Small values of the Euler function and the Riemann hypothesis

by

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À André Schinzel pour son 75ème anniversaire,
en très amical hommage

1. Introduction. Let \( \varphi \) be the Euler function. In 1903, it was proved by E. Landau (cf. [5, §59] and [4, Theorem 328]) that

\[
\limsup_{n \to \infty} \frac{n}{\varphi(n) \log \log n} = e^\gamma = 1.7810724179 \ldots
\]

where \( \gamma = 0.5772156649 \ldots \) is Euler’s constant.

In 1962, J. B. Rosser and L. Schoenfeld proved (cf. [9, Theorem 15])

\[
(1.1) \hspace{1cm} \frac{n}{\varphi(n)} \leq e^\gamma \log \log n + \frac{2.51}{\log \log n}
\]

for \( n \geq 3 \) and asked if there exist an infinite number of \( n \) such that \( n/\varphi(n) > e^\gamma \log \log n \). In [6] (cf. also [7]), I answered this question in the affirmative. Soon after, A. Schinzel told me that he had worked unsuccessfully on this question, which made me very proud to have solved it.

For \( k \geq 1 \), \( p_k \) denotes the \( k \)th prime and

\[ N_k = 2 \cdot 3 \cdot 5 \cdot \ldots \cdot p_k \]

the primorial number of order \( k \). In [6], it is proved that the Riemann hypothesis (for short RH) is equivalent to

\[
\forall k \geq 1, \quad \frac{N_k}{\varphi(N_k)} > e^\gamma \log \log N_k.
\]

The aim of the present paper is to make the results of [6] more precise by estimating the quantity

\[
(1.2) \hspace{1cm} c(n) = \left( \frac{n}{\varphi(n)} - e^\gamma \log \log n \right) \sqrt{\log n}.
\]
Let us denote by $\rho$ a generic root of the Riemann $\zeta$ function satisfying $0 < \Re(\rho) < 1$. Under RH, $1 - \rho = \bar{\rho}$. It is convenient to define (cf. [2, p. 159])

$$(1.3) \quad \beta = \sum_{\rho} \frac{1}{\rho(1 - \rho)} = 2 + \gamma - \log \pi - 2 \log 2 = 0.0461914179 \ldots .$$

We shall prove

**Theorem 1.1.** Under the Riemann hypothesis (RH) we have

$$\lim_{n \to \infty} \sup_n c(n) = e^\gamma (2 + \beta) = 3.6444150964 \ldots ,$$

(1.4)

$$\forall n \geq N_{120569} = 2 \cdot 3 \ldots \cdot 1591883, \quad c(n) < e^\gamma (2 + \beta),$$

(1.5)

$$\forall n \geq 2, \quad c(n) \leq c(N_{66}) = c(2 \cdot 3 \ldots \cdot 317) = 4.0628356921 \ldots ,$$

(1.6)

$$\forall k \geq 1, \quad c(N_k) \geq c(N_1) = c(2) = 2.2085892614 \ldots .$$

(1.7)

We keep the notation of [6]. For a real $x \geq 2$, the usual Chebyshev functions are denoted by

$$(1.8) \quad \theta(x) = \sum_{p \leq x} \log p \quad \text{and} \quad \psi(x) = \sum_{p^m \leq x} \log p.$$  

We set

$$(1.9) \quad f(x) = e^\gamma \log \theta(x) \prod_{p \leq x} (1 - 1/p).$$

Mertens’s formula yields $\lim_{x \to \infty} f(x) = 1$. In [6] Th. 3(c)] it is shown that, if RH fails, there exists $b$, $0 < b < 1/2$, such that

$$(1.10) \quad \log f(x) = \Omega_{\pm} (x^{-b}).$$

For $p_k \leq x < p_{k+1}$, we have $f(x) = e^\gamma \log \log(N_k) \frac{\varphi(N_k)}{N_k}$. When $k \to \infty$, by observing that the Taylor development about 1 yields $\log f(p_k) \sim f(p_k) - 1$, we get

$$\log f(p_k) \sim f(p_k) - 1 = \frac{\varphi(N_k)}{N_k} \frac{c(N_k)}{\log N_k} \sim \frac{e^{-\gamma}}{\log \log N_k} \frac{c(N_k)}{\sqrt{\log N_k}},$$

and it follows from (1.10) that, if RH does not hold, then

$$\lim_{n \to \infty} \inf c(n) = -\infty \quad \text{and} \quad \lim_{n \to \infty} \sup c(n) = +\infty.$$  

Therefore, from Theorem 1.1 we deduce:

**Corollary 1.1.** Each of the four assertions (1.4) to (1.7) is equivalent to the Riemann hypothesis.

**1.1. Notation and results used.** If $\theta(x)$ and $\psi(x)$ are the Chebyshev functions defined by (1.8), we set

$$(1.11) \quad R(x) = \psi(x) - x \quad \text{and} \quad S(x) = \theta(x) - x.$$
Under RH, we shall use the upper bound (cf. [10, (6.3)])

\[(1.12) \quad x \geq 599 \Rightarrow |S(x)| \leq T(x) := \frac{1}{8\pi} \sqrt{x \log^2 x}.\]

P. Dusart (cf. [11 Table 6.6]) has shown that

\[(1.13) \quad \theta(x) < x \quad \text{for } x \leq 8 \cdot 10^{11},\]

thus improving the result of R. P. Brent who has checked \[(1.13)\] for \(x < 10^{11}\) (cf. [10, p. 360]). We shall also use (cf. [9, Theorem 10])

\[(1.14) \quad \theta(x) \geq 0.84 x \geq \frac{4}{5} x \quad \text{for } x \geq 101.\]

As in [6], we define the integrals

\[(1.15) \quad K(x) = \int_x^\infty \frac{S(t)}{t^2} \left( \frac{1}{\log t} + \frac{1}{\log^2 t} \right) dt,\]

\[(1.16) \quad J(x) = \int_x^\infty \frac{R(t)}{t^2} \left( \frac{1}{\log t} + \frac{1}{\log^2 t} \right) dt,\]

and, for \(\Re(z) < 1,

\[(1.17) \quad F_z(x) = \int_x^\infty t^{z-2} \left( \frac{1}{\log t} + \frac{1}{\log^2 t} \right) dt.\]

We also set, for \(x \geq 1,

\[(1.18) \quad W(x) = \sum_{\rho} \frac{x^{\Im(\rho)}}{\rho(1-\rho)},\]

so that, under RH, from \[(1.3)\] we have

\[(1.19) \quad |W(x)| \leq \beta = \sum_{\rho} \frac{1}{\rho(1-\rho)}.\]

We often implicitly use the following result: for \(a\) and \(b\) positive, the function

\[(1.20) \quad t \mapsto \frac{\log^a t}{t^b} \quad \text{is decreasing for } t > e^{a/b}\]

and

\[(1.21) \quad \max_{t \geq 1} \frac{\log^a t}{t^b} = \left( \frac{a}{eb} \right)^a.\]

Organization of the article. In Section 2 the results of [6] about \(f(x)\) are revised so as to get effective upper and lower bounds for both \(\log f(x)\) and \(1/f(x) - 1\) under RH (cf. Proposition 2.1). In Section 3 we study \(c(N_k)\) and \(c(n)\) in terms of \(f(p_k)\). Section 4 is devoted to the proof of Theorem 1.1.
2. Estimate of $\log(f(x))$. The following lemma is Proposition 1 of [6].

**Lemma 2.1.** For $x \geq 121$, we have

\[ K(x) - \frac{S^2(x)}{x^2 \log x} \leq \log f(x) \leq K(x) + \frac{1}{2(x-1)} . \]  

The next lemma is a slight improvement of Lemma 1 of [6].

**Lemma 2.2.** Let $x$ be a real number, $x > 1$. For $\Re(z) < 1$, we have

\[ F_z(x) = \frac{x^{z-1}}{(1-z) \log x} + r_z(x) \quad \text{with} \quad r_z(x) = \int_x^{\infty} \frac{zt^{z-2}}{(1-z) \log^2 t} \, dt \]

and, if $\Re(z) = 1/2$,

\[ |r_z(x)| \leq \frac{1}{|1-z| \sqrt{x \log^2 x}} \left( 1 + \frac{4}{\log x} \right) . \]

Moreover, for $z = 1/2$, we have

\[ 2 \sqrt{x \log x} - \frac{2}{\sqrt{x \log^2 x}} \leq F_{1/2}(x) \leq \frac{2}{\sqrt{x \log x}} - \frac{2}{\sqrt{x \log^2 x}} + \frac{8}{\sqrt{x \log^3 x}} \]

and, for $z = 1/3$,

\[ 0 \leq F_{1/3}(x) \leq \frac{3}{2x^{2/3} \log x} . \]

**Proof.** The proof of (2.2) is easy by taking the derivative. By partial summation, we get

\[ r_z(x) = -\frac{z}{1-z} \left( \frac{x^{z-1}}{(1-z) \log^2 x} + \int_x^{\infty} \frac{2 t^{z-2}}{(z-1) \log^3 t} \, dt \right) . \]

If we assume $\Re(z) = 1/2$, we have $1 - z = \bar{z}$ and

\[ |r_z(x)| \leq \frac{1}{|1-z| \sqrt{x \log^2 x}} + \frac{2}{|1-z| \log^3 x} \int_x^{\infty} t^{-3/2} \, dt , \]

which yields (2.3). The estimates (2.4) follow from (2.2) and (2.6) by choosing $z = 1/2$, while (2.5) follows from (2.2) since $r_{1/3}$ is negative.

To estimate the difference $J(x) - K(x)$, we need Lemma 2.4 below, which, under RH, is an improvement of Propositions 3.1 and 3.2 of [1] (obtained without assuming RH). The following lemma will be useful for proving Lemma 2.4.

**Lemma 2.3.** Let $\kappa = \kappa(x) = \left\lfloor \frac{\log x}{\log 2} \right\rfloor$ the largest integer such that $x^{1/\kappa} \geq 2$. For $x \geq 16$, set

\[ H(x) = 1 + \sum_{k=4}^{\kappa} x^{1/k-1/3} , \]
and for \( x \geq 4 \),
\[
L(x) = \sum_{k=2}^K \ell_k(x) \quad \text{with} \quad \ell_k(x) = \frac{T(x^{1/k})}{x^{1/3}} = \frac{\log^2 x}{8\pi k^2 x^{1/3 - 1/(2k)}}.
\]

Then

(i) \( H(x) \leq H(2^j) \) for \( j \geq 9 \) and \( x \geq 2^j \).
(ii) \( L(x) \leq L(2^j) \) for \( j \geq 35 \) and \( x \geq 2^j \).

**Proof.** The function \( H \) is continuous and decreasing on \([2^j, 2^{j+1})\); so, to show (i), it suffices to prove that for \( j \geq 9 \),
\[
H(2^j) \geq H(2^{j+1}). \tag{2.7}
\]

If \( 9 \leq j \leq 19 \), we check (2.7) by computation. If \( j \geq 20 \), we have
\[
H(2^j) - H(2^{j+1}) = \sum_{k=4}^{j} 2^{j(1/3 - 1/k)} (1 - 2^{1/3 - 1/k}) - 2^{(j+1)(1/3 - 1/(j+1) - 1/k)}
\geq 2^{j(1/3 - 1/4)} (1 - 2^{1/3 - 1/4}) - 2^{(j+1)(1/3 - 1/(j+1) - 1/4)}
= 2^{-j/3} [(1 - 2^{-1/12})2^{j/4} - 2^{2/3}],
\]
which proves (2.7) since the above bracket is \( \geq (1 - 2^{-1/12})2^{20/4} - 2^{2/3} = 0.208\ldots \) and therefore positive.

Let us assume that \( j \geq 35 \) so that \( 2^j \geq e^{24} \). From (1.20), for each \( k \geq 2 \), \( x \mapsto \ell_k(x) \) is decreasing for \( x \geq 2^j \) so that \( L \) is decreasing on \([2^j, 2^{j+1})\), and to show (ii), it suffices to prove
\[
L(2^j) \geq L(2^{j+1}). \tag{2.8}
\]

We have
\[
L(2^j) - L(2^{j+1}) = \sum_{k=2}^{j} \{\ell_k(2^j) - \ell_k(2^{j+1})\} - \ell_{j+1}(2^{j+1})
\geq \ell_2(2^j) - \ell_2(2^{j+1}) - \ell_{j+1}(2^{j+1})
= \frac{\log^2 2}{32\pi} 2^{-j/3} \{2^{j/4}j^2 - 2^{-1/12}(j + 1)^2 \} - 4 \cdot 2^{1/6}.
\]

For \( j \geq 1/(2^{1/12} - 1) = 16.81\ldots \), the above square bracket is increasing in \( j \) and positive for \( j = 35 \). Therefore, the curly bracket is increasing for \( j \geq 35 \), and since its value for \( j = 35 \) is equal to 744.17\ldots \), (2.8) is proved for \( j \geq 35 \).

**Lemma 2.4.** Under RH, we have
\[
\psi(x) - \theta(x) \geq \sqrt{x} \quad \text{for} \quad x \geq 121, \tag{2.9}
\]
and, for $x \geq 1$,
\[
(2.10) \quad \frac{\psi(x) - \theta(x) - \sqrt{x}}{x^{1/3}} \leq 1.332768 \ldots \leq \frac{4}{3}.
\]

Proof. For $x < 599^3$, we check (2.9) by computation. Note that 599 is prime. Let $q_0 = 1$, and let $q_1 = 4, q_2 = 8, q_3 = 9, \ldots, q_{1922} = 599^3$ be the sequence of powers (with exponent $\geq 2$) of primes not exceeding 599$^3$. On the intervals $[q_i, q_{i+1})$, the function $\psi - \theta$ is constant and $x \mapsto (\psi(x) - \theta(x))/\sqrt{x}$ is decreasing. For $11 \leq i \leq 1921$ (i.e. $121 \leq q_i < q_{i+1} \leq 599^3$), we calculate $\delta_i = (\psi(q_i) - \theta(q_i))/\sqrt{q_{i+1}}$ and find that $\min_{11 \leq i \leq 1921} \delta_i = \delta_{1886} = 1.0379 \ldots$ $(q_{1886} = 206468161 = 14369^2)$ while $\delta_{10} = 0.9379 \ldots < 1$ ($q_{10} = 81$).

Now, we assume $x \geq 599^3$, so that, by (1.12),
\[
(2.11) \quad \psi(x) - \theta(x) \geq \theta(x^{1/2}) + \theta(x^{1/3}) \geq x^{1/2} + x^{1/3} - T(x^{1/2}) - T(x^{1/3}).
\]

By using (1.21), we get
\[
\frac{T(x^{1/2})}{x^{1/3}} + \frac{T(x^{1/3})}{x^{1/3}} = \frac{1}{8\pi} \left( \log^2 x + \log^2 x \right) \leq \frac{20}{\pi e^2} = 0.86157 \ldots,
\]
which, with (2.11), implies
\[
(2.12) \quad \psi(x) - \theta(x) \geq \sqrt{x} + \left( 1 - \frac{20}{\pi e^2} \right) x^{1/3} \geq \sqrt{x}.
\]

The inequality (2.10) is Lemma 3 of [8]. Below we give another proof by considering three cases according to the values of $x$.

Case 1: $1 \leq x < 2^{32}$. The largest $q_i$ smaller than $2^{32}$ is $q_{6947} = 4293001441 = 65521^2$. On the intervals $[q_i, q_{i+1})$, the function
\[
G(x) := \frac{\psi(x) - \theta(x) - \sqrt{x}}{x^{1/3}}
\]
is decreasing. By computing $G(q_0), G(q_1), \ldots, G(q_{6947})$ we get
\[
G(x) \leq G(q_{103}) = 1.332768 \ldots \quad [q_{103} = 80089 = 283^2].
\]

Case 2: $2^{32} \leq x < 64 \cdot 10^{22}$. By using (1.13), we get
\[
\psi(x) - \theta(x) = \sum_{k=2}^{\kappa} \theta(x^{1/k}) \leq \sum_{k=2}^{\kappa} x^{1/k}
\]
so that Lemma 2.3 implies $G(x) \leq H(x) \leq H(2^{32}) = 1.31731 \ldots$.

Case 3: $x \geq 64 \cdot 10^{22} \geq 2^{79}$. By (1.12) and (1.13), we get
\[
\psi(x) - \theta(x) = \sum_{k=2}^{\kappa} \theta(x^{1/k}) \leq \sum_{k=2}^{\kappa} \left\{ x^{1/k} + T(x^{1/k}) \right\},
\]
whence, from Lemma 2.3, $G(x) \leq H(x) + L(x) \leq H(2^{79}) + L(2^{79}) = 1.32386 \ldots$. 

Corollary 2.1. For \( x \geq 121 \), we have

\[
F_{1/2}(x) \leq J(x) - K(x) \leq F_{1/2}(x) + \frac{4}{3} F_{1/3}(x).
\]

The following lemma is an improvement of [6, Proposition 2].

Lemma 2.5. Assume that RH holds. For \( x > 1 \), we may write

\[
J(x) = -\frac{W(x)}{\sqrt{x} \log x} - J_1(x) - J_2(x)
\]

with

\[
0 < J_1(x) \leq \frac{\log(2\pi)}{x \log x} \quad \text{and} \quad |J_2(x)| \leq \frac{\beta}{\sqrt{x} \log^2 x} \left( 1 + \frac{4}{\log x} \right).
\]

Proof. In [6, (17)–(19)], for \( x > 1 \), it is proved that

\[
J(x) = -\sum_{\rho} \frac{1}{\rho} F_{\rho}(x) - J_1(x)
\]

with \( J_1 \) satisfying \( 0 < J_1(x) \leq \frac{\log(2\pi)}{x \log x} \).

Now, by Lemma 2.2, we have

\[
F_{\rho}(x) = \frac{x^{\rho-1}}{(1-\rho) \log x} + r_{\rho}(x),
\]

which yields (2.14) by setting \( J_2(x) = \sum_{\rho} (1/\rho) r_{\rho}(x) \). Further, from (2.3) and (1.3), we get the upper bound for \( |J_2(x)| \) given in (2.15).

Proposition 2.1. Under RH, for \( x \geq x_0 = 10^9 \), we have

\[
-\frac{2 + W(x)}{\sqrt{x} \log x} + \frac{0.055}{\sqrt{x} \log^2 x} \leq \log f(x) \leq -\frac{2 + W(x)}{\sqrt{x} \log x} + \frac{2.062}{\sqrt{x} \log^2 x}
\]

and

\[
\frac{2 + W(x)}{\sqrt{x} \log x} - \frac{2.062}{\sqrt{x} \log^2 x} \leq 1 \leq \frac{2 + W(x)}{\sqrt{x} \log x} - \frac{0.054}{\sqrt{x} \log^2 x}.
\]

Proof. By collecting the information from (2.1), (1.12), (2.13), (2.14), (2.15), (2.4) and (2.5), for \( x \geq 599 \), we get

\[
\log f(x) \geq -\frac{W(x) + 2}{\sqrt{x} \log x} + \frac{2 - \beta}{\sqrt{x} \log^2 x} - \frac{8 + 4\beta}{\sqrt{x} \log^3 x}
\]

\[
-\frac{\log(2\pi)}{x \log x} - \frac{2}{x^{2/3} \log x} - \frac{\log^3 x}{64\pi^2 x}
\]

and

\[
\log f(x) \leq -\frac{W(x) + 2}{\sqrt{x} \log x} + \frac{2 + \beta}{\sqrt{x} \log^2 x} + \frac{4\beta}{\sqrt{x} \log^3 x} + \frac{1}{2(x - 1)}.
\]
Since \( x \geq x_0 = 10^9 \), (2.18) and (2.19) imply respectively

\[
\log f(x) \geq - \frac{W(x) + 2}{\sqrt{x \log x}} + \frac{1}{\sqrt{x \log^2 x}} \left( 2 - \beta - \frac{8 + 4\beta}{\log x_0} \right) - \frac{\log(2\pi) \log x_0}{\sqrt{x_0}} - 2\log x_0 - \frac{\log^5 x_0}{64\pi^2 \sqrt{x_0}} \tag{2.20}
\]

and

\[
\log f(x) \leq - \frac{W(x) + 2}{\sqrt{x \log x}} + \frac{1}{\sqrt{x \log^2 x}} \left( 2 + \beta + \frac{4\beta}{\log x_0} + \sqrt{x_0 \log^2 x_0} \right), \tag{2.21}
\]

which proves (2.16).

Setting \( v = -\log f(x) \), it follows from (2.16), (1.19) and (1.3) that

\[
v \leq \frac{W(x) + 2}{\sqrt{x \log x}} \leq 2 + \beta \leq v_0 := \frac{2 + \beta}{\sqrt{x_0 \log x_0}} = 0.00000312 \ldots.
\]

By Taylor’s formula, we have \( e^v - 1 \geq v \), which, with (2.16), provides the lower bound of (2.17), and

\[
e^v - 1 - v \leq \frac{e^{v_0}}{2} v^2 \leq \frac{e^{v_0}(2 + \beta)^2}{2x \log^2 x} \leq \frac{e^{v_0}(2 + \beta)^2}{2\sqrt{x_0 \log^2 x}} = \frac{0.00000662 \ldots}{\sqrt{x \log^2 x}},
\]

which implies the upper bound in (2.17).

3. Bounding \( c(n) \)

**Lemma 3.1.** Let \( n \) and \( k \) be two integers satisfying \( n \geq 2 \) and \( k \geq 1 \). Assume that either the number \( j = \omega(n) \) of distinct prime factors of \( n \) is equal to \( k \), or \( N_k \leq n < N_{k+1} \). Then

\[
c(n) \leq c(N_k). \tag{3.1}
\]

**Proof.** It follows from our hypothesis that \( n \geq N_k \) and \( j \leq k \). Let us write \( n = q_1^{\alpha_1} \cdots q_j^{\alpha_j} \) (with \( q_1 < \cdots < q_j \) as defined in the proof of Lemma 2.4). We have

\[
\frac{n}{\varphi(n)} = \prod_{i=1}^{j} \frac{1}{1 - 1/q_i} \leq \prod_{i=1}^{j} \frac{1}{1 - 1/p_i} \leq \prod_{i=1}^{k} \frac{1}{1 - 1/p_i} = \frac{N_k}{\varphi(N_k)},
\]

which yields

\[
c(n) \leq \left( \frac{N_k}{\varphi(N_k)} - e^\gamma \log \log n \right) \sqrt{\log n} =: h(n) \tag{3.2}
\]

and \( h(n) \) can be extended to all real \( n \). Further,

\[
\frac{d}{dn} h(n) = \frac{1}{2n \sqrt{\log n}} \left( \frac{N_k}{\varphi(N_k)} - e^\gamma \log \log n - 2e^\gamma \right) \leq \frac{1}{2n \sqrt{\log n}} \left( \frac{N_k}{\varphi(N_k)} - e^\gamma \log \log N_k - 2e^\gamma \right).
\]

If \( k = 1 \) or \( 2 \), it is easy to see that the expression in parentheses is negative, while, if \( k \geq 3 \), by (1.1), it is smaller than \( \frac{2.51}{\log \log N_k} - 2e^\gamma \), which is also negative because \( \log \log N_k \geq \log \log 30 = 1.22 \ldots \). Therefore, \( h(n) \leq h(N_k) = c(N_k) \), which, with (3.2), completes the proof of Lemma 3.1.

**Proposition 3.1.** Assume that \( x_0 = 10^9 \leq p_k \leq x < p_{k+1} \). Under RH, we have

\[
(3.3) \quad c(N_k) \leq e^\gamma \left( 2 + W(x) \right) - \frac{0.07}{\log x} \leq e^\gamma (2 + \beta) - \frac{0.07}{\log x}
\]

and

\[
(3.4) \quad c(N_k) \geq e^\gamma \left( 2 + W(x) \right) - \frac{3.7}{\log x} \geq e^\gamma (2 - \beta) - \frac{3.7}{\log x}.
\]

**Proof.** From (1.2) and (1.9), we get

\[
(3.5) \quad c(N_k) = e^\gamma \sqrt{\theta(x)(\log \theta(x))} \left( \frac{1}{f(x)} - 1 \right).
\]

By the fundamental theorem of calculus, (1.14) and (1.12), we have

\[
|\sqrt{\theta(x)} \log \theta(x) - \sqrt{x} \log x| = \left| \int_x^{\theta(x)} \frac{\log t + 2}{2\sqrt{t}} \, dt \right| \leq |\theta(x) - x| \frac{\log(4x/5) + 2}{2\sqrt{4x/5}} \leq \frac{\sqrt{5}}{4} T(x) \frac{\log x + 2}{\sqrt{x}} = \frac{\sqrt{5}}{32\pi} (\log^2 x)(\log x + 2),
\]

whence

\[
\left| \frac{\sqrt{\theta(x)} \log \theta(x)}{\sqrt{x} \log x} - 1 \right| \leq \frac{\sqrt{5}(\log^2 x)(\log x + 2)}{32\pi \sqrt{x} \log x} \leq \frac{\sqrt{5}(\log^2 x_0)(\log x_0 + 2)}{32\pi \sqrt{x_0} \log x_0} \leq \frac{0.0069}{\log x}.
\]

Therefore, (3.5), (2.17) and (1.19) yield

\[
c(N_k) \leq e^\gamma \left( 2 + W(x) - \frac{0.054}{\log x} \right) \left( 1 + \frac{0.0069}{\log x} \right) \leq e^\gamma (2 + W(x)) - \frac{e^\gamma}{\log x} (0.054 - 0.0069(2 + \beta)),
\]

which proves (3.3). The proof of (3.4) is similar.

**4. Proof of Theorem 1.1.** It follows from (3.1), (3.3) and (3.4) that

\[
\limsup_{n \to \infty} c(n) = e^\gamma \left( 2 + \limsup_{x \to \infty} W(x) \right).
\]

As observed in [6, p. 383], by the pigeonhole principle (cf. [3, §2.11] or [4, §11.12]), one can show that \( \limsup_{x \to \infty} W(x) = \beta \), which proves (1.4).
To show the other items of Theorem 1.1 we first consider $k_0 = 50847534$, the number of primes up to $x_0 = 10^9$. For all $k \leq k_0$, we have calculated $c(N_k)$ in Maple with 30 decimal digits, so that we may think that the first ten are correct.

We have found that for $k_1 = 120568 < k \leq k_0$, we have $c(N_k) < e^{\gamma}(2 + \beta)$ (while $c(N_{k_1}) = 3.6444180 \ldots > e^{\gamma}(2 + \beta)$) and for $1 \leq k \leq k_0$, we have $c(N_1) = 3 < c(N_k) \leq c(N_{66})$.

Further, for $k > k_0$, (3.3) implies $c(N_k) < e^{\gamma}(2 + \beta) < c(N_{66})$, which, together with Lemma 3.1 proves (1.5) and (1.6).

As a challenge, for $k_1 = 120568$, I ask what is the largest number $M$ such that $M < N_{k_1} + 1$ and $c(M) \geq e^{\gamma}(2 + \beta)$. Note that $M > N_{k_1}$ since, for $n = N_{k_1-1}p_{k_1+1}$, we have $c(n) = 3.6444178 \ldots > e^{\gamma}(2 + \beta)$. Another challenge is to determine all the $n$'s satisfying $n < N_{k_1+1}$ and $c(n) > e^{\gamma}(2 + \beta)$.

Finally, for $k > k_0$, (3.4) implies
\[ c(N_k) \geq e^{\gamma}(2 - \beta) - \frac{3.7}{\log(10^9)} = 3.30 \ldots > c(2), \]
which completes the proof of (1.7) and of Theorem 1.1.

It is not known if $\liminf_{x \to \infty} W(x) = -\beta$. Let $\rho_1 = 1/2 + it_1$ with $t_1 = 14.13472 \ldots$ be the first zero of $\zeta$. By using a theorem of Landau (cf. [3] Th. 6.1 and §2.4), it is possible to prove that $\liminf_{x \to \infty} W(x) \leq -1/(\rho_1(1 - \rho_1)) = -0.00499 \ldots$. A smaller upper bound is desired.

An interesting question is the following: assume that RH fails. Is it possible to get an upper bound for $k$ such that $k > k_0$ and either $c(N_k) > e^{\gamma}(2 + \beta)$ or $c(N_k) < c(2)$?

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References

Small values of the Euler function


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