A Robin inequality for $n/\varphi(n)$

Jean-Louis Nicolas*

January 10, 2023

Abstract

Let $\varphi(n)$ be the Euler function, $\sigma(n) = \sum_{d|n} d$ the sum of divisors function and $\gamma = 0.577\dots$ the Euler constant. In 1982, Robin proved that, under the Riemann hypothesis, $\sigma(n)/n < e^\gamma \log\log n$ holds for $n > 5040$ and that this inequality is equivalent to the Riemann hypothesis. The aim of this paper is to give a similar equivalence for $n/\varphi(n)$.

2010 *Mathematics Subject Classification*: Primary 11N56; Secondary 11N37.

Keywords: Euler function, Robin inequality, Riemann hypothesis.

1 Introduction

Let n be a positive integer, $\varphi(n)$ the Euler function (i.e. the number of integers m satisfying $1 \leqslant m \leqslant n$ and coprime with n), $\sigma(n) = \sum_{d|n} d$ the sum of divisors of n and $\gamma = 0.577\dots$ the Euler constant.

When $n \to \infty$, Landau proved that

$$n/\varphi(n) \leqslant \big(1 + o(1)\big)e^\gamma \log\log n \tag{1.1}$$

(cf. [6] and [5, Theorem 328]), while in 1913, Gronwall proved that $\sigma(n)/n \leqslant \big(1 + o(1)\big)e^\gamma \log\log n$, (cf. [4] and [5, Theorem 323]). There are infinitely many n's such that $n/\varphi(n) > e^\gamma \log\log n$ (cf. [8, 9]) but there are infinitely many n's such that $\sigma(n)/n > e^\gamma \log\log n$ only if the Riemann hypothesis fails (cf. [15, 14, 11]).

*Research partially supported by CNRS, Institut Camille Jordan, UMR 5208.

In 1982, Robin proved that

$$\frac{\sigma(n)}{n} < e^\gamma \log\log n \quad \text{for} \quad n > 5040, \tag{1.2}$$

is equivalent to the Riemann hypothesis (cf. [15, 14]. The inequality (1.2) is called *Robin inequality*.

Let $f(n)$ be an arithmetical function, i.e. a function defined on the positive integers with positive real values. The integer n is said to be a f-champion if $1 \leqslant m < n$ implies $f(m) < f(n)$.

The champions for the number $d(n)$ of divisors of n are called *highly composite numbers*. They have been defined and studied by Ramanujan (cf. [12], [1, Sect. 4] and [10]). The champions for $\sigma(n)/n$ are said to be *superabundant* (cf. [13, Sect. 59], [1, Sect. 4] and [11, Sect. 3.4]).

An integer M is called super f-champion if there exists $\varepsilon > 0$ such that

$$\frac{f(n)}{n^\varepsilon} \leqslant \frac{f(M)}{M^\varepsilon} \quad \text{for} \quad n \in \mathbb{N}. \tag{1.3}$$

Let p_j denotes the jth prime and

$$M_{p_j} = p_1 p_2 \dots p_j$$

the jth primorial, i.e. the product of the first j primes. It is easy to see that, if $f(n) = n/\varphi(n)$ then the f-champions are the numbers M_{p_j} for $j \geqslant 1$. Indeed, if $n < M_{p_j}$ then the standard factorization of n can be written $n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_r^{\alpha_r}$ with $q_1 < q_2 < \dots < q_r$, $r < j$ and $q_i \geqslant p_i$ for $1 \leqslant i \leqslant r$. Therefore,

$$\frac{n}{\varphi(n)} = \prod_{i=1}^{r} \frac{q_i}{q_i - 1} \leqslant \prod_{i=1}^{r} \frac{p_i}{p_i - 1} < \prod_{i=1}^{j} \frac{p_i}{p_i - 1} = \frac{M_{p_j}}{\varphi\big(M_{p_j}\big)}. \tag{1.4}$$

It follows from (1.3) that a super champion is a champion. In Sect. 2, in the case of $f(n) = n/\varphi(n)$, it is proved that all the f-champions are super f-champions, i.e. that the set of super f-champions coincide with the set of primorials.

Let us set

$$\delta = e^\gamma(4 + \gamma - \log(4\pi)) = 3.6444150964\dots \tag{1.5}$$

and, if n is an integer $\geqslant 2$,

$$c(n) = \left(\frac{n}{\varphi(n)} - e^\gamma \log\log n\right)\sqrt{\log n}. \tag{1.6}$$

In [9, Theorem 1.1], it is proved that, under the Riemann hypothesis,

$$\limsup_{n\to\infty} c(n) = \delta.$$

Theorem 1.1. *Let*

$$k = 120568, \quad p_k = 1591873, \quad \log M_{p_k} = 1590171.635973\ldots \tag{1.7}$$

and

$$A = M_{p_k}\frac{p_{k+1}p_{k+2}}{p_k p_{k-10}} = M_{p_k}\frac{1591883\times 1591901}{1591873\times 1591697}, \log A = 1590171.636107\ldots \tag{1.8}$$

Then,

$$c(A) = 3.6444151157\ldots > \delta = 3.6444150964\ldots \tag{1.9}$$

and, under the Riemann hypothesis, for $n > A$,

$$c(n) < \delta = e^\gamma(4+\gamma-\log(4\pi)) = 3.6444150964\ldots \tag{1.10}$$

In other words, A is the largest number n such that (1.10) *holds.*

Moreover,

$$\frac{n}{\varphi(n)} < e^\gamma \log\log n + \frac{e^\gamma\bigl(4+\gamma-\log(4\pi)\bigr)}{\sqrt{\log n}} \quad \text{for} \quad n > A \tag{1.11}$$

is equivalent to the Riemann hypothesis.

In [9, cf. Theorem 1.1 and p. 320], it is proved that (1.10) holds for $n \geqslant M_{p_{k+1}}$, but not for $n = M_{p_k}$. So, to prove 1.10, it suffices to show that A is the largest number satisfying $M_{p_k} \leqslant A < M_{p_{k+1}}$ and $c(A) \geqslant \delta$. This will be done in Sect. 3 by using the method of benefits, cf. below, Sect. 2.1.

If the Riemann hypothesis does not hold, then (cf. [8, Theorem 3 (c)] and [9, p. 312])

$$\limsup_{n\to\infty} c(n) = +\infty \tag{1.12}$$

which contradicts (1.10) and proves the equivalence of (1.11) with the Riemann hypothesis.

1.1 Notation

- $p_1 = 2,\ p_2 = 3, \ldots, p_j$ is the jth prime.
- $\mathcal{P} = \{2, 3, 5, \ldots\}$ is the set of primes.

- $\theta(x) = \sum_{p \leqslant x} \log p$ is the Chebyshev function
- $M_{p_j} = p_1 p_2 \ldots p_j$ is the jth primorial. If p is the jth prime then $M_p = M_{p_j}$
- k and p_k are defined in (1.7).
- We use the following constants: γ is Euler constant, A is defined in (1.8), δ in (1.5) and λ in (3.2).
- All the computation have been carried out in Maple, cf. [16].

2 The super-champions for $n/\varphi(n)$

M is said to be a super champion (cf. (1.3)) for the function $n \mapsto n/\varphi(n)$ if there exists $\varepsilon > 0$ such that

$$\frac{n^{(1-\varepsilon)}}{\varphi(n)} \leqslant \frac{M^{(1-\varepsilon)}}{\varphi(M)} \tag{2.1}$$

for all positive integer n. The number ε is said to be a *parameter* of M. From (1.1), it follows that, for $\varepsilon > 0$, $\lim_{n\to\infty} n^{(1-\varepsilon)}/\varphi(n) = 0$ so that $n^{(1-\varepsilon)}/\varphi(n)$ has a maximum attained in one or several numbers, and all these numbers are super champions.

The study of these super champions is similar to the one of superior hignly composite numbers (cf. [12], [1], [2, Sect. 6.3] and [10, Sect. 4]) or of CA numbers (cf. [13, Sect. 59], [1], [3] or [11]), but much simpler. We consider the set of decreasing numbers

$$\widehat{\mathcal{E}} = \left\{ \widehat{\varepsilon}_0 = \infty > \widehat{\varepsilon}_1 = 1 > \widehat{\varepsilon}_2 = \frac{\log(3/2)}{\log 3} > \ldots > \widehat{\varepsilon}_i = \frac{\log\big(p_i/(p_i-1)\big)}{\log p_i} > \ldots \right\} \tag{2.2}$$

where p_i denotes the ith prime.

Proposition 2.1. *Let M be a super champion for the function $n \mapsto n/\varphi(n)$ with parameter ε. One defines $i \geqslant 1$ by $\widehat{\varepsilon}_i \leqslant \varepsilon < \widehat{\varepsilon}_{i-1}$ (cf. (2.2)).*

If ε satisfies $\widehat{\varepsilon}_i < \varepsilon < \widehat{\varepsilon}_{i-1}$ then there is one and only one super champion for the function $n \mapsto n/\varphi(n)$ with parameter ε. This super champion number M is equal to the primorial defined by

$$M = M_{p_{i-1}} = \prod_{p \leqslant p_{i-1}} p. \tag{2.3}$$

(By convention, $p_0 = 1$ and the empty product $M_1 = 1$).

If $\varepsilon = \widehat{\varepsilon}_i$, then there exist two super champions with parameter ε, namely

$$M_{p_{i-1}} = \prod_{p \leqslant p_{i-1}} p \quad \text{and} \quad M_{p_i} = \prod_{p \leqslant p_i} p. \tag{2.4}$$

Proof. Let $n = \prod_{j \geqslant 1} p_j^{a_j}$ (with only finitely many a_j's positive). We have to find the maximum of

$$\frac{n^{1-\varepsilon}}{\varphi(n)} = \prod_{j \geqslant 1} \frac{p_j^{a_j(1-\varepsilon)}}{\varphi\big(p_j^{a_j}\big)},$$

i.e. for each $j \geqslant 1$, to find the maximum on a_j of

$$\frac{p_j^{a_j(1-\varepsilon)}}{\varphi\big(p^{a_j}\big)} = \begin{cases} 1 & \text{if } \ a_j = 0 \\ \frac{p_j}{(p_j-1)p_j^{a_j\varepsilon}} = p_j^{\widehat{\varepsilon}_j - a_j\varepsilon} \leqslant p_j^{\widehat{\varepsilon}_j - \varepsilon} & \text{if } \ a_j \geqslant 1. \end{cases} \tag{2.5}$$

So, this maximum is attained for $a_j = 0$ or $a_j = 1$.

If $j \leqslant i-1$, then $\widehat{\varepsilon}_j \geqslant \widehat{\varepsilon}_{i-1}$, $\widehat{\varepsilon}_j - \varepsilon$ is positive and $p_j^{\widehat{\varepsilon}_j - \varepsilon} > 1$ holds so that from (2.5) the maximum on a_j of $p_j/\big((p_j-1)p_j^{a_j\varepsilon}\big)$ is attained for $a_j = 1$.

If $j \geqslant i+1$, then $\widehat{\varepsilon}_j < \widehat{\varepsilon}_i$, $\widehat{\varepsilon}_j - \varepsilon$ is negative and $p_j^{\widehat{\varepsilon}_j - \varepsilon} < 1$ holds so that the maximum on a_j of $p_j/\big((p_j-1)p_j^{a_j\varepsilon}\big)$ is attained for $a_j = 0$.

If $j = i$ and $\varepsilon \neq \widehat{\varepsilon}_i$, then $\widehat{\varepsilon}_j = \widehat{\varepsilon}_i$, $\widehat{\varepsilon}_j - \varepsilon$ is negative and $p_j^{\widehat{\varepsilon}_j - \varepsilon} < 1$ holds so that the maximum on a_j of $p_j/\big((p_j-1)p_j^{a_j\varepsilon}\big)$ is still attained for $a_j = 0$. Therefore, if $\varepsilon \neq \widehat{\varepsilon}_i$, the maximum on n of $n^{1-\varepsilon}/\varphi(n)$ is attained on $n = M_{p_{i-1}}$.

If $j = i$ and $\varepsilon = \widehat{\varepsilon}_i$, then the maximum of $p_j/\big((p_j-1)p_j^{a_j\varepsilon}\big)$ is equal to 1 and is attained on two points, namely $a_j = 0$ and $a_j = 1$ which implies that the maximum on n of $n^{1-\varepsilon}/\varphi(n)$ is attained on $n = M_{p_{i-1}}$ and $n = M_{p_i}$. □

From now on, we shall replace the expression "super-champion for the function $n \mapsto n/\varphi(n)$ with parameter ε" by "primorial with parameter ε" . The first primorial numbers are given in Figure 1.

i	p_i	$\widehat{\varepsilon}_i$	$M = M_{p_i}$	$M/\varphi(M)$	parameter
0	1	∞	1	1	$[\widehat{\varepsilon}_1, \widehat{\varepsilon}_0)$
1	2	1	2	2	$[\widehat{\varepsilon}_2, \widehat{\varepsilon}_1]$
2	3	$\log(3/2)/\log(3) = 0.369$	6	3	$[\widehat{\varepsilon}_3, \widehat{\varepsilon}_2]$
3	5	$\log(5/4)/\log(5) = 0.138$	30	15/4	$[\widehat{\varepsilon}_4, \widehat{\varepsilon}_3]$
4	7	$\log(7/6)/\log(7) = 0.079$	210	35/8	$[\widehat{\varepsilon}_5, \widehat{\varepsilon}_4]$
5	11	$\log(11/10)/\log(11) = 0.039$	2310	77/16	$[\widehat{\varepsilon}_6, \widehat{\varepsilon}_5]$
6	13	$\log(13/12)/\log(13) = 0.031$	30030	1001/192	$[\widehat{\varepsilon}_7, \widehat{\varepsilon}_6]$
7	17	$\log(17/16)/\log(17) = 0.021$	510510	17017/3072	$[\widehat{\varepsilon}_8, \widehat{\varepsilon}_7]$

Figure 1: The first primorial numbers

2.1 Benefit

Definition 2.2. *Let ε be a positive real number and M a primorial of parameter ε. For a positive integer n, we introduce the benefit of n*

$$\operatorname{ben}_\varepsilon(n) = \log\left(\frac{M^{1-\varepsilon}}{\varphi(M)}\right) - \log\left(\frac{n^{1-\varepsilon}}{\varphi(n)}\right) = \log\left(\frac{\varphi(n)}{\varphi(M)}\right) + (1-\varepsilon)\log\left(\frac{M}{n}\right). \quad (2.6)$$

Note that that, if $\widetilde{M}$ is another primorial of parameter ε, then (2.1) yields $\widetilde{M}^{1-\varepsilon}/\varphi(\widetilde{M}) \leqslant M^{1-\varepsilon}/\varphi(M)$ and $M^{1-\varepsilon}/\varphi(M) \leqslant \widetilde{M}^{1-\varepsilon}/\varphi(\widetilde{M})$, which implies $M^{1-\varepsilon}/\varphi(M) = \widetilde{M}^{1-\varepsilon}/\varphi(\widetilde{M})$ so that (2.6) returns the same value for $\operatorname{ben}_\varepsilon(n)$ if M is replaced by $\widetilde{M}$.

This notion of benefit has been used in [7, 3] for theoretical results and, for computation, in [10, Sect. 3.5] and [11, Sect. 4.6].

From (2.1), it follows that, for any n,

$$\operatorname{ben}_\varepsilon(n) \geqslant 0 \quad (2.7)$$

holds. Let M be a primorial of parameter ε. Let us write

$$M = \prod_{p\in\mathcal{P}} p^{a_p} \quad \text{and} \quad n = \prod_{p\in\mathcal{P}} p^{b_p}, \quad (2.8)$$

(with only finitely many b_p's positive). For $p \in \mathcal{P}$, (2.6) yields

$$\operatorname{ben}_\varepsilon\left(Mp^{b_p-a_p}\right) = \log\left(\frac{\varphi\left(p^{b_p}\right)}{\varphi\left(p^{a_p}\right)}\right) + (1-\varepsilon)\left(a_p - b_p\right)\log p \geqslant 0. \quad (2.9)$$

As $\varphi(n)$ is multiplicative, (2.6) and (2.9) give

$$\operatorname{ben}_\varepsilon(n) = \sum_{p\in\mathcal{P}} \operatorname{ben}_\varepsilon\left(M p^{b_p-a_p}\right) \tag{2.10}$$

and, from (2.9),

$$\operatorname{ben}_\varepsilon\left(M p^{b_p-a_p}\right) = \begin{cases} 0 & \text{if} \quad a_p = b_p \\ \log\left(p/(p-1)\right) - \varepsilon \log p & \text{if} \quad a_p = 1, b_p = 0 \\ \log\left((p-1)/p\right) + \varepsilon b_p \log p & \text{if} \quad a_p = 0, b_p \geqslant 1 \\ \left(b_p - 1\right)\varepsilon \log p & \text{if} \quad a_p = 1, b_p \geqslant 1. \end{cases} \tag{2.11}$$

Note that, if $a_p = 1$ and $b_p = 0$, then

$$\operatorname{ben}_\varepsilon(M/p) = \log\left(p/(p-1)\right) - \varepsilon \log p \quad \text{ is decreasing on } p \tag{2.12}$$

while, if $a_p = 0$ and $b_p = 1$ then, from (2.11),

$$\operatorname{ben}_\varepsilon(Mp) = \log\left((p-1)/p\right) + \varepsilon \log p \quad \text{ is increasing on } p. \tag{2.13}$$

3 Proof of Theorem 1.1

In this section, k and p_k are defined by (1.7). The benefit (cf. Sect. 2.1) is defined relatively to the primorial M_{p_k} with the parameter (cf. (2.2))

$$\varepsilon = \widehat{\varepsilon}_{k+1} = \frac{\log\left(p_{k+1}/(p_{k+1}-1)\right)}{\log p_{k+1}} = 4.39893721125\ldots \times 10^{-8}, \tag{3.1}$$

which is the common parameter of the primorials M_{p_k} and $M_{p_{k+1}}$. Note that

$$\log M_{p_k} = \theta\left(p_k\right) = 1590171.6359\ldots, \; M_{p_k}/\varphi\left(M_{p_k}\right) = 25.43545096\ldots$$

and

$$\log M_{p_{k+1}} = \theta\left(p_{k+1}\right) = 1590185.9164\ldots, M_{p_{k+1}}/\varphi\left(M_{p_{k+1}}\right) = 25.43546694\ldots$$

From (1.5), we also introduce the notation

$$\lambda = \delta e^{-\gamma} = 4 + \gamma - \log(4\pi) = 2.046191417932\ldots \tag{3.2}$$

Lemma 3.1. *The function*

$$g(t) = \varepsilon t - \log\left(\log t + \lambda/\sqrt{t}\right), \tag{3.3}$$

with ε defined by (3.1) *and λ by* (3.2)*, is convex for $t > 2.36$. Moreover, g is decreasing on t for* $\log M_{p_k} \leqslant t \leqslant \log M_{p_{k+1}}$.

Proof. We have

$$g'(t) = \varepsilon - \frac{1/t - \lambda/\left(2t^{3/2}\right)}{\log t + \lambda/\sqrt{t}}, \tag{3.4}$$

$$g''(t) = \frac{1/t^2 - 3\lambda/\left(4t^{5/2}\right)}{\log t + \lambda/\sqrt{t}} + \frac{\left(1/t - \lambda/(2t^{3/2})\right)^2}{(\log t + \lambda/\sqrt{t})^2}. \tag{3.5}$$

The second fraction of (3.5) is clearly non-negative while the first one is positive for $t > 9\lambda^2/16 = 2.35513\ldots$, which proves the convexity of g. Therefore, $g'(t)$ is increasing on t for $t > 2.36$. As

$$g'\left(\log M_{p_{k+1}}\right) = -9.49208\ldots \times 10^{-12} < 0 \quad \text{and} \quad \lim_{t\to+\infty} g'(t) = \varepsilon > 0,$$

$g(t)$ is decreasing for $2.36 \leqslant t \leqslant \log M_{p_{k+1}}$ and since $\log M_{p_k} > 2.36$ holds, $g(t)$ is decreasing for $\log M_{p_k} \leqslant t \leqslant \log M_{p_{k+1}}$. In fact, the minimum of $g(t)$ is attained for $t = 1590506.7305\ldots$ (cf. [16]). □

Lemma 3.2. *Let n satisfy $M_{p_k} < n < M_{p_{k+1}}$ and*

$$c(n) \geqslant \delta = \lambda e^{\gamma}, \tag{3.6}$$

where $c(n)$ is defined by (1.6)*, δ by* (1.5) *and λ by* (3.2)*. If ε is defined by* (3.1)*, then*

$$\operatorname{ben}_\varepsilon(n) \leqslant \beta = 9.1 \times 10^{-11}. \tag{3.7}$$

Proof. As ε is a parameter of the primorial M_{p_k}, from (2.6) and (1.6),

$$\begin{aligned}\operatorname{ben}_\varepsilon(n) &= -\log\frac{n}{\varphi(n)} + \varepsilon\log n + \log\frac{M_{p_k}}{\varphi(M_{p_k})} - \varepsilon\log M_{p_k}\\ &= -\log\left(e^{\gamma}\left(\log\log n + \frac{c(n)/e^{\gamma}}{\sqrt{\log n}}\right)\right) + \varepsilon\log n + \log\frac{M_{p_k}}{\varphi(M_{p_k})} - \varepsilon\log M_{p_k},\end{aligned}$$

which, from (3.6) and Lemma 3.1, implies

$$\begin{aligned}\operatorname{ben}_\varepsilon(n) &\leqslant g(\log n) - \gamma + \log\frac{M_{p_k}}{\varphi(M_{p_k})} - \varepsilon\log M_{p_k}\\ &\leqslant g\big(\log M_{p_k}\big) - \gamma + \log\frac{M_{p_k}}{\varphi(M_{p_k})} - \varepsilon\log M_{p_k} = 9.0974000017\ldots\times 10^{-11},\end{aligned} \tag{3.8}$$

which proves (3.7). □

Lemma 3.3. *Let n be an integer satisfying $M_{p_k} < n < M_{p_{k+1}}$ and $\operatorname{ben}_\varepsilon(n) \leqslant \beta = 9.1\times 10^{-11}$. Then there there exist primes $q_1, q_2, \ldots, q_r, q'_1, q'_2, \ldots, q'_r$ such that*

$$n = \frac{q_1 q_2 \ldots q_r}{q'_1 q'_2 \ldots q'_r} M_{p_k} \quad \textit{with} \quad 1 \leqslant r \leqslant 4, \tag{3.9}$$

$$p_{k+1} \leqslant q_1 < q_2 < \ldots < q_r \leqslant p_{k+14} = 1592081$$

and

$$p_k \geqslant q'_1 > q'_2 > \ldots > q'_r \geqslant p_{k-10} = 1591697.$$

Proof. Let us write $n = \prod_{p\in\mathcal{P}} p^{b_p}$ and $M_{p_k} = \prod_{p\in\mathcal{P}} p^{a_p}$ with $a_p = 1$ if $p \leqslant p_k$ and $a_p = 0$ if $p > p_k$. From (2.10),

$$\operatorname{ben}_\varepsilon(n) = \sum_{p\in\mathcal{P}} \operatorname{ben}_\varepsilon\big(M_{p_k} p^{b_p - a_p}\big). \tag{3.10}$$

From (2.7), each term of the above sum is non-negative and our hypothesis, $\operatorname{ben}_\varepsilon(n) \leqslant \beta$, implies

$$0 \leqslant \operatorname{ben}_\varepsilon\big(M_{p_k} p^{b_p - a_p}\big) \leqslant \beta \quad \text{for } p \in \mathcal{P}. \tag{3.11}$$

• If $a_p = 1$ and $b_p = 0$, then, $p \leqslant p_k$ and from (2.11) and (2.12),

$$\operatorname{ben}_\varepsilon\big(M_{p_k}/p\big) = \log\big(p/(p-1)\big) - \varepsilon\log p$$

is decreasing on p. From (2.11)

$$\operatorname{ben}_\varepsilon\big(M_{p_k}/p_{k-11}\big) = \operatorname{ben}_\varepsilon\big(M_{p_k}/1591663\big) = 9.29\ldots\times 10^{-11} > \beta,$$

so that

$$p \in \{p_{k-10}, p_{k-9}, \ldots, p_k\}. \tag{3.12}$$

• If $a_p = 1$ and $b_p \geqslant 2$, then, from (2.11),

$$\begin{aligned} \mathrm{ben}_\varepsilon\left(M_{p_k} p^{b_p - a_p}\right) &= (b_p - 1)\varepsilon \log p \geqslant \varepsilon \log p \\ &\geqslant \varepsilon \log 2 = 3.049 \ldots \times 10^{-8} > \beta. \end{aligned} \tag{3.13}$$

Consequently, from (3.11), such a p does not divide n.
• If $a_p = 0$ and $b_p \geqslant 2$, then $p \geqslant p_{k+1}$ holds and from (2.11),

$$\begin{aligned} \mathrm{ben}_\varepsilon(M_{p_k} p^{b_p}) &= \log\big((p-1)/p\big) + \varepsilon b_p \log p \\ &\geqslant \log\big((p_{k+1} - 1)/p_{k+1}\big) + 2\varepsilon \log p_{k+1} \\ &= 6.28 \ldots \times 10^{-7} > \beta \end{aligned} \tag{3.14}$$

so that such a p does not divide n.
• If $a_p = 0$ and $b_p = 1$ then $p \geqslant p_{k+1}$ holds and, from (2.11) and (2.13),

$$\mathrm{ben}_\varepsilon\left(M_{p_k} p^{b_p - a_p}\right) = \mathrm{ben}_\varepsilon\left(p M_{p_k}\right) = \log\big((p-1)/p\big) + \varepsilon \log p$$

is increasing on p. From (2.11),

$$\mathrm{ben}_\varepsilon\left(p_{k+15} M_{p_k}\right) = \log\big((p_{k+15} - 1)/p_{k+15}\big) + \varepsilon \log p_{k+15} = 9.119 \ldots \times 10^{-11} > \beta,$$

so that

$$p \in \{p_{k+1}, p_{k+2}, \ldots, p_{k+14}\}. \tag{3.15}$$

From (3.12) – (3.15), it follows that n should be equal to

$$n = \frac{q_1 q_2 \ldots q_r}{q'_1 q'_2 \ldots q'_s} M_{p_k} \tag{3.16}$$

with $r \geqslant 0$, $s \geqslant 0$, $p_{k+1} \leqslant q_1 < q_2 < \ldots < q_r \leqslant p_{k+14}$ and $p_k \geqslant q'_1 > q'_2 > \ldots > q'_s \geqslant p_{k-10}$. Let us prove that $r = s$. Ad absurdum, if $r > s$, then, from (3.16), we would have

$$n \geqslant M_{p_k} \frac{p_{k+1}^r}{p_k^s} \geqslant M_{p_k} p_{k+1}^{r-s} \geqslant M_{p_k} p_{k+1} = M_{p_{k+1}},$$

which contradicts our hypothesis $n < M_{p_{k+1}}$. Similarly, if $s > r$, then we would have

$$\frac{n}{M_{p_k}} \leqslant \frac{p_{k+14}^r}{p_{k-10}^s} = \left(\frac{p_{k+14}}{p_{k-10}}\right)^r \frac{1}{p_{k-10}^{s-r}} \leqslant \left(\frac{p_{k+14}}{p_{k-10}}\right)^{14} \frac{1}{p_{k-10}} = \frac{1.00183 \ldots}{1591697} < 1,$$

which contradicts $n > M_{p_k}$.

It remains to show that $1 \leqslant r \leqslant 4$. If $r = 0$, $n = M_{p_k}$ and we have supposed $n > M_{p_k}$. If $r \geqslant 5$, (2.10), (2.12), (2.13) and (2.11) imply

$$\begin{aligned}\operatorname{ben}_\varepsilon(n) = \operatorname{ben}_\varepsilon\Big(\frac{q_1 q_2 \dots q_r}{q_1' q_2' \dots q_r'} M_{p_k}\Big) &= \sum_{i=1}^{r} \Big(\operatorname{ben}_\varepsilon\big(q_i M_{p_k}\big) + \operatorname{ben}_\varepsilon\Big(\frac{M_{p_k}}{q_i'}\Big)\Big) \\ &\geqslant \sum_{i=1}^{5} \Big(\operatorname{ben}_\varepsilon\big(p_{k+i} M_{p_k}\big) + \operatorname{ben}_\varepsilon\Big(\frac{M_{p_k}}{p_{k-i+1}}\Big)\Big) = 1.21\dots 10^{-10} > \beta,\end{aligned}$$

which completes the proof of Lemma 3.3. □

After the statement of Theorem 1.1, we have seen that, to prove it, it suffices to show that A is the largest number satisfying $M_{p_k} < A < M_{p_{k+1}}$ and $c(A) \geqslant \delta$. Let n be an integer satisfying $M_{p_k} < n < M_{p_{k+1}}$ and $c(n) > \delta$. Lemma 3.2 implies $\operatorname{ben}_\varepsilon(n) \leqslant \beta$ defined by (3.7). From Lemma 3.3, we compute the numbers n described in (3.9) and satisfying $\operatorname{ben}_\varepsilon(n) \leqslant \beta$, cf. [16]. There are 882 such numbers and all of them satisfy $c(n) > \delta$ and $M_{p_k} < n < M_{p_{k+1}}$. Moreover, if we order these 882 numbers in a decreasing sequence $n_1 > n_2 > \dots > n_{882}$ then the sequence $\operatorname{ben}_\varepsilon(n_i)$ is decreasing while the sequences $n_i/\varphi(n_i)$ and $c(n_i)$ are increasing. The largest number is $n_1 = A$ (defined in (1.8)), which completes the proof of Theorem 1.1.

References

[1] Alaoglu, L., Erdős, P.: On highly composite and similar numbers. Trans. Amer. Math. Soc. **56**, 448–469 (1944)

[2] Broughan, K.: Equivalents of the Riemann Hypothesis, vol. 1. Encyclopedia of Mathematics and its Applications, 164, Cambridge University Press, (2017)

[3] Erdős, P., Nicolas, J.-L.: Répartition des nombres superabondants. Bull. Soc. Math. France **103**, 65–90 (1975)

[4] Gronwall, T.H.: Some asymptotic expressions in the theory of numbers. Trans. Amer. Math. Soc. **14**, 113–122 (1913)

[5] Hardy, G.H., Wright, E.M.: An introduction to the theory of numbers. 4th edition, Oxford, Clarendon Press, (1960)

[6] Landau, E.: Über den Verlauf der zahlentheoretischen Funktion $\varphi(x)$. Archiv. der Mathematik und Physik, Ser. 3, **5**, 86–91 (1903). Handbuch der Lehre von der Verteilung der Primzahlen, I, 2nd ed., Chelsea, 216–219 (1953)

[7] Nicolas, J.-L.: Ordre maximum d'un élément du groupe de permutations et highly composite numbers, Bull. Soc. Math. France **97**, 129–191 (1969)

[8] Nicolas, J.-L.: Petites valeurs de la fonction d'Euler. J. Number Theory **17**, 375–388 (1983)

[9] Nicolas, J.-L.: Small values of the Euler function and the Riemann hypothesis. Acta Arithmetica **155**(3), 311–321 (2012)

[10] Nicolas, J.-L.: Highly Composite Numbers and the Riemann hypothesis, Ramanujan J. **57**, 507-550 (2022)

[11] Nicolas, J.-L.: The sum of divisors function and the Riemann hypothesis. The Ramanujan J. **58**, 1113-1157 (2022)

[12] S. Ramanujan. Highly composite numbers. Proc. London Math. Soc., Serie 2, **14**, 347–409 (1915) In: Collected papers of Srinivasa Ramanujan, 78–128. Cambridge University Press (1927)

[13] Ramanujan, S.: Highly composite numbers, annotated and with a foreword by J.-L. Nicolas and G. Robin. The Ramanujan J. **1**, 119–153 (1997)

[14] Robin, G.: Sur l'ordre maximum de la fonction somme des diviseurs. Seminar on number theory, Paris (1981–82), 233–244. Progr. Math., 38, Birkhäuser, Boston, (1983)

[15] Robin, G.: Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. J. Math. Pures Appl. **63**, 187–213 (1984)

[16] `http://math.univ-lyon1.fr/homes-www/nicolas/calculphisig2.html`

Jean-Louis Nicolas,
Univ. Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208
Institut Camille Jordan, Mathématiques, Bât. Doyen Jean Braconnier,
43 Bd du 11 Novembre 1918, F-69622 Villeurbanne cedex, France.
`nicolas@math.univ-lyon1.fr`,
`http://math.univ-lyon1.fr/homes-www/nicolas/`