```
> ###### COMPUTATION of the paper
> ###### Estimates of Li(theta(x))-pi(x) and the Riemann hypothesis
```

```
> ###### Formula (2.6)
```

```
> L:=proc(N,t) local k;
      Li(t)-sum(factorial(k-1)*t/log(t)^k, k = (1 .. N));
  end;
```

$L :=$ **proc**(N, t) **local** k; $\mathrm{Li}(t) - (sum(\mathrm{factorial}(k-1) * t / \log(t) \hat{} k, k = 1 .. N))$ **end proc**

```
> for N from 1 to 5 do
     print("N=",N,"L(N,t)=",L(N,t),"L'(N,t)=",diff(L(N,t),t));
  od;
```

$$\text{"N=", } 1, \text{ "L(N,t)=", } \mathrm{Li}(t) - \frac{t}{\ln(t)}, \text{ "L'(N,t)=", } \frac{1}{\ln(t)^2}$$

$$\text{"N=", } 2, \text{ "L(N,t)=", } \mathrm{Li}(t) - \frac{t}{\ln(t)} - \frac{t}{\ln(t)^2}, \text{ "L'(N,t)=", } \frac{2}{\ln(t)^3}$$

$$\text{"N=", } 3, \text{ "L(N,t)=", } \mathrm{Li}(t) - \frac{t}{\ln(t)} - \frac{t}{\ln(t)^2} - \frac{2\,t}{\ln(t)^3}, \text{ "L'(N,t)=", } \frac{6}{\ln(t)^4}$$

$$\text{"N=", } 4, \text{ "L(N,t)=", } \mathrm{Li}(t) - \frac{t}{\ln(t)} - \frac{t}{\ln(t)^2} - \frac{2\,t}{\ln(t)^3} - \frac{6\,t}{\ln(t)^4}, \text{ "L'(N,t)=", } \frac{24}{\ln(t)^5}$$

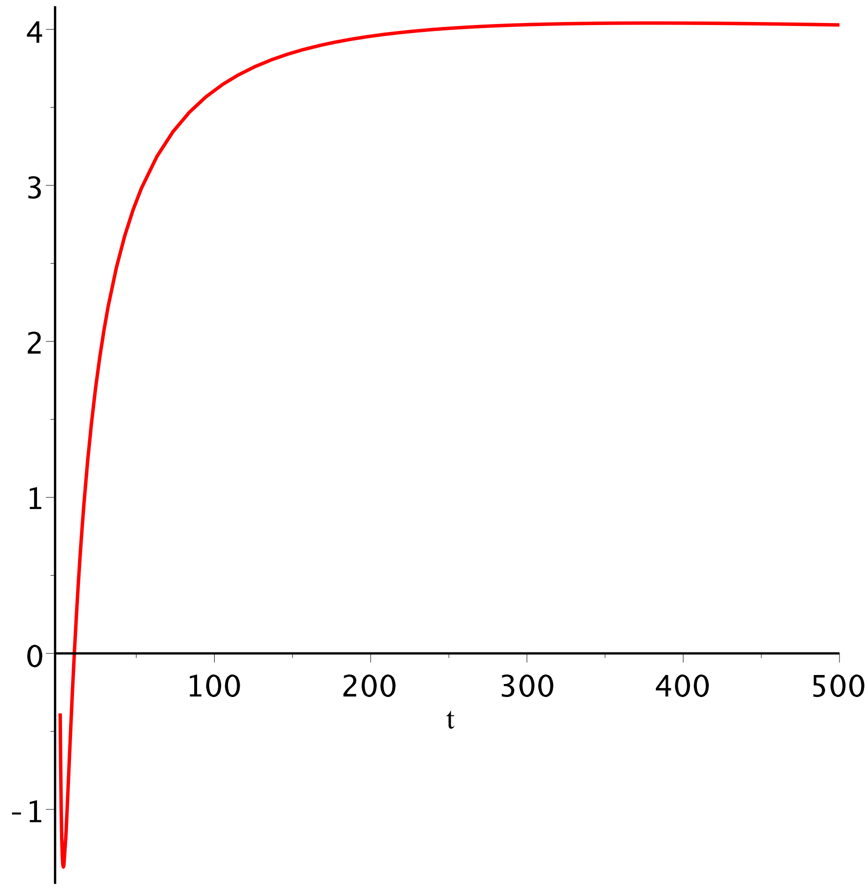
$$\text{"N=", } 5, \text{ "L(N,t)=", } \mathrm{Li}(t) - \frac{t}{\ln(t)} - \frac{t}{\ln(t)^2} - \frac{2\,t}{\ln(t)^3} - \frac{6\,t}{\ln(t)^4} - \frac{24\,t}{\ln(t)^5}, \text{ "L'(N,t)=", } \frac{120}{\ln(t)^6}$$

```
> ######  STUDY of F2 (Lemma 2.2)
  ######################################
```

```
> F2:=L(2,t)*log(t)^3/t; # Study of F2 on (1,infinity)
```

$$F2 := \frac{\left(\mathrm{Li}(t) - \frac{t}{\ln(t)} - \frac{t}{\ln(t)^2}\right) \ln(t)^3}{t}$$

```
> plot(F2,t=1..500);
```




```
> limit(F2,t=1,right); limit(F2,t=infinity);
```

$$0$$

$$2$$

```
> f1:=expand(t^2*diff(F2,t)/\log(t)^2); # f1 and diff(F2,t) have the same sign
```

$$f1 := -\frac{t}{\ln(t)^2} + 3\,\mathrm{Li}(t) - \frac{2\,t}{\ln(t)} - \ln(t)\,\mathrm{Li}(t) + t$$

```
> f2:=expand(t*(diff(f1,t)));  # f2 and diff(f1,t) have the same sign
```

$$f2 := \frac{t}{\ln(t)^2} + \frac{2\,t}{\ln(t)^3} + \frac{t}{\ln(t)} - \mathrm{Li}(t)$$

```
> f3:=diff(f2,t); # f3 is negative, thus f2 is decreasing
```

$$f3 := -\frac{6}{\ln(t)^4}$$

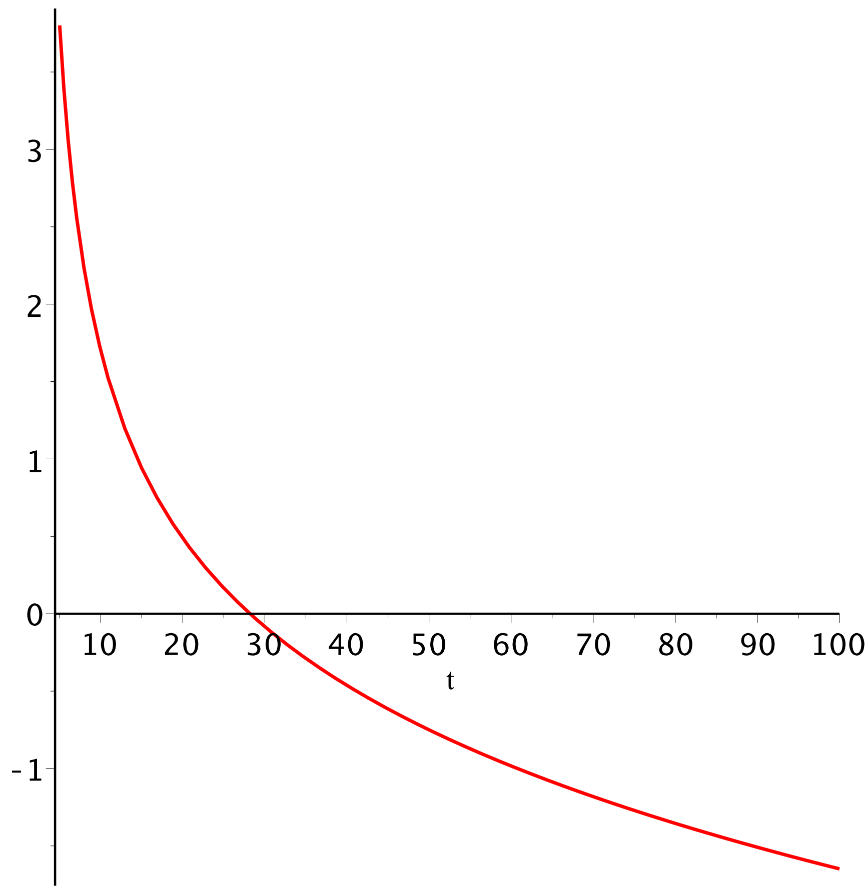
```
> limit(f2,t=1,right); limit(f2,t=infinity); # f2 vanishes in t2
```

$$\infty$$

$-\infty$

```
> plot(f2,t=5..100); t2:=fsolve(f2=0,t=20..40);
  # f2 is positive for t < 28.19...   and negative for t > 28.19...

  # f1 is increasing for t < 28.19... and decreasing for t > 28.19.
  ..
```



$$t2 := 28.19524628$$

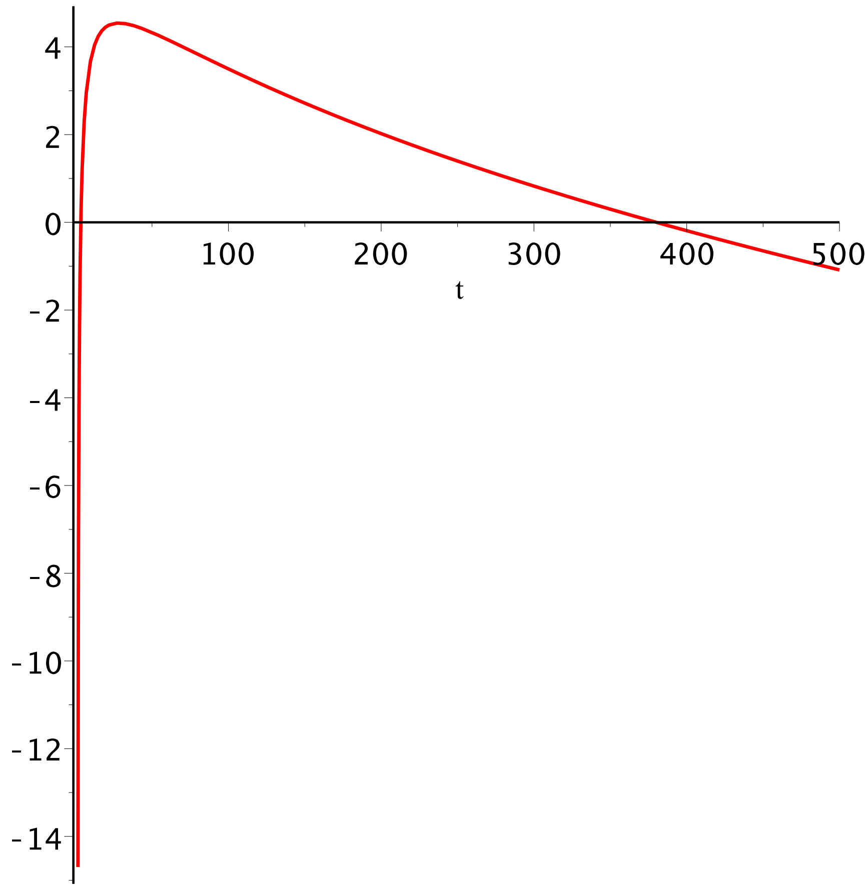
```
> maxf1:=evalf(subs(t=t2,f1)); limit(f1,t=1,right); limit(f1,t=
  infinity);
  # f1 has two zeros
```

$$maxf1 := 4.54378772$$

$$-\infty$$

$$-\infty$$

```
> plot(f1,t=1.5..500); t3:=fsolve(f1=0,t=1.5..24);t4:=fsolve(f1=0,
  t=80..1000);
  # F2 is decreasing for t < t3 = 3.38, increasing up to t = t4 =
  380.15
  # and decreasing for  t > 380.15
```



$$t3 := 3.384879470$$

$$t4 := 380.1544505$$

```
> limit(F2,t=1,right); minF2:=evalf(subs(t=t3,f)); maxF2:=evalf
  (subs(t=t4,f)); limit(F2,t=infinity);
```

$$0$$

$$minF2 := f$$

$$maxF2 := f$$

$$2$$

```
> # Conclusion : for t > 1, one has
  #         Li(t) < t/log(t) + t/log^2(t) + 4.05 t/log^3(t).
  # For t > t0 > 381, one has
  #         Li(t) < t/log(t) + t/log^2(t) +c t/log^3(t)
  # with    c = F2(t0) = (Li(t0) - t0/log(t0)-t0/log^2(t0))/
  (t0/log^3(t0)).
```

```
> fsolve(F2=2,t=10..100); # Observe that it is t2
```

$$28.19524628$$

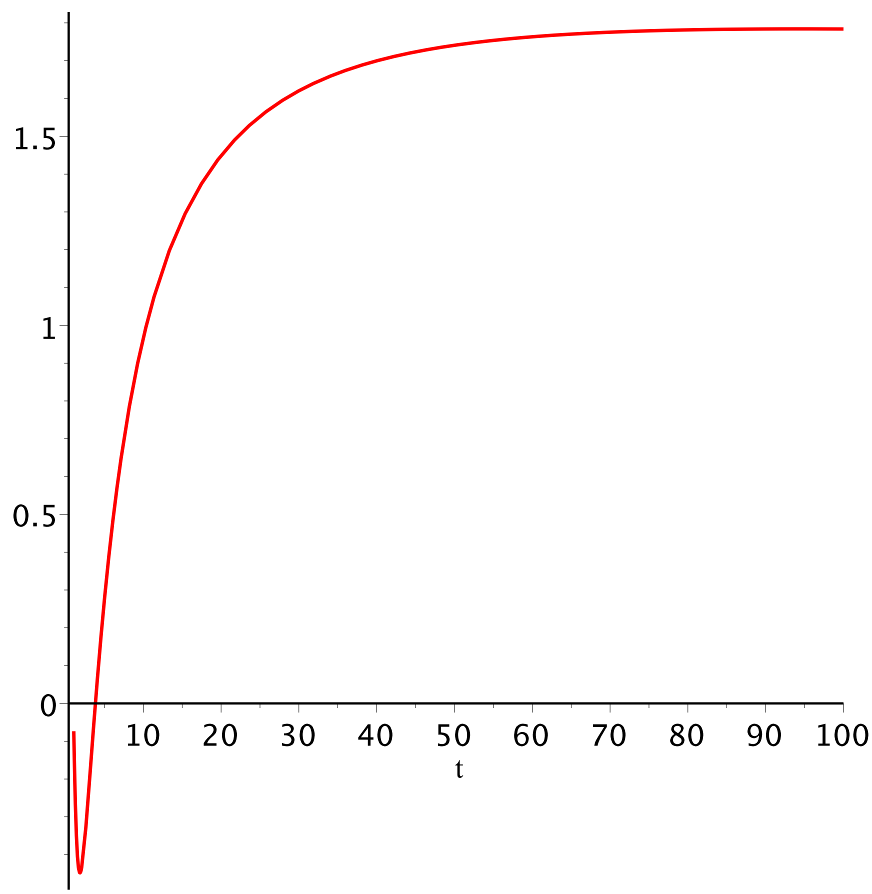
```
> # Lower bound : for t > 29, one has
  #         Li(t) > t/log(t) + t/log^2(t) + 2t/log^3(t).
```

```
> ##### STUDY OF F1  ############################################
```

```
> F1:=(L(1,t))*log(t)^2/t; # Study of F1 on (1,infinity)
```

$$F1 := \frac{\left(\mathrm{Li}(t) - \frac{t}{\ln(t)}\right)\ln(t)^2}{t}$$

```
> plot(F1,t=1..100);
```




```
> limit(F1,t=1,right); limit(F1,t=infinity);
```

$$0$$

$$1$$

```
> f1:=expand(t^2*diff(F1,t)/\log(t)); # f1 and diff(F1,t) have the
  same sign
```

$$f1 := -\frac{t}{\ln(t)} + 2\,\mathrm{Li}(t) - \ln(t)\,\mathrm{Li}(t) + t$$

```
> f2:=expand(t*(diff(f1,t)));  # f2 and diff(f1,t) have the same
  sign
```

$$f2 := \frac{t}{\ln(t)} + \frac{t}{\ln(t)^2} - \mathrm{Li}(t)$$

```
> f3:=diff(f2,t); # f3 is negative, thus f2 is decreasing
```

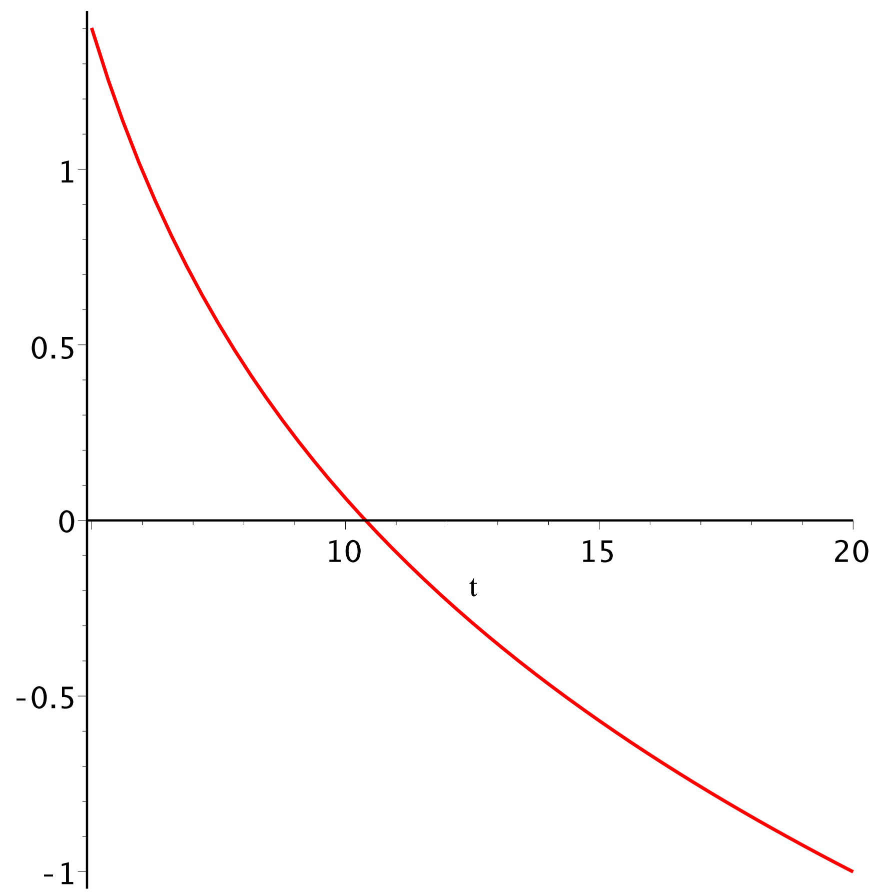
$$f3 := -\frac{2}{\ln(t)^3}$$

```
> limit(f2,t=1,right); limit(f2,t=infinity); # f2 vanishes in t2
```

$$\infty$$

$$-\infty$$

```
> plot(f2,t=5..20); t2:=fsolve(f2=0,t=10..12);
  # f2 is positive for t < 10.39 and negative for t > 10.39.
  # f1 is increasing for t < 10.39 and decreasing for t > 10.39.
```



$$t2 := 10.39730042$$

```
> maxf1:=evalf(subs(t=t2,f1)); limit(f1,t=1,right); limit(f1,t=
```

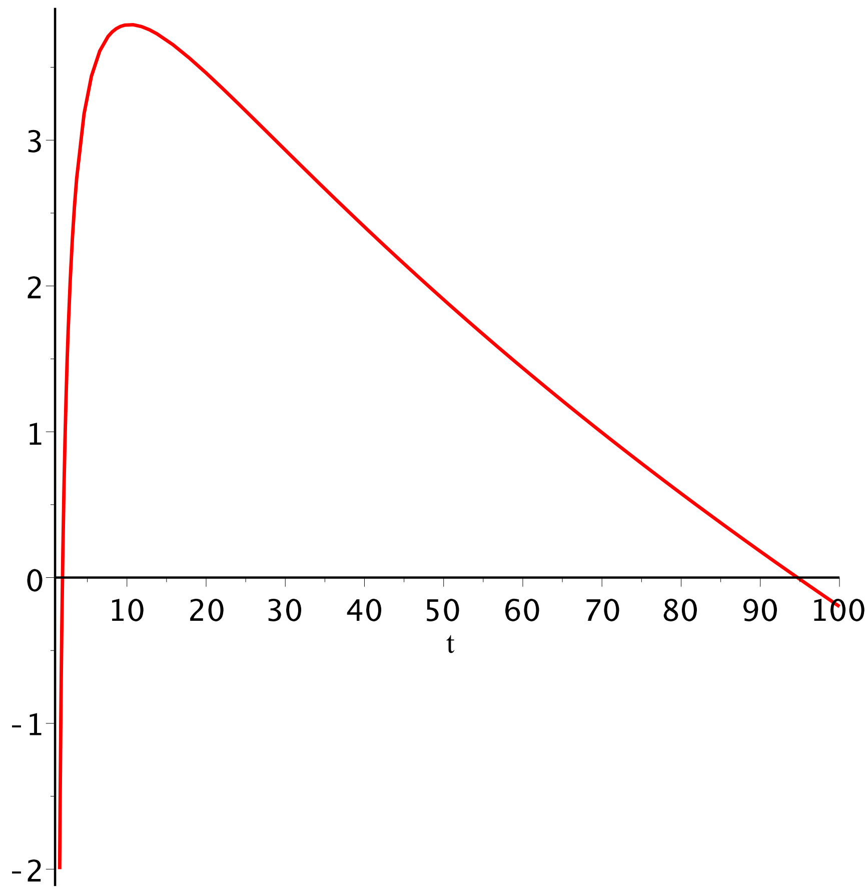
```
infinity);
# f1 has two zeros
```

$$maxf1 := 3.792670618$$

$$-\infty$$

$$-\infty$$

```
> plot(f1,t=1.5..100); t3:=fsolve(f1=0,t=1.5..4);t4:=fsolve(f1=0,t=
80..100);
# F1 is decreasing for t < t3 = 1.85, increasing up to t = t4 =
94.69 and decreasing for  t > 94.69
```



$$t3 := 1.857239995$$

$$t4 := 94.69496961$$

```
> limit(F1,t=1,right); minF1:=evalf(subs(t=t3,F1)); maxF1:=evalf
(subs(t=t4,F1)); limit(F1,t=infinity);
```

$$0$$

$$minF1 := -0.4483218962$$

$$maxF1 := 1.784110509$$

$$1$$

```
> # Conclusion : for t > 1, one has
  #               Li(t) < t/log(t)  + 1.785 t/log^2(t).
  # For t > t0 > 95, one has
  #               Li(t) < t/log(t)  +  c t/log^2(t)
  # with   c = F1(t0) = (Li(t0) - t0/log(t0))/(t0/log^2(t0)).
```

```
> fsolve(F1=1,t=4..20); # Observe that it is t2
```

10.39730042

```
> # Lower bound : for t > 11, one has
  #               Li(t) > t/log(t) + t/log^2(t).
```

```
> read "lithetax0.mpl": #### reading the Maple code
> valA(26):
```

"VALUES of A(p)"

"p=", 2, −1.76199358637, "   p=", 3, −1.28143753012
"p=", 5, −0.49101363449, "   p=", 7, −0.15416569740
"p=", 11, 0.13013061815, "   p=", 13, 0.29932110395
"p=", 17, 0.45224501392, "   p=", 19, 0.55128200174
"p=", 23, 0.64424524881, "   p=", 29, 0.7525373544
"p=", 31, 0.8292166098, "   p=", 37, 0.9152123365
"p=", 41, 0.9919784456, "   p=", 43, 1.0481569703
"p=", 47, 1.0990087170, "   p=", 53, 1.1551108052
"p=", 59, 1.2144821349, "   p=", 61, 1.2597979598
"p=", 67, 1.3080919862, "   p=", 71, 1.3516132764
"p=", 73, 1.3843640927, "   p=", 79, 1.4197394660
"p=", 83, 1.4515166795, "   p=", 89, 1.4851882541
"p=", 97, 1.5249690320, "   p=", 101, 1.5610057927

"p=", 401, 2.5141881093, "   p=", 409, 2.5266906222
"p=", 3643, 4.545342085, "   p=", 33647, 7.94512429

```
> lem33():
```

"NUMERICAL VALUES in LEMMA 3.3"

"y0=", 8.3, "B(y0)-L_1(y0)=", -0.00137957661

"y1=", 599, "B(y1)-L_1(y1)-sqrt(y1)/(4*pi)=", -4.80566101966

"y2=", 2903, "B(y2)-L_1(y2)+sqrt(y2)/(4*pi)=", 0.00671140798

```
> coro31():
```

"NUMERICAL VALUE in COROLLARY 3.1"

"7.993-log(10^8)^3/8/pi/(10^8)^(1/4)-18/10000*log(10^8)^5/sqrt(10^8)=", 5.12422288905

(1)

```
> Q(5,10^8): ##### Proposition 3.4
```

"CALCULATION of Q(kappa_1,x) by FORMULA (3.21)"

"ka1=", 5, "x=", 100000000, "log x=", 18.4206807440, "rac x=", 10000.0000

"T1=", 13.2882262576

"T2=", $\frac{1}{150}$

"T3=", 1.90641245384

"k=", 3, "T4[k]=", 4.25018754232

"k=", 4, "T4[k]=", 1.31439598814

"k=", 5, "T4[k]=", 0.654506404340

"T4=", 6.21908993479

"T5=", 3.59789355388

"T6=", 0.00277023429986

"T7=", 0.190884598098

"T=Q(kappa_1,x)=", 25.2119436992 **(2)**

```
> Qminka1(10^8):
```

"DETERMINATION of kappa MINIMIZING Q(kappa_1,x) for x=", 100000000

"kappa_1=", 3, "T=Q(kappa_1,x)=", 28.7059933149

"kappa_1=", 4, "T=Q(kappa_1,x)=", 25.5867437409

"kappa_1=", 5, "T=Q(kappa_1,x)=", 25.2119436992

"kappa_1=", 6, "T=Q(kappa_1,x)=", 25.4036264172

"kappa_1=", 7, "T=Q(kappa_1,x)=", 25.8047000159

"x=", 100000000, "log10(x)=", 8.00, "minimum of Q=", 25.2119436992, "for ka1=", 5

(3)

```
> prop36i(): ##### Proposition 3.6
```

"SOLUTION of PROPOSITION 3.6 (i)"

"A(11)=", 0.13013061815, "A(7)=", -0.15416569740

(4)

```
> prop36ii(10^8,10^7,25.22):
```

"SOLUTION of PROPOSITION 3.6 (ii), (iii) and (v)"

"pmax=", 100000000, "log10(pmax)=", 8., "pas=", 10000000, "const=", 25.22

"p=", 9999991, "pp=", 10000019, "A(p)=", 40.3154817357

"C(p)=", 20.4035449096, "ctilda(p)=", 21.8925278242, "time=", 841.549

"p=", 19999999, "pp=", 20000003, "A(p)=", 50.6337982521

"C(p)=", 19.3938820562, "ctilda(p)=", 20.9469484886, "time=", 1634.142

"p=", 29999999, "pp=", 30000001, "A(p)=", 58.2924552897

"C(p)=", 19.0841037472, "ctilda(p)=", 20.6746307381, "time=", 2408.632

"p=", 39999983, "pp=", 40000003, "A(p)=", 64.2853343576

"C(p)=", 18.6985342868, "ctilda(p)=", 20.3156295027, "time=", 3178.803

"p=", 49999991, "pp=", 50000017, "A(p)=", 69.0536288220

"C(p)=", 18.1320334192, "ctilda(p)=", 19.7697428521, "time=", 3943.884

"p=", 59999999, "pp=", 60000011, "A(p)=", 73.5594785313

"C(p)=", 17.9085957060, "ctilda(p)=", 19.5631548918, "time=", 4704.649

"p=", 69999989, "pp=", 70000027, "A(p)=", 77.8563013689

"C(p)=", 17.8888915033, "ctilda(p)=", 19.5576845680, "time=", 5460.342

"p=", 79999987, "pp=", 80000023, "A(p)=", 81.0864249388

"C(p)=", 17.3955923927, "ctilda(p)=", 19.0767233509, "time=", 6213.625

"p=", 89999999, "pp=", 90000049, "A(p)=", 84.6224105076

"C(p)=", 17.3267723337, "ctilda(p)=", 19.0187822335, "time=", 6961.626

"p=", 99999989, "pp=", 100000007, "A(p)=", 88.2750462350

"C(p)=", 17.4843691600, "ctilda(p)=", 19.1861201754, "time=", 7709.820

"SOLUTION of PROPOSITION 3.6 (ii)"

"max C(p)=", 27.7269854510, "for p=", 33647, "A(p)=", 7.94512398978

"SOLUTION of PROPOSITION 3.6 (iii)"

"p0=", 520867, "p0+=", 520889, "A(p0)=", 16.5032241037, "C(p0)=", 25.2203285672

"lim_{x->p0+} C(x)=", 25.2196427714, "t=", $5.20877540142\ 10^{5}$

"SOLUTION of PROPOSITION 3.6 (v) for 409 ! = p ! 10^8"

"Minimum of ctilda(p) for 409 <= p < 10^8=", 15.3735871153, "for p=", 409

"Total time=", 7709.848

(5)

```
> ###### Study of f(x) in Proposition 3.6 (v)
> f:=sqrt(x)/log(x)^2*(2-lambda+5.12/log(x));
```

$$f := \frac{\sqrt{x}\left(1.9538085820677579324 + \frac{5.12}{\ln(x)}\right)}{\ln(x)^2}$$

```
> fprime:=normal(diff(f,x));
```

$$fprime := \frac{0.97690429103387896620\ \ln(x)^2 - 1.3476171641355158648\ \ln(x) - 15.36}{\sqrt{x}\ \ln(x)^4}$$

```
> lis:=[solve(numer(fprime),x)]; limit(f,x=1,right);limit(f,x=
  infinity);
```

$$lis := [111.55540231124777551, 0.035613012726284959345]$$

$$Float(\infty)$$

$$Float(\infty)$$

```
> x1:=lis[1]; fx1:=evalf(subs(x=x1,f)); # fx1 = min of f on (1,
  infinity)
```

$$x1 := 111.55540231124777551$$

$$fx1 := 1.4444989490349352398$$

```
> ######f is decreasing for 1 < x < x1 and increasing for x > x1 =
```

111.55

> prop36v(401):

"SOLUTION of PROPOSITION 3.6 (v) for 83 ! = p !   409"

"A(83)=", 1.45151667946, "f(83)=", 1.45221005086, "f(89)=", 1.44894527824

"f(t0)=A(83) for t0=", 84.1093794368

"Minimum for 89 <= p < 409 of A(p)-max(f(p),f(p+))=", 0.0362429758225, "for p=", 89

(6)

> prop36iv(10000):

"SOLUTION of PROPOSITION 3.6 (iv)"

"max for 59 <= p < 10000 of A(p)*log^2(p)/sqrt(p)=", 5.06435691382, "for p =", 3643

"SOLUTION of PROPOSITION 3.6 (vi)"

"2-lambda=", 1.95380858207

"p=", 11, "pp=", 13, "A(p)phi(p)=", 0.225602326138, "A(p)phi(p+)=", 0.237446299796

"p=", 13, "pp=", 17, "A(p)phi(p)=", 0.546164227858, "A(p)phi(p+)=", 0.582735444824

"p=", 17, "pp=", 19, "A(p)phi(p)=", 0.880456459204, "A(p)phi(p+)=", 0.899501938591

"p=", 19, "pp=", 23, "A(p)phi(p)=", 1.09648357419, "A(p)phi(p+)=", 1.13011308587

"p=", 23, "pp=", 29, "A(p)phi(p)=", 1.32068520995, "A(p)phi(p+)=", 1.35648429650

"p=", 29, "pp=", 31, "A(p)phi(p)=", 1.58449768255, "A(p)phi(p+)=", 1.59383940751

"p=", 31, "pp=", 37, "A(p)phi(p)=", 1.75624253387, "A(p)phi(p+)=", 1.77747033795

"p=", 37, "pp=", 41, "A(p)phi(p)=", 1.96180679680, "A(p)phi(p+)=", 1.97112265687

"p=", 41, "pp=", 43, "A(p)phi(p)=", 2.13645632951, "A(p)phi(p+)=", 2.14003506480

"p=", 43, "pp=", 47, "A(p)phi(p)=", 2.26123125947, "A(p)phi(p+)=", 2.26637701077

"p=", 47, "pp=", 53, "A(p)phi(p)=", 2.37633118071, "A(p)phi(p+)=", 2.37962257710

"p=", 53, "pp=", 59, "A(p)phi(p)=", 2.50109731490, "A(p)phi(p+)=", 2.50030826356

"p=", 59, "pp=", 61, "A(p)phi(p)=", 2.62882115228, "A(p)phi(p+)=", 2.62781335732

"p=", 61, "pp=", 67, "A(p)phi(p)=", 2.72586463884, "A(p)phi(p+)=", 2.72102169466

"p=", 67, "pp=", 71, "A(p)phi(p)=", 2.82533135211, "A(p)phi(p+)=", 2.82081463548

"p=", 71, "pp=", 73, "A(p)phi(p)=", 2.91466544545, "A(p)phi(p+)=", 2.91204861246

"p=", 73, "pp=", 79, "A(p)phi(p)=", 2.98261019306, "A(p)phi(p+)=", 2.97365035016

"p=", 79, "pp=", 83, "A(p)phi(p)=", 3.04963757920, "A(p)phi(p+)=", 3.04289020092

"p=", 83, "pp=", 89, "A(p)phi(p)=", 3.11099746546, "A(p)phi(p+)=", 3.09995846898