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EXISTENCE OF UNITY IN LEHMER'S ψ-PRODUCT RING-II

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Let Z^+ denote the set of positive integers and T be a non-empty subset of $Z^+ \times Z^+$. Let $\psi : T \to Z^+$ be a mapping such that for each n in Z^+, $\psi(x, y) = n$ has a finite number of solutions. If F denotes the set of arithmetic functions, for f, g in F, $f \psi g$ is defined by $(f \psi g)(n) = \sum_{\psi(x, y) = n} f(x) g(y)$ for each n in Z^+. If ψ satisfies the max condition viz., $\psi(x, y) \geq \{x, y\}$ for all $(x, y) \in T$, in 1989, V. Sitaramaiah proved that the commutative ring $(F, +, \psi)$ possesses unity if and only if ψ is onto. In the present paper, we show that this result holds well irrespective of the max condition. We also show that if ψ is multiplicativity preserving and onto $(F, +, \psi)$ possesses unity which is also multiplicative. This answers a question of V. Sitaramaiah and M. V. Subbarao raised in 1994.

Key Words : Arithmetic Function; Lehmer's-ψ Product;Unity; Multiplicativity Preserving

1. INTRODUCTION

As usual, an arithmetic function is a complex-valued function defined on the set of positive integers Z^+. The set of all arithmetic functions will be denoted by F. Let T be a non-empty subset of $Z^+ \times Z^+$ and $\psi : T \to Z^+$ be a mapping satisfying the following :

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For each $n \in Z^+$, $\psi(x, y) = n$ has a finite number of solutions. ... (1.1)

The statements "$(y, z) \in T$ and $(x, \psi(y, z)) \in T$" and "$(x, y) \in T$ and $(\psi(x, y), z) \in T$" are equivalent, when one of these conditions hold, we have

$$\psi(x, \psi(y, z)) = \psi(\psi(x, y), z).$$

If

$$(x, y) \in T, \text{ then } (y, x) \in T \text{ and } \psi(x, y) = \psi(y, x), \quad \text{... (1.3)}$$

for $f, g \in F$, the ψ-product of f and g denoted by $f \psi g$ is defined by

$$(f \psi g)(n) = \sum_{\psi(x, y) = n} f(x) g(y),$$

for each $n \in Z^+$. The binary operation ψ in (1.4) is due to D. H. Lehmer[1].

Using (1.1)-(1.3) it is not difficult to show that $(F, +, \psi)$ is a commutative ring, where '+' denotes the pointwise addition.

In 1989, V. Sitaramaiah proved the following.

Theorem 1.5[3] (Theorem 2.2)) — *Let ψ satisfy (1.1)-(1.3) and $\psi(x, y) \geq \max\{x, y\}$ for all $(x, y) \in T$. Then the commutative ring $(F, +, \psi)$ possesses the unity if and only if for each $k \in Z^+$, $\psi(x, k) = k$ has a solution. In such a case, if g stands for the unity, then*

$$g(k) = 1 - \sum_{\substack{\psi(x, k) = k \\ x < k}} g(x), \text{ if } \psi(k, k) = k, \quad \text{... (1.6)}$$

and

$$g(k) = 0, \text{ if } \psi(k, k) \neq k. \quad \text{... (1.7)}$$

In this paper, we prove (see Theorem 3.1) the above result without assuming the condition "$\psi(x, y) \geq \max\{x, y\}$ for all $(x, y) \in T$" (this condition will be referred to as max condition in the sequel). The proof of Theorem 1.5 was divided into two cases (see[3]); in case 1 the max condition played an important role; in the present paper, we suppress this by using complete induction on an appropriate 'candidate'. The second case of the proofs of Theorem 1.5 and 3.1 is same except for few modifications.

As usual, an arithmetic function f which is not identically zero is said to be multiplicative if $f(mn) = f(m) f(n)$ for all positive integers m and n which are relatively prime. If ψ satisfies (1.1)-(1.3), ψ in (1.4) is said to be multiplicativity preserving (see [4] and [5]) if $f \psi g$ is multiplicative whenever f and g are.

In [5, Section 4], V. Sitaramaiah and M. V. Subbarao expected that if ψ is multiplicativity preserving and onto, then the commutative ring $(F, +, \psi)$ would possess a unity which is multiplicative and that it would be possible to find necessary and sufficient conditions for a multiplicative function to be invertible with respect to ψ; in this paper we answer the first part of the question in affirmative (see Theorem 3.12) and give a partial solution to the second part (see section 4).

Section 2 contains preliminaries. The main results are included in sections 3 and 4.

2. PRELIMINARIES

First we have

Lemma 2.1 (cf. [3], Lemma 2.1) — $g \in F$ is an identity with respect to ψ, that is $f \psi g = f$ for all $f \in F$ if and only if, for any fixed $k, n \in Z^+$,

$$\sum_{\psi(x,n)=k} g(x) = \begin{cases} 1 & \text{if } n=k, \\ 0 & \text{if } n \neq k \end{cases} \qquad \text{... (2.2)}$$

For each $k \in Z^+$, let S_k denote the set defined by

$$S_k = \{x : \psi(x,k) = k\}. \qquad \text{... (2.3)}$$

Lemma 2.4 (cf. [3], Lemma 2.1) — We have

(*i*) $a, b \in S_k \Rightarrow \psi(a,b) \in S_k$.

(*ii*) If $a \in S_k$, then $S_a \subseteq S_k$.

(*iii*) $\psi(a,b) = k \Rightarrow S_a \subseteq S_k$ and $S_b \subseteq S_k$.

Remark 2.5 : If S_k is non-empty, for $a, b \in S_k$ if we define $a \psi b = \psi(a,b)$, then (i) of Lemma 2.4 shows that ψ is a binary operation on S_k. Also, by (1.1) and (1.2) it follows that (S_k, ψ) is a finite semi-group. Thus (S_k, ψ) contains an idempotent element if $S_k \neq \phi$; this means that we can find a positive integer t in S_k such that $t \in S_t$.

Lemma 2.6 (cf. [5], Theorem 3.1) — Let ψ be multiplicativity preserving and for each $k \in Z^+$, $\psi(x,k) = k$ has a solution. Let $x = \prod_{i=1}^{r} p_i^{\alpha_i}$ and $y = \prod_{i=1}^{r} p_i^{\beta_i}$, where $p_1, p_2, \ldots, p_r$ are distinct primes, α_i and β_i are non-negative integers for $i = 1, 2, \ldots, r$. Then we have

$$(x,y) \in T \text{ and only if } (p_i^{\alpha_i}, p_i^{\beta_i}) \in T \text{ for } i = 1, 2, \ldots, r. \qquad \text{... (2.7)}$$

If

$$(x,y) \in T, \text{ then } \psi(x,y) = \prod_{i=1}^{r} p_i^{\theta_{p_i}(\alpha_i, \beta_i)}, \qquad \text{... (2.8)}$$

where for any prime p,

$\theta_p(\alpha, \beta)$ is a unique non-negative integer defined for non-negative integers α, β such that $(p^\alpha, p^\beta) \in T$ and satisfies

$$p^{\theta_p(\alpha,\beta)} = \psi(p^\alpha, p^\beta). \qquad \text{... (2.9)}$$

For each integer

$$\gamma \geq 0,\ \theta_p(\alpha, \beta) = \gamma \text{ has a finite number of solutions.} \quad \text{... (2.10)}$$

$$\theta_p(\alpha, \beta) = 0 \text{ if and only if } \alpha = \beta = 0. \quad \text{... (2.11)}$$

For each $\gamma \geq 0$,

$$\theta_p(\alpha, \gamma) = \gamma \text{ has a solution.} \quad \text{... (2.12)}$$

$$\theta_p(\alpha, \beta) = \theta_p(\beta, \alpha). \quad \text{... (2.13)}$$

For non-negative integers α, β, γ the statements "$(p^\alpha, p^\gamma) \in T, (p^\alpha, p^{\theta_p(\beta, \gamma)}) \in T$" and "$(p^\alpha, p^\beta) \in T$ and $(p^{\theta_p(\alpha, \beta)}, p^\gamma) \in T$" are equivalent; when one of these conditions holds, we have

$$\theta_p(\alpha, \theta_p(\beta, \gamma)) = \theta_p(\theta_p(\alpha, \beta), \gamma). \quad \text{... (2.14)}$$

Lemma 2.15 (cf. [5], Theorem 3.2) — Let $\phi \neq T \subseteq Z^+ \times Z^+$ be such that

$$(x, y) \in T \text{ if and only if } (y, x) \in T. \quad \text{... (2.15)}$$

If x and y are as given in Lemma 2.6, then $(x, y) \in T$ if and only if

$$(p_i^{\alpha_i}, p_i^{\beta_i}) \in T \text{ for } i = 1, 2, \ldots, r. \quad \text{... (2.16)}$$

Further, for each prime p and non-negative integers α and β such that $(p^\alpha, p^\beta) \in T$, let $\theta_p(\alpha, \beta)$ be a non-negative integer satisfying (2.9)-(2.14). If for $(x, y) \in T$, $\psi(x, y)$ is defined by (2.8), then ψ is multiplicativity preserving and for each $k \in Z^+$, $\psi(x, k) = k$ has a solution.

3. Existence of the Unity

In this section we prove

Theorem 3.1 —*The commutative ring $(F, +, \psi)$ possesses unity if and only if for each $k \in Z^+$, $\psi(x, k) = k$ has a solution.*

PROOF : If $(F, +, \psi)$ possesses unity, by (2.2) for each $k \in Z^+$ the equation $\psi(x, k) = k$ must have a solution. Conversely, we assume that for each $k \in Z^+$, $\psi(x, k) = k$ has a solution so that the set S_k defined in (2.3) is non-empty. We define the arithmetic function g as follows :

$$g(k) = 0 \text{ if } k \notin S_k. \quad \text{... (3.2)}$$

If $k \in S_k$, we define $g(k)$ by complete induction on the cardinality of the sets S_k. For each $n \in Z^+$, let $P(n)$ denote the proposition that g can be defined on sets $S_k \supseteq \{k\}$ such that $\sum_{x \in S_k} g(x) = 1$, whenever $|S_k| = n$.

If $n = 1$ so that $S_k = \{k\}$, we define $g(k) = 1$, and hence P (1) is true. Assume that $P(m)$ is true for $1 \le m \le n$. Let $|S_k| = n+1$ and $S_k \supseteq \{k\}$. We define $g(k)$ by the identity

$$1 = g(k) + \sum_{\substack{x \in S_k \cap S_x \\ x \neq k}} g(x). \quad \text{... (3.3)}$$

By Lemma 2.4, $x \in S_k$ implies $S_x \subseteq S_k$. Als $k \notin S_x$ if $x \in S_k$ and $x \neq k$. For, if $k \in S_x$, we must have $\psi(x, k) = x$. But $x \in S_k$ implies that $\psi(x, k) = k$ so that $x = k$. This is a contradiction since $x \neq k$. Thus if $x \in S_k$ and $x \neq k$, $|S_x| < |S_k|$. By our induction hypothesis $g(x)$ in the sum on the right hand side of (2.2) is a known quantity and hence $g(k)$ can be found from (3.3). Again by (3.2) and (3.3),

$$1 = \sum_{x \in S_k} g(x). \quad \text{... (3.4)}$$

The induction is complete.

Having defined g, we now claim that g is the unity in $(F, +, \psi)$. By Lemma 2.1 and the identity (3.4), we need to only show that if $n \neq k$, then

$$\sum_{\psi(x, n) = k} g(x) = 0. \quad \text{... (3.5)}$$

Let $n \in Z^+$ and $n \neq k$. We distinguish two cases:

Case 1 — Let $n \in S_n$. Let $P(m)$ denote the proposition that (3.5) holds for all $k \neq n$ such that $|S_k| = m$. Let $|S_k| = 1$. If the sum in (3.5) is non-empty, we can assume that x appearing in this sum is in S_x since $g(x) = 0$ if $x \notin S_x$. Also, by Lemma 2.4, $\psi(x, n) = k$ implies $S_x \subseteq S_k$ and $S_n \subseteq S_k$.

Since $x \in S_x$ and $n \in S_n$, both x and n are in S_k so that by Lemma 2.4, $k = \psi(x, n) \in S_k$. Since $|S_k| = 1$, we must have $S_k = \{k\}$. Since $S_n \subseteq S_k = \{k\}$, $n \in S_n$ implies that $n = k$. This is a contradiction to $n \neq k$. Thus the sum in (3.5) is an empty sum so that (3.5) holds. Thus P (1) is true. We assume that $P(m)$ is true for $1 \le m \le r$. Let $|S_k| = r+1$. If $\psi(x,n) = k$ has a solution, as above it follows that $k \in S_k$ and $n \in S_n \subseteq S_k$. Hence we can assume that

$$S_k = \{n, k, x_1, x_2, \ldots, x_{r-1}\}, \qquad \ldots (3.6)$$

where $\quad x_i \notin \{n, k\}$ for $i = 1, 2, \ldots, r-1$.

If $x \in S_k$, then since $n \in S_k$, $\psi(x,n) \in S_k$. Hence by (3.4) and (3.6), we have

$$1 = \sum_{x \in S_k} g(x) = \sum_{\substack{x \in S_k \\ \psi(x,n)=n}} g(x) + \sum_{\substack{x \in S_k \\ \psi(x,n)=k}} g(x) + \sum_{j=1}^{r-1} \sum_{\substack{x \in S_k \\ \psi(x,n)=x_j}} g(x) . \qquad \ldots (3.7)$$

In each sum on the right-hand side of (3.7), by (3.2) we can assume that $x \in S_x$. Also, if $a \in S_k$, $\psi(x,n) = a$ implies $x \in S_x \subseteq S_k$ so that $x \in S_k$. Therefore

$$1 = \sum_{x \in S_k} g(x) = \sum_{\psi(x,n)=n} g(x) + \sum_{\psi(x,n)=k} g(x) + \sum_{j=1}^{r-1} \sum_{\psi(x,n)=x_j} g(x)$$

$$= \sum_{x \in S_n} g(x) + \sum_{\psi(x,n)=k} g(x) + \sum_{j=1}^{r-1} \sum_{\psi(x,n)=x_j} g(x)$$

$$= 1 + \sum_{\psi(x,n)=k} g(x) + \sum_{j=1}^{r-1} \sum_{\psi(x,n)=x_j} g(x) . \qquad \ldots (3.8)$$

Now, $x_j \ne n$ and $k \in S_{xj}$ for $1 \le j \le r-1$. Since $k \in S_k$ and $S_{xj} \subseteq S_k$, it follows that $|S_{xj}| < |S_k| = r+1$. By the induction hypothesis, $\sum_{\psi(x,n)=x_j} g(x) = 0$, for $1 \le j \le r-1$; (3.5) follows from this and (3.8).

Case 2 — Let $n \notin S_n$ and $n \ne k$. Since S_n is non-empty, by Remark 2.5 we can find $t \in S_n$ such that $t \in S_t$. Clearly, $t \ne n$. The remaining part of the proof of this case is almost same as that in [3, Theorem 2.3]. In (3.5), we can assume that $x \in S_x$ so that $\psi(x,n) = k$ implies $x \in S_x \subseteq S_k$ and $S_n \subseteq S_k$ (by Lemma 2.4). Hence we can assume that x appearing in the sum in (3.5) is in S_k. Since $t \in S_t \subseteq S_n \subseteq S_k$, $x \in S_k$ implies that $\psi(x,t) \in S_k$. Moreover, if $\psi(x,n) = k$, then we also have $k = \psi(x,n) = \psi(x, \psi(t,n)) = \psi(\psi(x,t), n)$, so that $\psi(x,t)$ is also a solution of $\psi(y,n) = k$.

Let $x_1, x_2, \ldots, x_r$ be all the elements of S_k satisfying $\psi(x_i, n) = k$ for $i = 1, 2, \ldots, r$. Since $t \in S_n$, $\psi(t, n) = n$. This together with $n \neq k$ implies that $x_i \neq t$ for $i = 1, 2, \ldots r$. We have

$$\sum_{\psi(x, n) = k} g(x) = \sum_{i=1}^{r} \sum_{\substack{\psi(x, n) = k \\ \psi(x, t) = x_i}} g(x) . \qquad \ldots (3.9)$$

Also, for $i = 1, 2, \ldots, r$, $\psi(x, t) = x_i$ and $\psi(x_i, n) = k$ implies

$$k = \psi(x_i, n) = \psi(\psi(x, t), n) = \psi(x, \psi(t, n)) = \psi(x, n),$$

so that from (3.9) we obtain

$$\sum_{\psi(x, n) = k} g(x) = \sum_{i=1}^{r} \sum_{\psi(x, t) = x_i} g(x) . \qquad \ldots (3.10)$$

Since $t \in S_t$ and $t \neq x_i$ for $i = 1, 2, \ldots, r$ it follows from Case 1 that each term on the right-hand side of (3.10) vanishes. Hence (3.5) follows. The proof of Theorem 3.1 is complete.

It has been shown in [5, Lemma 2.2] that the statements (a) ψ is onto (b) Given $k \in Z^+$, $\psi(x, k) = k$ has a solution, are equivalent, where ψ satisfies (1.1)-(1.3). Hence we can restate Theorem 3.1 as

Theorem 3.11 — *The commutative ring $(F, +, \psi)$ possesses unity if and only if ψ is onto.*

Theorem 3.12 — *If ψ is multiplicativity preserving and onto, then the commutative ring $(F, +, \psi)$ possesses unity which is multiplicative.*

PROOF : By Theorem 3.11, $(F, +, \psi)$ possesses unity. Only we need to show that the unity is multiplicative. Instead of proving that the unity is multiplicative, we use the method of proof of Theorem 3.1 to construct a unity, which is multiplicative. By Lemmas 2.6 and 2.15, we can assume that T, ψ and θ_p are as given in Lemma 2.5. We fix a prime p and write $\theta_p = \theta$ for convenience. For integers $\alpha \geq 0$, define the sets E_α by $E_\alpha = \{\beta : \theta(\beta, \alpha) = \alpha\}$. Since ψ is onto, E_α is not empty for each $\alpha \geq 0$. In view of (2.10)-(2.14), Lemma 2.4 and Remark 2.5 remain valid with S_k replaced by E_α and ψ replaced by θ. We define the multiplicative function g by defining $g(p^\alpha)$ at any $\alpha \geq 0$. If $\alpha \notin E_\alpha$, we define $g(p^\alpha) = 0$. If $\alpha \in E_\alpha$, $g(p^\alpha)$ is defined by using complete induction on the sets E_α so that

$$\sum_{\theta(\beta, \alpha) = \alpha} g(p^\beta) = 1. \qquad \ldots (3.13)$$

As in the proof of Theorem 3.1, we can show that for non-negative integers α and γ

$$\sum_{\theta(\beta,\alpha)=\gamma} g(p^{\beta})=0, \text{ if } \alpha\neq\gamma. \qquad \text{... (3.14)}$$

If $n=p_1^{\alpha_1}\ldots p_r^{\alpha_r}$ and $k=p_1^{\gamma_1}\ldots p_r^{\gamma_r}$, where $p_1,\ldots,p_r$ are distinct primes, αi's and βi's are non-negative integers, we have

$$\sum_{\psi(x,n)=k} g(x)=\prod_{i=1}^{r}\sum_{\theta(\beta_i,\alpha_i)=\gamma_i} g(p^{\beta_i})=\begin{cases}1 & \text{if } n=k,\\ 0 & \text{if } n\neq k,\end{cases}$$

by (3.13) and (3.14). By Lemma 2.2, it follows that g is the unity in $(F,+,\psi)$ and since g is also multiplicative, we obtain Theorem 3.12.

4. Units in $(F,+,\psi)$

After finding necessary and sufficient conditions for the existence of unity in the commutative ring $(F,+,\psi)$, it remains to investigate necessary and sufficient conditions for the invertibility of an arithmetic function with respect to ψ. In the presence of max condition, the following theorem due to V. Sitaramaiah gives such conditions:

***Theorem* 4.1** — (Cf. [2], Theorem 2.2) — *Let $(F,+,\psi)$ be a commutative ring with unity g. Let $\psi(x,y)\geq\max\{x, y\}$ for all $(x,y)\in T$. Then $f\in F$ is invertible with respect to ψ if and only if*

$$S_f(k):=\sum_{\psi(x,k)=k} f(x)\neq 0 \qquad \text{... (4.2)}$$

for all $k\in Z^+$. In such a case, f^{-1} can be computed inductively by

$$f^{-1}(1)=1/f(1),$$

and for $k\geq 2$,

$$f^{-1}(k)=(S_f(k))^{-1}\left\{g(k)-\sum_{\substack{\psi(x,y)=k\\ y<k}} f(x)f^{-1}(y)\right\}.$$

In a way, Theorem 4.2 is a partial solution to the problem mentioned in the beginning of section 4; Yet, some thing is remaining viz., to find necessary and sufficient conditions for the invertibility of $f\in F$ with respect to ψ without assuming the max condition. We shall make an attempt to answer this question partially.

Suppose that we are given the commutative ring $(F,+,\psi)$ with unity say g so that ψ satisfies the hypothesis of Theorem 3.1. A natural way to try for $f^{-1}(n)$ is to consider the identity $(f\psi f^{-1})(n)=g(n)$ for $n\in Z^+$. Let us fix a positive integer n. We have

$$g(n) = (f\psi f^{-1})(n) = \sum_{\psi(x,y)=n} f(x)f^{-1}(y)$$

$$= f^{-1}(n)\, S_f(n) = \sum_{\substack{\psi(x_1,y_1)=n \\ y_1 \neq n}} f(x_1)f^{-1}(y_1). \qquad \text{... (4.3)}$$

Let $y_1 \neq n$ and $\psi(x_1, y_1) = n$ so that $S_{y_1} \subseteq S_n$. If $f^{-1}(y_1)$ is known, $f^{-1}(n)$ can be found from (4.3) if $S_f(n) \neq 0$. To find $f^{-1}(y_1)$ we proceed as above. We have

$$g(y_1) = (f\psi f^{-1})(y_1) = f^{-1}(y_1)\, S_f(y_1) + \sum_{\psi(x_2,y_2)=y_1} f(x_2)f^{-1}(y_2)$$

$$= f^{-1}(y_1)\, S_f(y_1) + f^{-1}(n) \sum_{\psi(x_2,n)=y_1} f(x_2) + \sum_{\substack{\psi(x_2,y_2)=y_1 \\ y_2 \neq y_1,\, y_2 \neq n}} f(x_2)f^{-1}(y_2). \qquad \text{... (4.4)}$$

In (4.4), $\psi(x_2, n) = y_1$ implies that $S_n \subseteq S_{y_1}$ that $S_{y_1} = S_n$. Hence $S_f(y_1) = S_f(n)$. Using this in (4.4), we obtain

$$g(y_1) = f^{-1}(y_1)\, S_f(n) + f^{-1}(n)\, h(n, y_1) + H_2(y_1), \qquad \text{... (4.5)}$$

where
$$h(a, b) = \sum_{\psi(x,a)=b} f(x), \qquad \text{... (4.6)}$$

and
$$H_2(y_1) = \sum_{\substack{\psi(x_2,y_2)=y_1 \\ y_2 \neq y_1,\, y_2 \neq n}} f(x_2)f^{-1}(y_2). \qquad \text{... (4.7)}$$

Assuming $S_f(n) \neq 0$, we have from (4.5),

$$f^{-1}(y_1) = (S_f(n))^{-1}\,(g(y_1) - f^{-1}(n)\, h(n, y_1) - H_2(y_1))$$

Using this in (4.3), we obtain after simplification,

$$g(n) = f^{-1}(n)\,(S_f(n) - (S_f(n))^{-1} \sum_{\substack{\psi(x_1,y_1)=n \\ y_1 \neq n}} f(x_1)\, h(n, y_1))$$

$$+ (S_f(n))^{-1} \sum_{\substack{\psi(x_1,y_1)=n \\ y_1 \neq n}} f(x_1)\, g(y_1) \qquad \text{... (4.8)}$$

$$-(S_f(n))^{-1} \sum_{\substack{\psi(x_1, y_1)=n \\ y_1 \neq n}} f(x_1) H_2(y_1). \quad \text{... (4.9)}$$

At this stage, we can impose some sufficient conditions to find $f^{-1}(n)$. For example, if the sum in (4.9) vanishes, $S_f(n) \neq 0$ and the coefficient of $f^{-1}(n)$ in (4.8) is non-zero, we can find $f^{-1}(n)$. However, to calculate $f^{-1}(n)$ these need not be the only sufficient conditions. If we continue further, that is, finding $f^{-1}(y_2)$ from $g(y_2) = (f \psi f^{-1})(y_2)$, substituting this into (4.7) and the resulting expression for $H_2(y_1)$ in (4.9), we obtain after simplification,

$$g(n) = f^{-1}(n)\,(S_f(n) - (S_f(n))^{-1} \sum_{\substack{\psi(x_1, y_1)=n \\ y_1 \neq n}} f(x_1)\, h(n, y_1)$$

$$+ (S_f(n))^{-2} \sum_{\substack{\psi(x_1, y_1)=n \\ y_1 \neq n}} \sum_{\substack{\psi(x_2, y_2)=y_1 \\ y_2 \notin (y_1, n)}} f(x_1) f(x_2)\, h(n, y_2))$$

$$+ (S_f(n))^{-1} \left\{ \sum_{\substack{\psi(x_1, y_1)=n \\ y_1 \neq n}} f(x_1)\, g(y_1) - (S_f(n))^{-1} \sum_{\substack{\psi(x_1, y_1)=n \\ y_1 \neq n}} \sum_{\substack{\psi(x_2, y_2)=y_1 \\ y_2 \notin (y_1, n)}} f(x_1) f(x_2)\, g(y_2) \right\}$$

$$+ S_f(n))^{-2} \left\{ \sum_{\substack{\psi(x_1, y_1)=n \\ y_1 \neq n}} f(x_1) f^{-1}(y_1) \sum_{\substack{\psi(x_2, y_2)=y_1 \\ y_2 \notin (y_1, n)}} f(x_2)\, h(y_1, y_2) \right\}$$

$$+ (S_f(n))^{-2} \sum_{\substack{\psi(x_1, y_1)=n \\ y_1 \neq n}} f(x_1) \sum_{\substack{\psi(x_2, y_2)=y_1 \\ y_2 \notin (y_1, n)}} f(x_2)\, H_3(y_2), \quad \text{... (4.10)}$$

where
$$H_3(y_2) = \sum_{\substack{\psi(x_3, y_3)=y_2 \\ y_3 \notin (y_1, y_2, n)}} f(x_3) f^{-1}(y_3).$$

If $L = \{y : \psi(x, y) = n\}$, y_1, y_2, y_3 and n appearing in the above sums are pairwise distinct elements of L. Hence $|L| \geq 4$. Thus if $m = |L|$, proceeding as above and since $H_m(y_{m-1}) = 0$, we obtain after simplification,

$$g(n)=f^{-1}(n)\ \ \{S_f(n)+\sum_{j=1}^{m-1}(-S_f(n))^{-j}\sum_{\substack{\psi(x_k,y_k)=y_{k-1}\\ y_k\notin(y_0,y_1,\dots,y_{k-1})\\ 1\le k\le j}} f(x_1)f(x_2)\dots f(x_j))\,h(n,y_j)$$

$$+\sum_{j=1}^{m-1}(-1)^{j-1}(S_f(n))^{-j}\sum_{\substack{\psi(x_k,y_k)=y_{k-1}\\ y_k\notin(y_0,y_1,\dots,y_{k-1})\\ 1\le k\le j}}$$

$$(f(x_1)f(x_2)\dots f(x_j))\,g(y_j)\} \qquad \dots (4.11)$$

$$+\sum_{r=2}^{m-1}\sum_{j=r}^{m-1}(-S_f(n))^{-j}\sum_{\substack{\psi(x_k,y_k)=y_{k-1}\\ y_k\notin(y_0,y_1\dots y_{k-1})\\ 1\le k\le j}}$$

$$(f(x_1)\dots f(x_j))f^{-1}(y_{r-1})\,h(y_{r-1},y_j), \qquad \dots (4.12)$$

where $y_0=n$. Hence it would be possible to find $f^{-1}(n)$ if

(*a*) $S_f(n)\neq 0$. ... (4.13)

(*b*) The coefficient of $f^{-1}(n)$ in (4.11) is non-zero. ... (4.14)

(*c*) The coefficient of $f^{-1}(y_{r-1})$ in (4.12) vanishes for $r=2, 3, \dots, m-1$. ... (4.15)

Although we could arrive at a set of sufficient conditions, honestly speaking, it is not easy to verify them. At present we do not know whether the conditions (4.13)-(4.15) are also necessary. As mentioned in the introduction, Sitaramaiah and Subbarao (cf. [5] Section 4) expected that if ψ is multiplicativity preserving and onto it would be possible to find necessary and sufficient conditions for a multiplicative function to be invertible with respect to ψ. One may note that a partial solution has been given above not only for a multiplicative function but for any arithmetic function.

To find f^{-1} in particular examples, it would be convenient to use the procedure to arrive at (4.11) than using (4.11). We illustrate this in the following two examples discussed in [3, Remarks 2.1 and 2.2].

Example 4.16 — Let $T=\{(1,2),(2,1)\}\cup\{(k,k):k\geq 2\}$ and ψ on T be defined by $\psi(1,2)=\psi(2,1)=1$ and $\psi(k,k)=k$ for $k\geq 2$. Then ψ satisfies (1.1)-(1.3) and ψ does not satisfy the max condition. Further ψ is onto. Also, $S_1=\{2\}$ and $S_k=\{k\}$ for $k\geq 2$. Thus if g is the arithmetic function defined by $g(1)=0$ and $g(k)=1$ for $k\geq 2$, then g is the unity in $(F,+,\psi)$. If f is in F, we have

$$0 = g(1) = (f \psi f^{-1})(1) = \sum_{\psi(x, y) = 1} f(x) f^{-1}(y) = f(1) f^{-1}(2) + f(2) f^{-1}(1); \quad \ldots (4.17)$$

and in a similar way for $k \geq 2$,

$$1 = g(k) = f(k) f^{-1}(k). \quad \ldots (4.18)$$

From (4.17) and (4.18) it is clear that f^{-1} exists if and only if $f(k) \neq 0$ for all $k \geq 2$.

Remark 4.19 : In [3, Remark 2.1] it has been mentioned that the commutative ring $(F, +, \psi)$ in example 4.16 does not contain unity which is incorrect.

Example 4.20 — Let $T = \{(2k, 2k), (2k, 2k-1), (2k-1, 2k) : k \in Z^+\}$ and ψ on T be defined by $\psi(x, y) = \min\{x, y\}$. It can be shown that ψ satisfies (1.1)-(1.3) and ψ is onto. Also, $S_{2k} = S_{2k-1} = \{2k\}$. Therefore $g \in F$ defined by $g(2k = 1$ and $g(2k - 1) = 0$ for $k \in Z^+$ is the unity in $(F, +, \psi)$. It can also be shown that $f \in F$ is invertible with respect to ψ if and only if $f(2k) \neq 0$ for all $k \in Z^+$.

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