2-\varepsilon)((\log 2)(\log X)\log(1/z))^{1/2}). \tag{9}$$

In particular,

$$F(X, 1/2) > \exp((1-\varepsilon)(\log 2)(\log X)^{1/2}). \tag{10}$$

While we need a very strong hypothesis to prove (9) for all X, we will show without any unproved hypothesis that, for fixed z and with another constant in the exponent, it holds for infinitely many $X \in \mathbf{N}$:

Theorem 2. *If z is a fixed real number with $0 < z < 1$, and $\varepsilon > 0$, then for infinitely many $X \in \mathbf{N}$ we have*

$$F(X, z) > \exp((1-\varepsilon)(\log 4 \log X \log(1/z)^{1/2}) \tag{11}$$

so that, in particular

$$F(X, 1/2) > \exp((1-\varepsilon)\sqrt{2}\ \log 2(\log X)^{1/2}). \tag{12}$$

We remark that the constant factor $\sqrt{2}\log 2$ on the right hand side could be improved by the method used in [12] but here we will not work out the details of this. It would also be possible to extend Theorem 2 to all z depending on X and satisfying (6).

3. Upper bounds

We will show that:

Theorem 3. *There exists a positive real number γ such that, for $z \geq 1 - (\log X)^{-\gamma}$, as $X \to +\infty$ we have*

$$\log F(X, z) = O\big((\log X)^{(1-\gamma)/2}\big), \tag{13}$$

and if λ, η are two real numbers, $0 < \lambda < 1$, $0 < \eta < \gamma$, we have for

$$1 - (\log X)^{-\gamma+\eta} \geq z \geq \exp(-(\log X)^{\lambda}), \tag{14}$$

and X large enough:

$$F(X, z) \leq \exp\left(\frac{24}{\sqrt{1-\gamma}}(\log(1/z)\log X)^{1/2}\right). \tag{15}$$

The constant γ will be defined in Lemma 5 below. One may take $\gamma = 0.03$. Then for $z = 1/2$, (15) yields

$$\log F(X, 1/2) \leq 21(\sqrt{\log X})$$

which, together with the results of Section 2, shows that the right order of magnitude of $\log F(X, 1/2)$ is, probably, $\sqrt{\log X}$.

4. The cases $z = 1$ and z close to 1

Let us first define an integer n to be largely composite (l.c.) if $m \le n \Rightarrow d(m) \le d(n)$. S. Ramanujan has built a table of l.c. numbers (see [14, p. 280 and 15, p. 150]). The distribution of l.c. numbers has been studied in [9], where one can find the following results:

Proposition 1. *Let $Q_\ell(X)$ be the number of l.c. numbers up to X. There exist two real numbers $0.2 < b_1 < b_2 < 0.5$ such that for X large enough the following inequality holds*:

$$\exp((\log X)^{b_1}) \le Q_\ell(X) \le \exp((\log X)^{b_2}).$$

We may take any number $<(1 - \frac{\log 3/2}{\log 2})/2 = 0.20752$ *for b_1, and any number* $>(1-\gamma)/2$ *with $\gamma > 0.03$ defined in Lemma 5, for b_2.*

From Proposition 1, it is easy to deduce:

Theorem 4. *There exists a constant $b_2 < 0.485$ such that for all X large enough we have*

$$F(X, 1) \le \exp((\log X)^{b_2}). \tag{16}$$

There exists a constant $b_1 > 0.2$ such that, for a sequence of X tending to infinity, we have

$$F(X, 1) \ge \exp((\log X)^{b_1}). \tag{17}$$

Proof: $F(X, 1)$ is exactly the number of l.c. numbers n such that $n_k \le n \le X$. Thus $F(X, 1) \le Q_\ell(X)$ and (16) follows from Proposition 1.

The proof of Proposition 1 in [9, Section 3] shows that for any $b_1 < 0.207$, there exists an infinite number of h.c. numbers n_j such that the number of l.c. numbers between n_{j-1} and n_j (which is exactly $F(n_j - 1, 1)$) satisfies $F(n_j - 1, 1) \ge \exp((\log n_j)^{b_1})$ for n_j large enough, which proves (17). □

We shall now prove:

Theorem 5. *Let (n_j) be the sequence of h.c. numbers. There exists a positive real number a, such that for infinitely many n_j's, the following inequality holds:*

$$\frac{d(n_j)}{d(n_{j-1})} \ge 1 + \frac{1}{(\log n_j)^a}. \tag{18}$$

One may take any $a > 0.71$ in (18).

Proof: Let X tend to infinity, and define $k = k(X)$ by $n_k \le X < n_{k+1}$. By [8], the number $k(X)$ of h.c. numbers up to X satisfies

$$k(X) \le (\log X)^\mu \tag{19}$$

for X large enough, and one may choose for μ the value $\mu = 1.71$, cf. [10, p. 411 or 11, p. 224]. From (19), the proof of Theorem 5 follows by an averaging process: one has

$$\prod_{\sqrt{X}<n_j\le X} \frac{d(n_j)}{d(n_{j-1})} = \frac{D(X)}{D(\sqrt{X})}.$$

The number of factors in the above product is $k(X) - k(\sqrt{X}) \le k(X)$ so that there exists j, $k(\sqrt{X}) + 1 \le j \le k(X)$, with

$$\frac{d(n_j)}{d(n_{j-1})} \ge \left(\frac{D(X)}{D(\sqrt{X})}\right)^{1/k(X)}. \tag{20}$$

But it is well known that $\log D(X) \sim \frac{(\log 2)(\log X)}{\log\log X}$, and thus

$$\log(D(X)/D(\sqrt{X})) \sim \frac{\log 2}{2}\frac{\log X}{\log\log X}.$$

Observing that $X < n_j^2$, it follows from (19) and (20) for X large enough:

$$\begin{aligned}\frac{d(n_j)}{d(n_{j-1})} &\ge \exp\left(\frac{1}{3}\frac{1}{(\log X)^{\mu-1}\log\log X}\right)\\ &\ge \exp\left(\frac{1}{3}\frac{1}{(2\log n_j)^{\mu-1}\log(2\log n_j)}\right)\\ &\ge \exp\left(\frac{1}{(\log n_j)^a}\right) \ge 1 + \frac{1}{(\log n_j)^a}\end{aligned}$$

for any $a > \mu - 1$, which completes the proof of Theorem 5. □

A completely different proof can be obtained by choosing a superior h.c. number for n_j and following the proof of Theorem 8 in [7, p. 174], which yields $a = \frac{\log(3/2)}{\log 2} = 0.585\ldots$ See also [10, Proposition 4].

Corollary 1. *For $c > 0.71$, there exists a sequence of values of X tending to infinity such that $F(X, z) = 1$ for all z, $1 - 1/(\log X)^c < z \le 1$.*

Proof: Let us choose $X = n_j$, with n_j satisfying (18), and $c > a$. For all $n < X$, we have

$$d(n) \le d(n_{j-1}) \le \frac{d(n_j)}{1 + (\log n_j)^{-a}} = \frac{D(X)}{1 + (\log X)^{-a}} < zD(X).$$

Thus $S(X, z) = \{n_j\}$, and $F(X, z) = 1$. □

5. Proofs of the lower estimates

Proof of Theorem 1: Let us denote by α_i/β_i the convergents of θ, defined by (4). It is known that θ cannot be too well approximated by rational numbers and, more precisely, there exists a constant κ such that

$$|q\theta - p| \gg q^{-\kappa} \tag{21}$$

for all integers $p, q \neq 0$ (cf. [4]). The best value of κ

$$\kappa = 7.616 \tag{22}$$

is due to G. Rhin (cf. [16]). It follows from (21) that

$$\beta_{i+1} = O\big(\beta_i^{\kappa}\big). \tag{23}$$

Let us introduce a positive real number δ which will be fixed later, and define $j = j(X, \delta)$ so that

$$\beta_j \leq (\log X)^{\delta} < \beta_{j+1}. \tag{24}$$

By Kronecker's theorem (cf. [6], Theorem 440), there exist two integers α and β such that

$$\left|\beta\theta - \alpha - \frac{\log z}{\log 2} - \frac{2}{\beta_j}\right| < \frac{2}{\beta_j} \tag{25}$$

and

$$\frac{\beta_j}{2} \leq \beta \leq \frac{3\beta_j}{2}. \tag{26}$$

Indeed, as α_j and β_j are coprime, one can write B, the nearest integer to $(\beta_j \frac{\log z}{\log 2} + 2)$, as $B = u_1\alpha_j - u_2\beta_j$ with $|u_1| \leq \beta_j/2$, and then $\alpha = \alpha_j + u_2$ and $\beta = \beta_j + u_1$ satisfy (25).

With the notation of Section 1, we write

$$\hat{n} = n_k \frac{p_{m+1} p_{m+2} \cdots p_{m+\beta}}{p_\ell p_{\ell-1} \cdots p_{\ell-\alpha+1}} \tag{27}$$

for X large enough. By (26), (24), and (6), (25) yields

$$\alpha \leq \beta\theta + \frac{\log(1/z)}{\log 2} \ll \max((\log X)^{\delta}, (\log X)^{\lambda}) \tag{28}$$

and

$$\alpha \geq \beta\theta - \frac{\log z}{\log 2} - \frac{4}{\beta_j} > \beta\theta - \frac{6}{\beta} + \frac{\log(1/z)}{\log 2} > 0$$

for X large enough. Thus, if we choose $\delta < 1$, from (3) and (1) we have $r_\ell = r_{\ell-1} = \cdots = r_{\ell-\alpha+1} = 1$. By (1) and the prime number theorem, we also have

$$p_\ell \sim \log X \tag{29}$$

and by (3), we have $r_{m+1} = r_{m+2} = \cdots = r_{m+\beta} = 1$ so that, by (25),

$$d(\hat{n}) = d(n_k)\frac{(3/2)^\beta}{2^\alpha} = d(n_k)\exp(\log 2(\beta\theta - \alpha)) \geq zd(n_k) = zD(X). \tag{30}$$

Now we need an upper bound for $\hat{n}/n_k$. First, it follows from (5) that for $i = o(m)$ we have

$$p_{m+i} - p_m \leq \max\left(p_{m+i}^\tau, \frac{i}{A}\log p_{m+i}\right) \tag{31}$$

and consequently,

$$\begin{aligned}\prod_{i=1}^{\beta}\frac{p_{m+i}}{p_m} &= \exp\left(\sum_{i=1}^{\beta}\log\frac{p_{m+i}}{p_m}\right) \leq \exp\left(\sum_{i=1}^{\beta}\frac{p_{m+i}-p_m}{p_m}\right)\\ &\leq \exp\left(\frac{\beta}{p_m}\max\left(p_{m+\beta}^\tau, \frac{\beta}{A}\log p_{m+\beta}\right)\right)\\ &\leq \exp\big(O\big(\max\big((\log X)^{\delta+\theta(\tau-1)}, (\log X)^{2\delta-\theta}\log\log X\big)\big)\big)\end{aligned} \tag{32}$$

by (26), (24), (3) and (1). Similarly, we get

$$\begin{aligned}\prod_{i=0}^{\alpha-1}\frac{p_\ell}{p_{\ell-i}} &\leq \exp\left(\frac{\alpha}{p_{\ell-\alpha+1}}\max\left(p_l^\tau, \frac{\alpha}{A}\log p_\ell\right)\right)\\ &\leq \exp\left(O\left(\max\left(\frac{(\log X)^\delta - \log z}{(\log X)^{1-\tau}}, \frac{((\log X)^\delta - \log z)^2}{\log X}\log\log X\right)\right)\right)\end{aligned} \tag{33}$$

by (28). Further, it follows from (3) and (25) that

$$\begin{aligned}\frac{p_m^\beta}{p_\ell^\alpha} &= p_\ell^{\beta\theta-\alpha}\big(1 + O\big(p_\ell^{(\tau-1)\theta}\big)\big)^\beta \leq p_\ell^{\frac{\log z}{\log 2}+\frac{4}{\beta_j}}\exp\big(O\big(\beta p_\ell^{(\tau-1)\theta}\big)\big)\\ &\leq \exp\left\{\left(\frac{\log z}{\log 2}\log p_\ell\right) + \frac{4\log p_\ell}{\beta_j} + \frac{\beta}{p_\ell^{(1-\tau)\theta}}\right\}.\end{aligned} \tag{34}$$

It follows from (23) and (24) that

$$\beta_j \gg (\log X)^{\delta/\kappa}. \tag{35}$$

Multiplying (32), (33) and (34), we get from (27) and (29):

$$\hat{n}/n_k \leq \exp\left\{(1 + o(1))\frac{\log z\log\log X}{\log 2}\right\} \tag{36}$$

if we choose δ in such a way that the error terms in (32), (33) and (34) can be neglected. More precisely, from (6) and (36), δ should satisfy:

$$\begin{aligned} \delta + \theta(\tau - 1) &< -\lambda_1 \\ 2\delta - \theta &< -\lambda_1 \\ \kappa\lambda_1 &< \delta < 1. \end{aligned}$$

It is possible to find such a δ if λ_1 satisfies

$$\lambda_1 < \min\left(\frac{(1-\tau)\theta}{1+\kappa}, \frac{\theta}{1+2\kappa}\right).$$

(4), (8) and (22) yield $\lambda_1 < 0.03157$.

For convenience, let us write

$$\hat{n} = p_1^{\hat{r}_1} p_2^{\hat{r}_2} \cdots p_t^{\hat{r}_t} \tag{37}$$

with, by (27), $t = \ell - \alpha$. It follows from (1) and (28) that

$$t = (1 + o(1))\frac{\log X}{\log\log X}; \quad p_t \sim \log X \tag{38}$$

and from (24) and (26) that

$$\hat{r}_i = 1 \quad \text{for } i \ge t - t^{9/10}. \tag{39}$$

Now, consider the integers v satisfying

$$P(t, v) \stackrel{\text{def}}{=} \frac{p_{t+1} p_{t+2} \cdots p_{t+v}}{p_{t-v+1} p_{t-v+2} \cdots p_t} \le \exp\left((1-\varepsilon)\frac{\log(1/z)\log X}{\log 2}\right) \tag{40}$$

and

$$v \le t^{9/10}. \tag{41}$$

By a calculation similar to that of (32) and (33), by (5) and the prime number theorem, for all v satisfying (41) and for all $1 \le i \le v$ we have:

$$\begin{aligned} \frac{p_{t+i}}{p_{t-v+i}} &= 1 + \frac{p_{t+i} - p_{t-v+i}}{p_{t-v+i}} \le 1 + (1+o(1))\frac{1}{p_t}\max\left(p_{t+v}^{\tau}, \frac{v}{A}\log p_{t+v}\right) \\ &= 1 + (1+o(1))\frac{1}{t}\max\left(t^{\tau}(\log t)^{\tau-1}, \frac{v}{A}\right) \end{aligned}$$

so that, by (38), the left hand side of (40) is

$$\begin{aligned} P(t, v) &= \prod_{i=1}^{v} \frac{p_{t+i}}{p_{t-v+i}} \\ &\leq \exp\left(v(1+o(1))\frac{1}{t}\max\left(t^{\tau}(\log t)^{\tau-1}, \frac{v}{A}\right)\right) \\ &= \exp\left((1+o(1))v\frac{\log\log X}{\log X}\max\left(\frac{(\log X)^{\tau}}{\log\log X}, \frac{v}{A}\right)\right) \\ &= \exp\left((1+o(1))v \max\left((\log X)^{\tau-1}, \frac{v}{A}\frac{\log\log X}{\log X}\right)\right). \end{aligned} \tag{42}$$

By (42), (40) follows from

$$\exp\left((1+o(1))v \max\left((\log X)^{\tau-1}, \frac{v}{A}\frac{\log\log X}{\log X}\right)\right) < \exp\left(\left(1-\frac{\varepsilon}{2}\right)\frac{\log(1/z)\log X}{\log 2}\right). \tag{43}$$

An easy computation shows that with

$$\left(1-\frac{5\varepsilon}{6}\right)\min\left(\left(\frac{A\log X}{\log 2}\log(1/z)^{1/2}\right), (\log X)^{1-\tau}\frac{\log\log X}{\log 2}\log(1/z)\right)$$

in place of v both (41) and (43) hold. Thus fixing v now as the greatest integer v satisfying (41) and (43), we have

$$v > \left(1-\frac{3\varepsilon}{4}\right)\min\left(\left(\frac{A\log X}{\log 2}\log(1/z)^{1/2}\right), (\log X)^{1-\tau}\frac{\log\log X}{\log 2}\log(1/z)\right). \tag{44}$$

Then it follows from (39) and (41) that

$$\hat{r}_{t-v+i} = 1 \quad \text{for } i = 1, 2, \ldots, v. \tag{45}$$

Let now $\mathcal{A}$ denote the set of the integers a of the form

$$a = 2^{\hat{r}_1} p_2^{\hat{r}_2} \cdots p_{t-v}^{\hat{r}_{t-v}} p_{i_1} \cdots p_{i_v} \quad \text{where } t-v+1 \leq i_1 < i_2 < \cdots < i_v \leq t+v. \tag{46}$$

Then, by (37), (46) and (30) we have

$$d(a) = d(\hat{n}) \geq zD(X). \tag{47}$$

Moreover, by (40) and (36) such an a satisfies

$$a = \frac{p_{i_1}p_{i_2}\cdots p_{i_v}}{p_{t-v+1}p_{t-v+2}\cdots p_t}\hat{n} \leq P(t, v)\hat{n} \leq n_k. \tag{48}$$

It follows from (47) and (48) that $a \in S(X, z)$ and

$$F(X, z) \geq |\mathcal{A}|. \tag{49}$$

The numbers $i_1, i_2, \ldots, i_v$ in (46) can be chosen in $\binom{2v}{v}$ ways so that

$$|\mathcal{A}| = \binom{2v}{v} > \exp\left(\left(1-\frac{\varepsilon}{8}\right)(\log 4)v\right). \tag{50}$$

Now (7) follows from (44), (49) and (50), and this completes the proof of Theorem 1. □

Proof of Theorem 2: By a theorem of Selberg [19, 9], if the real function $f(x)$ is increasing, $f(x) > x^{1/6}$ and $\frac{f(x)}{x} \searrow 0$, then there are infinitely many integers y such that

$$\pi(y + f(y)) - \pi(y) \sim \frac{f(y)}{\log y} \quad \text{and} \quad \pi(y) - \pi(y - f(y)) \sim \frac{f(y)}{\log y}. \tag{51}$$

We use this result with $f(y) = (1 - \frac{\varepsilon}{3}) \log y (\frac{y \log(1/z)}{\log 4})^{1/2}$ and for a y value satisfying (51), define t by

$$p_t \leq y < P_{t+1}. \tag{52}$$

Further, we define β_j (instead of (24)) so that $\beta_j \geq \frac{4 \log 2}{\varepsilon \log(1/z)}$ and α, β by (25) and (26); we set $\ell = t + \alpha$ and choose $X = n_k$ a h.c. number whose greatest prime factor is p_ℓ (such a number exists, see [13] or (59), (60) below). We define $\hat{n}$ by (27), and (30) and (38) still hold, while (36) becomes

$$\begin{aligned} \frac{\hat{n}}{n_k} &\leq \exp\left((1 + o(1)) \log\log X \left(\frac{\log z}{\log 2} + \frac{4}{\beta_j} \right) \right) \\ &\leq \exp\left((1 + o(1)) \frac{\log\log X}{\log 2} \log z (1 - \varepsilon) \right) \\ &\leq \exp\left(\frac{\log\log X}{\log 2} \log z \left(1 - \frac{\varepsilon}{2} \right) \right) \end{aligned} \tag{53}$$

for X large enough. Let v denote the greatest integer with

$$p_{t+v} \leq y + f(y) \quad \text{and} \quad p_{t-v} \geq y - f(y), \tag{54}$$

so that by the definition of y we have

$$v \sim \frac{f(y)}{\log y}. \tag{55}$$

By (38) and (52), we have

$$y \sim \log x. \tag{56}$$

Moreover, by (38), (54) and (55), we have

$$\begin{aligned} P(t, v) &\stackrel{\text{def}}{=} \prod_{i=1}^{v} \frac{p_{t+i}}{p_{t-v+i}} \leq \left(\frac{y + f(y)}{y - f(y)} \right)^v \\ &\leq \exp\left((1 + o(1)) \frac{f(y)}{\log\log X} \log\left(1 + 2\frac{f(y)}{y} \right) \right) \\ &= \exp\left((2 + o(1)) \frac{f^2(y)}{y \log\log X} \right) = \left(\frac{1}{\log 2} + o(1) \right) \left(1 - \frac{\varepsilon}{3} \right)^2 \log\log X \log(1/z). \end{aligned} \tag{57}$$

It follows from (53) and (57) that $P(t, v) < n_k/\hat{n}$ for X large enough and ε small enough.

Again, as in the proof of Theorem 1, we consider the set $\mathcal{A}$ of the integers a of the form (48). Then as in the proof of Theorem 1, by using (38) and (55) finally we obtain

$$F(X, z) \geq |\mathcal{A}| = \binom{2v}{v} > \exp\left(\left(1 - \frac{\varepsilon}{3}\right)(\log 4)v\right)$$
$$> \exp((1-\varepsilon)(\log 4)^{1/2}(\log X)^{1/2}(\log(1/z))^{1/2})$$

which completes the proof of Theorem 2. □

6. Superior highly composite numbers and benefits

Following Ramanujan (cf. [13]) we shall say that an integer N is superior highly composite (s.h.c.) if there exists $\varepsilon > 0$ such that for all positive integer M the following inequality holds:

$$d(M)/M^{\varepsilon} \leq d(N)/N^{\varepsilon}. \tag{58}$$

Let us recall the properties of s.h.c. numbers (cf. [13], [7, p. 174], [8–11]). To any ε, $0 < \varepsilon < 1$, one can associate the s.h.c. number:

$$N_{\varepsilon} = \prod_{p \leq x} p^{\alpha_p} \tag{59}$$

where

$$x = 2^{1/\varepsilon}, \quad \varepsilon = (\log 2)/\log x \tag{60}$$

and

$$\alpha_p = \left\lfloor \frac{1}{p^{\varepsilon} - 1} \right\rfloor. \tag{61}$$

For $i \geq 1$, we write

$$x_i = x^{\log(1+1/i)/\log 2} \tag{62}$$

and then (61) yields:

$$\alpha_p = i \Longleftrightarrow x_{i+1} < p \leq x_i. \tag{63}$$

A s.h.c. number is h.c. thus from (1) we deduce:

$$x \sim \log N_{\varepsilon}. \tag{64}$$

Let $P > x$ be the smallest prime greater than x. There is a s.h.c. number N' such that $N' \leq NP$ and $d(N') \leq 2d(N)$.

Definition. Let ε, $0 < \varepsilon < 1$, and N_ε satisfy (58). For a positive integer M, let us define the benefit of M by

$$\text{ben } M = \varepsilon \log \frac{M}{N_\varepsilon} - \log \frac{d(M)}{d(N_\varepsilon)}. \tag{65}$$

From (58), we have ben $M \geq 0$. Note that ben N depends on ε, but not on N_ε: If $N^{(1)}$ and $N^{(2)}$ satisfy (58), (65) will give the same value for ben M if we set $N_\varepsilon = N^{(1)}$or $N_\varepsilon = N^{(2)}$.

Now, let us write a generic integer:

$$M = \prod_p p^{\beta_p},$$

for $p > x$, let us set $\alpha_p = 0$, and define:

$$\text{ben}_p(M) = \varepsilon(\beta_p - \alpha_p) \log p - \log\left(\frac{\beta_{p+1}}{\alpha_{p+1}}\right). \tag{66}$$

From the definition (61) of α_p, we have $\text{ben}_p(M) \geq 0$, and (65) can be written as

$$\text{ben } M = \sum_p \text{ben}_p(M). \tag{67}$$

If $\beta_p = \alpha_p$, we have $\text{ben}_p(M) = 0$. If $\beta_p > \alpha_p$, let us set

$$\varphi_1 = \varphi_1(\varepsilon, p, \alpha_p, \beta_p) = (\beta_p - \alpha_p)\left(\varepsilon \log p - \log \frac{\alpha_p + 2}{\alpha_p + 1}\right) = (\beta_p - \alpha_p)\varepsilon \log\left(\frac{p}{x_{\alpha_p+1}}\right)$$

$$\psi_1 = \psi_1(\alpha_p, \beta_p) = (\beta_p - \alpha_p) \log\left(1 + \frac{1}{\alpha_p + 1}\right) - \log\left(1 + \frac{\beta_p - \alpha_p}{\alpha_p + 1}\right).$$

We have

$$\text{ben}_p(M) = \varphi_1 + \psi_1,$$

$\varphi_1 \geq 0$, $\psi_1 \geq 0$ and $\psi_1(\alpha_p, \alpha_p + 1) = 0$. Similarly, for $\beta_p < \alpha_p$, let us introduce:

$$\varphi_2 = \varphi_2(\varepsilon, p, \alpha_p, \beta_p) = (\alpha_p - \beta_p)\left(\log \frac{\alpha_p + 1}{\alpha_p} - \varepsilon \log p\right) = (\alpha_p - \beta_p)\, \varepsilon \log\left(\frac{x_{\alpha_p}}{p}\right)$$

$$\psi_2 = \psi_2(\alpha_p, \beta_p) = (\alpha_p - \beta_p)\log\left(1 - \frac{1}{\alpha_p + 1}\right) - \log\left(1 - \frac{\alpha_p - \beta_p}{\alpha_p + 1}\right).$$

We have $\varphi_2 \geq 0$, $\psi_2 \geq 0$, $\psi_2(\alpha_2, \alpha_p - 1) = 0$. Moreover, observe that ψ_1 is an increasing function of $\beta_p - \alpha_p$, and ψ_2 is an increasing function of $\alpha_p - \beta_p$, for α_p fixed.

We will prove:

Theorem 6. *Let* $x \to +\infty$, ε *be defined by* (60) *and* N_ε *by* (59). *Let* $\lambda < 1$ *be a positive real number*, μ *a positive real number not too large* ($\mu < 0.16$) *and* $B = B(x)$ *such that*

$x^{-\mu} \leq B(x) \leq x^{\lambda}$. *Then the number of integers M such that the benefit of M* (*defined by* (65)) *is smaller than B*, *satisfies*

$$\nu \leq \exp\left(\frac{23}{\sqrt{1-\mu}}\sqrt{Bx}\right) \tag{68}$$

for x large enough.

In [9], an upper bound for ν was given, with $B = x^{-\gamma}$. In order to prove Theorem 6, we shall need the following lemmas:

Lemma 1. *Let* $p_1 = 2, p_2 = 3, \dots, p_k$ *be the kth prime. For* $k \geq 2$ *we have* $k \log k \geq 0.46 p_k$.

Proof: By [18] for $k \geq 6$ we have

$$p_k \leq k(\log k + \log\log k) \leq 2k \log k$$

and the lemma follows after checking the cases $k = 2, 3, 4, 5$. □

Lemma 2. *Let* $p_1 = 2, p_2 = 3, \dots, p_k$ *be the kth prime. The number of solutions of the inequality*

$$p_1 x_1 + p_2 x_2 + \cdots + p_k x_k + \cdots \leq x \tag{69}$$

in integers $x_1, x_2, \dots,$ *is* $\exp((1 + o(1))\frac{2\pi}{\sqrt{3}}\sqrt{\frac{x}{\log x}})$.

Proof: The number $T(n)$ of partitions of n into primes satisfies (cf. [5]) $\log T(n) \sim \frac{2\pi}{\sqrt{3}}\sqrt{\frac{n}{\log n}}$, and the number of solutions of (69) is $\sum_{n \leq x} T(n)$. □

Lemma 3. *The number of solutions of the inequality*

$$x_1 + x_2 + \cdots + x_r \leq A \tag{70}$$

in integers $x_1, \dots, x_r$ *is* $\leq (2r)^A$.

Proof: Let $a = \lfloor A \rfloor$. It is well known that the number of solutions of (70) is

$$\binom{r+a}{a} = \frac{r+a}{a}\,\frac{r+a-1}{a-1}\cdots\frac{r+2}{2}\,\frac{r+1}{1} \leq (r+1)^a \leq (2r)^a.$$

□

Proof of Theorem 6: Any integer M can be written as

$$M = \frac{A}{D} N_\varepsilon, \ (A, D) = 1 \text{ and } D \text{ divides } N_\varepsilon.$$

First, we observe that, if p^y divides A and ben $M \le B$, we have for x large enough:

$$y \le x. \tag{71}$$

Indeed, by (61), we have

$$\alpha_p \le \frac{1}{p^\varepsilon - 1} \le \frac{1}{\varepsilon \log p} = \frac{\log x}{\log 2 \log p} \le \frac{\log x}{(\log 2)^2} \le 3 \log x.$$

It follows that

$$\begin{aligned} B \ge \text{ben}\, M \ge \text{ben}_p(AN_\varepsilon) \ge \psi_1(\alpha_p, \alpha_p + y) \\ = y \log\left(1 + \frac{1}{\alpha_p + 1}\right) - \log\left(1 + \frac{y}{\alpha_p + 1}\right) \\ \ge \frac{y}{\alpha_p} - \log(1+y) \ge \frac{y}{3 \log x} - \log(1+y), \end{aligned}$$

and since $B \le x^\lambda$, this inequality does not hold for $y > x$ and x large enough.
Further we write $A = A_1 A_2 \cdots A_6$ with $(A_i, A_j) = 1$ and

$$\begin{aligned} p \mid A_1 &\Longrightarrow p > 2x \\ p \mid A_2 &\Longrightarrow x < p \le 2x \\ p \mid A_3 &\Longrightarrow 2x_2 < p \le x \\ p \mid A_4 &\Longrightarrow x_2 < p \le 2x_2 \\ p \mid A_5 &\Longrightarrow 2x_3 < p \le x_2 \\ p \mid A_6 &\Longrightarrow p \le 2x_3, \end{aligned}$$

where x_2 and x_3 are defined by (62). Similarly, we write $D = D_1 D_2 \ldots D_5$, with $(D_i, D_j) = 1$ and

$$\begin{aligned} p \mid D_1 &\Longrightarrow x/2 < p \le x \\ p \mid D_2 &\Longrightarrow x_2 < p \le x/2 \\ p \mid D_3 &\Longrightarrow x_2/2 < p \le x_2 \\ p \mid D_4 &\Longrightarrow 2x_3 < p \le x_2/2 \\ p \mid D_5 &\Longrightarrow p \le 2x_3. \end{aligned}$$

We have

$$\text{ben}\, M = \sum_{i=1}^{6} \text{ben}(A_i N_\varepsilon) + \sum_{i=1}^{5} \text{ben}(N_\varepsilon / D_i),$$

and denoting by ν_i (resp. ν_i') the number of solutions of

$$\text{ben}(A_i N_\varepsilon) \le B \quad (\text{resp. } \text{ben}(N_\varepsilon / D_i) \le B),$$

we have

$$\nu \leq \prod_{i=1}^{6} \nu_i \prod_{i=1}^{5} \nu_i'. \tag{72}$$

In (72), we shall see that the main factors are ν_2 and ν_1' and the other ones are negligible.

Estimation of ν_2. Let us denote the primes between x and $2x$ by $x < P_1 < P_2 < \cdots < P_r \leq 2x$, and let

$$A_2 = P_1^{y_1} P_2^{y_2} \cdots P_r^{y_r}, \quad y_i \geq 0.$$

From the Brun-Titchmarsh inequality, it follows for $i \geq 2$ that

$$i = \pi(P_i) - \pi(x) \leq 2\frac{P_i - x}{\log(P_i - x)} \leq 2\frac{P_i - x}{\log 2(i-1)}$$

and it follows from Lemma 1:

$$P_i - x \geq \frac{i}{2}\log 2(i-1) \geq \frac{i \log i}{2} \geq 0.23 p_i.$$

By (60) and (61) we have $\alpha_{P_i} = 0$ and

$$\begin{aligned}\operatorname{ben}(A_2 N_\varepsilon) &\geq \sum_{i=2}^{r} \varphi_1(\varepsilon, P_i, 0, y_i) = \sum_{i=2}^{r} \varepsilon y_i \log(P_i/x) \\ &\geq \sum_{i=2}^{r} \varepsilon y_i \frac{P_i - x}{P_i} \geq \sum_{i=2}^{r} \frac{\varepsilon y_i}{2x}(P_i - x) \geq \sum_{i=2}^{r} 0.115 \frac{\varepsilon y_i}{x} p_i.\end{aligned}$$

By (71), the number of possible choices for y_1 is less than $(x+1)$, so that ν_2 is certainly less than $(x+1)$ times the number of solutions of:

$$\sum_{i=2}^{\infty} p_i y_i \leq \frac{Bx}{\varepsilon(0.115)} \leq 12.6 Bx \log x,$$

and, by Lemma 2,

$$\nu_2 \leq (x+1)\exp\left\{(1+o(1))\frac{2\pi}{\sqrt{3}}\sqrt{\frac{12.6 Bx \log x}{\log(Bx)}}\right\} \leq \exp\left(\frac{13\sqrt{Bx}}{\sqrt{1-\mu}}\right).$$

Estimation of ν_1. First we observe that, if a large prime P divides M and $\operatorname{ben} M \leq B$ then we have:

$$B \geq \operatorname{ben} M \geq \operatorname{ben}_p(M) \geq \varphi_1(\varepsilon, P, 0, \beta_p) \geq \varepsilon \log(P/x),$$

so that

$$P \le x \exp(B/\varepsilon) = x \exp\left(\frac{B \log x}{\log 2}\right).$$

If λ is large, we divide the interval $[0, \lambda]$ into equal subintervals: $[\lambda_i, \lambda_{i+1}]$, $0 \le i \le s-1$, such that $\lambda_{i+1} - \lambda_i < \frac{1-\lambda}{2}$. We set $T_0 = 2x$, $T_i = x \exp(x^{\lambda_i})$ for $1 \le i \le s-1$, and $T_s = x \exp(\frac{B \log x}{\log 2})$. If $\lambda < \frac{1}{3}$, there is just one interval in the subdivision. Further, we write $A_1 = a_1 a_2 \dots a_s$ with $p \mid a_i \Longrightarrow T_{i-1} < p \le T_i$, and if we denote the number of solutions of ben $(a_i N_\varepsilon) \le B$ by $\nu_1^{(i)}$ clearly we have

$$\nu_1 \le \prod_{i=1}^{s} \nu_1^{(i)}.$$

To estimate $\nu_1^{(i)}$ let us denote the primes between T_{i-1} and T_i by $T_{i-1} < P_1 < \cdots < P_r \le T_i$, and let $a_i = P_1^{y_1} \cdots P_r^{y_r}$. We have

$$B \ge \text{ben}(a_i N_\varepsilon) \ge \sum_{i=1}^{r} \varphi_1(\varepsilon, P_i, 0, y_i) = \sum_{i=1}^{r} \varepsilon y_i \log \frac{P_i}{x}$$
$$\ge \sum_{i=1}^{r} \varepsilon y_i \log \frac{T_{i-1}}{x}.$$

If $i = 1$, $T_0 = 2x$, this implies $\sum_{i=1}^{r} y_i \le \frac{B(\log x)}{(\log 2)^2} \le 3B \log x$, and by Lemma 3,

$$\nu_1^{(1)} \le \exp(3B \log x \log(2r)) \le \exp(3B \log x \log T_1) \le \exp((1 + o(1))Bx^{\lambda_1}).$$

If $i > 1$, we have $\sum_{i=1}^{r} y_i \le \frac{B}{\varepsilon x^{\lambda_{i-1}}}$, and by Lemma 3,

$$\nu_1^{(i)} \le \exp\left(\frac{B}{\varepsilon x^{\lambda_{i-1}}} \log T_i\right) \le \exp\left\{(1 + o(1))Bx^{\lambda_i - \lambda_{i-1}}\right\},$$

and from the choice of the λ_i's, one can easily see that, for $B \le x^\lambda$, $\nu_1 = \prod_{i=1}^{s} \nu_1^{(i)}$ is negligible compared with ν_2.

The other factors of (72) are easier to estimate:

Estimation of ν_3. Let us denote the primes between $2x_2$ and x by $2x_2 < P_r < P_{r-1} < \cdots < P_1 \le x$. By (62) and (4), $x_2 = x^\theta$, and by (63), $\alpha_{P_i} = 1$. Let us write $A_3 = P_1^{y_1} \cdots P_r^{y_r}$. We have

$$B \ge \text{ben}(A_3 M) \ge \sum_{i=1}^{r} \varphi_1(\varepsilon, P_i, 1, 1 + y_i) = \sum_{i=1}^{r} \varepsilon y_i \log \frac{P_i}{x_2} \ge \sum_{i=1}^{r} \frac{(\log 2)^2}{\log x} y_i.$$

So, $\sum_{i=1}^{r} y_i \le B \log x/(\log 2)^2 \le 3B \log x$, and by Lemma 3,

$$\nu_3 \le \exp(3B \log x \log(2r)) \le \exp(3B(\log x)^2).$$

Estimation of ν_4. Replacing x by x_2 the upper bound obtained for ν_2 becomes:

$$\nu_2 = \exp(O(\sqrt{Bx_2})) = \exp(O(\sqrt{Bx^\theta})).$$

Estimation of ν_5. Replacing x by x_2, the upper bound obtained for ν_3 becomes:

$$\nu_5 \le \exp(3B\log x \log x_2) = \exp(3\theta B(\log x)^2).$$

Estimation of ν_6. Let $p_1, p_2, \dots, p_r \le 2x_3$ be the first primes and write $A_6 = p_1^{y_1} p_2^{y_2} \cdots p_r^{y_r}$. By (71), $y_i \le x$, and thus by (62),

$$\nu_6 \le (x+1)^r \le (x+1)^{x_3} = \exp(x^{1-\theta}\log(x+1))$$

and for $B \ge x^{-\mu}$ and $\mu < 0.16$, this is negligible compared with ν_2.

Estimation of ν_1'. Let us denote the primes between $\frac{x}{2}$ and x by $\frac{x}{2} < P_r < P_{r-1} < \cdots < P_1 \le x$, and let $D_1 = P_1^{y_1} \cdots P_r^{y_r}$. We have $\alpha_{P_i} = 1$ and since D_1 divides N_ε, $y_i = 0$ or 1. By a computation similar to that of ν_2, we obtain

$$B \ge \operatorname{ben}\frac{N_\varepsilon}{D_1} \ge \sum_{i=2}^{r}\varphi_2(\varepsilon, P_i, 1, y_i) = \sum_{i=2}^{r}\varepsilon y_i \log\frac{x}{P_i} \ge \sum_{i=2}^{r}\varepsilon y_i \frac{x-P_i}{x},$$

and by using the Brun-Titchmarsch inequality and Lemma 1, it follows that

$$\sum_{i=2}^{r} p_i y_i \le \frac{Bx}{0.23\varepsilon} \le 6.3\, Bx\log x.$$

Thus, as y_1 can only take 2 values, by Lemma 2 we have

$$\nu_1' \le 2\exp((1+o(1))\frac{2\pi}{\sqrt{3}}\sqrt{\frac{6.3Bx\log x}{\log(Bx)}} \le \exp(9.2\sqrt{Bx}).$$

Estimation of ν_2'. By an estimation similar to that of ν_3, replacing φ_1 by φ_2 and using Lemma 3, we get

$$\nu_2' \le \exp(3B\log^2 x).$$

Estimation of ν_3'. Replacing x by x_2, it is similar to that of ν_1' and we get

$$\nu_3' = \exp(O(\sqrt{Bx_2})).$$

Estimation of ν_4'. Replacing x by x_2, we get, as for ν_2',

$$\nu_4' \le \exp(3B\log x\log x_2) = \exp(3\theta B\log^2 x).$$

Estimation of ν_5'. As we have seen for ν_6, we have

$$D_5 = p_1^{y_1} \cdots p_r^{y_r}$$

with $y_i \le \alpha_{p_i} \le 3 \log x$ and $r \le \pi(2x_3) \le x_3$. Thus

$$\nu_5' \le (1 + 3 \log x)^r \le \exp(x^{1-\theta} \log(1 + 3 \log x)).$$

By formula (68) and the estimates of ν_i and ν_i', the proof of Theorem 6 is completed. □

By a more careful estimate, it would have been possible to improve the constant in (68). However, using the Brun-Titchmarsch inequality we loose a factor $\sqrt{2}$, and we do not see how to avoid this loss. A similar method was used in [3]. Also, the condition $\mu < 0.16$ can be replaced easily by $\mu < 1$.

7. Proof of Theorem 3

We shall need the following lemmas:

Lemma 4. *Let n_j the sequence of h.c. numbers. There exists a positive real number c such that for j large enough, the following inequality holds*:

$$\frac{n_{j+1}}{n_j} \le 1 + \frac{1}{(\log n_j)^c}.$$

Proof: This result was first proved by Erdős in [2]. The best constant c is given in [8]:

$$c = \frac{\log(15/8)}{\log 8}(1 - \tau_0) = 0.1405\ldots$$

with the value of τ_0 given by (8). □

Lemma 5. *Let n_j be a h.c. number, and N_ε the superior h.c. number preceding n_j. Then the benefit of n_j (defined by* (65)) *satisfies*:

$$\text{ben } n_j = O((\log n_j)^{-\gamma}).$$

Proof: This is Theorem 1 of [8]. The value of γ is given by

$$\gamma = \theta(1 - \tau_0)/(1 + \kappa) = 0.03157\ldots$$

where θ, τ_0 and κ are defined by (4), (8) and (22). □

To prove Theorem 3, first recall that n_k is defined so that

$$n_k \le X < n_{k+1}. \tag{73}$$

We define N_ε as the largest s.h.c. number $\le n_k$. Now let $n \in S(X, z)$. We get from (65):

$$\text{ben } n = \varepsilon \log \frac{n}{N_\varepsilon} - \log \frac{d(n)}{d(N_\varepsilon)},$$
$$\text{ben } n_k = \varepsilon \log \frac{n_k}{N_\varepsilon} - \log \frac{d(n_k)}{d(N_\varepsilon)}$$

and, subtracting,

$$\text{ben } n = \text{ben } n_k + \varepsilon \log \frac{n}{n_k} - \log \frac{d(n)}{d(n_k)}.$$

But $n \in S(X, z)$ so that $n \le X$ and $d(n) \ge z d(n_k)$. Thus

$$\text{ben } n \le \text{ben } n_k + \varepsilon \log \frac{X}{n_k} + \log(1/z).$$

By (73) and Lemma 4, we have $n_k \sim X$, and by (60), (64), (73) and Lemma 4, we have

$$\varepsilon \log \frac{X}{n_k} \le \varepsilon \log \frac{n_{k+1}}{n_k} \le \frac{1}{(\log X)^{c+o(1)}}.$$

By Lemma 5,

$$\text{ben } n \le B = \log \frac{1}{z} + O(\log X)^{-\gamma}.$$

Applying Theorem 6 completes the proof of Theorem 3. □

8. An upper bound for $d(n_{j+1})/d(n_j)$

We will prove:

Theorem 7. *There exists a constant $c > 0$ such that for n_j large enough, the inequality*

$$\frac{d(n_{j+1})}{d(n_j)} \le 1 + \frac{1}{(\log n_j)^c}$$

holds. Here c can be chosen as any number less than γ defined in Lemma 5.

Proof: Let N_ε the s.h.c. number preceding n_j. We have by Lemma 5 $\text{ben}(n_j) = O((\log n_j)^{-\gamma})$ and $\text{ben}(n_{j+1}) = O((\log n_j)^{-\gamma})$. Further, it follows from (65) that

$$\log \frac{d(n_{j+1})}{d(n_j)} = \varepsilon \log \frac{n_{j+1}}{n_j} + \text{ben}(n_{j+1}) - \text{ben}(n_j) \le \log \frac{n_{j+1}}{n_j} + \text{ben}(n_{j+1})$$

which, by using Lemma 4 and Lemma 5, completes the proof of Theorem 7. □

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