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STRATIFIED SETS

by

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Abstract. — A set $\mathcal{A}$ of integers is said "stratified" if, for all t, $0 \le t < \operatorname{Card} \mathcal{A}$, the sum of any t distinct elements of $\mathcal{A}$ is smaller than the sum of any $t+1$ distinct elements of $\mathcal{A}$. That implies that all elements of $\mathcal{A}$ should be positive. It is proved that the number of stratified sets with maximal element equal to N is exactly the number $p(N)$ of partitions of N.

1. Introduction

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the set of positive integers. After Erdős and Straus (see **[3]** and **[7]**), a set $\mathcal{A} \subset \mathbb{N}$ is said admissible if for any pairs $\mathcal{A}_1, \mathcal{A}_2$ of subsets of $\mathcal{A}$, one has

$$\left(\sum_{a \in \mathcal{A}_1} a = \sum_{a \in \mathcal{A}_2} a \right) \Rightarrow \mid \mathcal{A}_1 \mid = \mid \mathcal{A}_2 \mid .$$

Here $\mid \mathcal{A} \mid$ will denote the number of elements of $\mathcal{A}$.

Straus has observed that, if $k = \lfloor 2\sqrt{N + 1/4} - 1 \rfloor$, then the set $\mathcal{A} = \{N, N - 1, \dots, N - k + 1\}$ is admissible, On the other hand, he proved (cf. **[7]**) that if $N = \max_{a \in \mathcal{A}} a$, and $\mathcal{A}$ is admissible, then $\mid \mathcal{A} \mid \le \left(\frac{4}{\sqrt{3}} + o(1) \right) \sqrt{N}$. The constant $4/\sqrt{3}$ has been improved in **[4]**, and in **[1]**, J.M. Deshouillers and G.A. Freiman have replaced it by the best possible constant 2. In **[2]**, they prove that for N large enough, the above example of Straus is the greatest possible admissible set with maximal element N.

Definition 1. — *A set $\mathcal{A} \subset \mathbb{Z}$ is stratified, if for $0 \le t < t'$ the sum of any t distinct elements of $\mathcal{A}$ is strictly smaller than the sum of any t' distinct elements of $\mathcal{A}$.*

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Note that from the above definition all the elements of $\mathcal{A}$ should be positive (choose $t = 0$ and $t' = 1$), and so $\mathcal{A}$ is included in $\mathbb{N}$.

Clearly a stratified set is admissible. The above example of Straus is stratified, and in the table at the end of [4], it can be seen that most of large admissible sets are stratified.

In this paper, stratified sets will be described in terms of partitions (Theorem 1). Further, we shall reformulate some of the conjectures about admissible sets given in [4] in terms of stratified sets. Finally, we shall show that the number of stratified sets with maximal element N is equal to the number of partitions of N (Theorem 2) and a one to one correspondence associating such a stratified set to a partition of N is explicited. As a corollary, the lower bound given in [4] for the total number of admissible sets with maximal element N will be improved.

It is possible to extend the notion of stratified set to subsets in arithmetic progression and in this way to describe some other classes of admissible sets. For instance a subset of odd numbers $\mathcal{A}$ which satisfies that the sum of any t distinct elements is smaller than the sum of any $t + 2$ distinct elements will certainly be admissible, since the sum of t elements and the sum of $(t+1)$ elements are of different parity and therefore are unequal. I hope to return to this question in an other paper.

At the end of this article, a table of the numbers $p(N)$ of stratified sets and $a(N)$ of admissible sets with largest element N is given. The table of $a(N)$ given in [4] is erroneous.

This work has started in September 1991, when G. Freiman was visiting me in Lyon. At that time he was trying to understand the structure of a large admissible set (it was before getting the result of [1]), and he wrote on the blackboard many equations like (11) or (12) below. So an important part of this paper is due to G. Freiman, and I thank him strongly.

I thank also very much Marc Deléglise for calculating the values of $a(N)$, and for listing stratified sets which drove me to Theorem 2. I thank also Paul Erdős, András Sárközy, Etienne Fouvry and Jean-Marc Deshouillers for fruitful discussions on this subject.

Notation. — $t^\wedge\mathcal{A}$ will denote the set of the sums of t distinct elements of $\mathcal{A}$.

2. Description of a stratified set

First it will be proved:

Proposition 1. — *Let $\mathcal{A} = \{a_1 < a_2 < \cdots < a_k = N\}$ be a set of positive integers, and $t_0 = \lfloor (k-1)/2 \rfloor$. Then $\mathcal{A}$ is stratified if and only if*

$$\max t_0{}^\wedge\mathcal{A} < \min (t_0+1)^\wedge\mathcal{A}. \tag{1}$$

Proof. — From the definition, $\mathcal{A}$ is stratified if for all $t, 1 \le t \le k-1$,

$$\max t^\wedge\mathcal{A} < \min (t+1)^\wedge\mathcal{A}. \tag{2}$$

Let us first prove (2) for $t \leq t_0$. From (1), one has:

$$a_k + a_{k-1} + \cdots + a_{k-t_0+1} < a_1 + a_2 + \cdots + a_{t_0+1}$$

which implies

$$(3) \qquad a_k + a_{k-1} + \cdots + a_{k-t+1} < a_1 + \cdots + a_{t+1} + \sum_{i=1}^{t_0-t} (a_{t+1+i} - a_{k-t+1-i}).$$

But for $1 \leq i \leq t_0 - t$, we have $t+1+i < k-t+1-i$ since $2i \leq 2(t_0 - t) \leq k-1-2t < k-2t$; thus, the last sum in (3) is non-positive and (3) yields (2). Let us now suppose that $t > t_0$ and set $S = \sum_{i=1}^{k} a_i$ and $t' = k-t-1$. We have $k/2 - 1 \leq t_0 \leq (k-1)/2$, so that

$$t' = k-t-1 < k-t_0-1 \leq k-(k/2-1)-1 = k/2 \leq t_0+1,$$

and so, $t' \leq t_0$. From the above proof, one gets

$$(4) \qquad \max(t')^\wedge \mathcal{A} < \min(t'+1)^\wedge \mathcal{A},$$

and from the definition of t',

$$(5) \qquad \max t^\wedge \mathcal{A} = S - \min(t'+1)^\wedge \mathcal{A}$$

and

$$(6) \qquad \min(t+1)^\wedge \mathcal{A} = S - \max(t')^\wedge \mathcal{A}.$$

(4), (5) and (6) prove (2), and this completes the proof of Proposition 1.

Theorem 1

(a) Let k be even. There is a one to one correspondence between the stratified sets $\mathcal{A} \subset \mathbb{Z}$ with $\max \mathcal{A} = N$ and $|\mathcal{A}| = k$ and the solutions of the inequality

$$(7) \qquad x_1 + 2(x_2 + x_{k-1}) + 3(x_3 + x_{k-2}) + \cdots + \frac{k}{2}(x_{k/2} + x_{k/2+1}) \leq N - \frac{k^2}{4} - \frac{k}{2}.$$

where the x_i's are non negative integers.

(b) Let k be odd. There is a one to one correspondence between the stratified sets $\mathcal{A} \subset \mathbb{Z}$ with $\max \mathcal{A} = N$ and $|\mathcal{A}| = k$ and the solutions of the inequality:

$$(8) \qquad x_1 + 2(x_2 + x_{k-1}) + \cdots + \frac{k-1}{2}(x_{(k-1)/2} + x_{(k+3)/2}) + \frac{k+1}{2}\, x_{(k+1)/2} \leq N - \frac{(k+1)^2}{4}.$$

Proof. — Let $\mathcal{A} = \{a_1, a_2, \ldots, a_k\} \subset \mathbb{Z}$ with

$$(9) \qquad a_1 < a_2 < \cdots < a_{k-1} < a_k = N.$$

be a stratified set. Let us introduce the new variables

$$x_i = a_{i+1} - a_i - 1, \quad 1 \leq i \leq k-1.$$

From (9), one has

$$x_i \geq 0, \quad 1 \leq i \leq k-1. \tag{10}$$

Conversely, (9) is clearly equivalent to $a_k = N$, and (10). Now we have to express (1) in terms of the $x_i's$, and this explains the role played by the parity of k.

Let us suppose k is even. From the definition of x_i, one has

$$x_1 + 2x_2 + \cdots + tx_t = -\frac{t(t+1)}{2} - a_1 - a_2 - \cdots - a_t + ta_{t+1} \tag{11}$$

$$\begin{aligned} 2x_{k-1} + 3x_{k-2} + \cdots + (u+1)x_{k-u} &= N + a_k + a_{k-1} + \cdots + a_{k-u+1} \\ &\quad -(u+1)a_{k-u} \\ &\quad -\frac{(u+1)(u+2)}{2} + 1. \end{aligned} \tag{12}$$

One chooses $t = t_0 + 1 = k/2$ in (11) and $u = t_0 = \frac{k}{2} - 1$ in (12) and then (11) and (12) give

$$\begin{aligned} \max t_0{}^\wedge\mathcal{A} - \min(t_0+1)^\wedge\mathcal{A} &= a_k + a_{k-1} + \cdots + a_{k-t_0+1} \\ &\quad -a_1 - a_2 - \cdots - a_{t_0+1} \\ &= x_1 + 2(x_2 + x_{k-1}) + \cdots \\ &\quad + \frac{k}{2}(x_{k/2} + x_{k/2+1}) \\ &\quad -N + \frac{k^2}{4} + \frac{k}{2} - 1. \end{aligned}$$

The last term -1 allows us to transform the strict inequality (2) in inequality (7) with $\leq$ sign.

The proof of (8) when k is odd is similar.

Corollary 1. — *Let us denote the number of stratified sets with k elements, and maximal element N by $S_k(N)$. The generating functions are:*
for k even

$$\sum_{N=0}^{\infty} S_k(N)x^N = x^{k^2/4+k/2} \prod_{i=1}^{k/2} \frac{1}{(1-x^i)^2}, \tag{13}$$

for k odd:

$$\sum_{N=0}^{\infty} S_k(N)x^N = \frac{x^{(k+1)^2/4}}{1-x^{(k+1)/2}} \prod_{i=1}^{(k-1)/2} \frac{1}{(1-x^i)^2}. \tag{14}$$

Proof. — It follows easily from the theorem, by the classical method of generating series. For k even, the generating series of the number of solutions of

$$x_1 + 2(x_2 + x_{k-1}) + \cdots + \frac{k}{2}(x_{k/2} + x_{(k/2+1)}) = n \tag{15}$$

is

$$\frac{1}{1-x}\prod_{i=2}^{k/2}\frac{1}{(1-x^i)^2},$$

and if in (15) " $= n$ " is replaced by " $\leq n$ ", the generating function must be multiplied by $1/(1-x)$. At last, in (7), the right hand side is $N - k^2/4 - k/2$, which explains the factor $x^{k^2/4+k/2}$ in (13).
For k odd, the proof is similar.

Corollary 2. — *Let $p(n)$ denote the classical partition function, i.e. the number of ways of writing $n = n_1 + n_2 + \cdots + n_k, n_1 \geq n_2 \cdots \geq n_k \geq 1$, and let us define $P(n) = \sum_{i=0}^{n} p(i)p(n-i)$. So, the generating function of $P(n)$ is*

$$\sum_{n=0}^{\infty} P(n)x^n = \prod_{i=1}^{\infty}\frac{1}{(1-x^i)^2}. \tag{16}$$

The number of stratified sets $\mathcal{A}$ with largest element N and with a maximal number of elements is given by

$$P(N-m^2) \text{ } if \text{ } m^2 \leq N < m(m+1)$$

and by

$$P(N-m^2-m) \text{ } if \text{ } m(m+1) \leq N < (m+1)^2.$$

Proof. — Let us suppose first that $m^2 \leq N < m(m+1)$. For $k = 2m-1$, one has

$$N - \frac{(k+1)^2}{4} = N - m^2 \leq m(m+1) - 1 - m^2 = m - 1 = \frac{k-1}{2}, \tag{17}$$

But by (14), (16) and (17), the number of stratified sets, $S_k(N)$, is equal to $P(N-m^2)$. For $k = 2m$, one has $N - \frac{k^2}{4} - \frac{k}{2} = N - m(m+1) < 0$, and from (13) there is no stratified sets with k elements.

The proof of the second case, $m(m+1) \leq N < (m+1)^2$ is similar.

Remark. — It follows from theorem 1 and the above proof, that the maximal number of elements of a stratified set $\mathcal{A}$ with maximal element N is $\lfloor 2\sqrt{N+1/4} - 1 \rfloor$, that is $2m-1$ if $m^2 \leq N < m(m+1)$ and $2m$ if $m(m+1) \leq N < (m+1)^2$.

Table of $P(N)$:

N	=	0	1	2	3	4	5	6	7	8	9	10	11
$P(N)$	=	1	2	5	10	20	36	65	110	185	300	481	752

This table has to be compared with the column $p(N)$ in the table of [4].

3. A conjecture about admissible sets with maximal size

Let m be an integer, $N = m^2 + m - 2, k = 2m - 1$, and let us consider the set $\{N, N-1, \ldots, N-m+2, N-m, N-m-1, \ldots, N-2m+1\}$. If the elements of this set are denoted by $a_1 < a_2 < \cdots < a_k$, and if we set $x_i = a_{i+1} - a_i - 1$, we have $x_i = 0$ for all i but

$$x_{m-1} = x_{(k-1)/2} = 1.$$

So, (8) writes:

$$m - 1 = \frac{k-1}{2} \leq N - \frac{(k+1)^2}{4} = m^2 + m - 2 - m^2 = m - 2$$

which does not hold. Therefore the set is not stratified. It is easy to see that $t_0 = \lfloor (k-1)/2 \rfloor = m - 1$,

$$\max t_0{}^\wedge\mathcal{A} = \min(t_0 + 1)^\wedge\mathcal{A} + 1$$

but the second largest term of $t_0{}^\wedge\mathcal{A}$ is smaller than all elements of $(t_0+1)^\wedge\mathcal{A}$, and the set is admissible.

A similar counter example admissible but not stratified does exist for $N = m^2 + m - 1, k = 2m - 1$, omitting $N - 2m$ instead of $N - 2m + 1$.
These two counterexamples will be said quasistratified.
Now, conjectures 1 to 4 of [**4**] can be reformulated in the following terms:
Conjecture 1 of [4], that the maximal number of elements of an admissible set with greatest element N is $\lfloor 2\sqrt{N + 1/4} - 1 \rfloor$, has been proved by J.M. Deshouillers and G.A. Freiman in [**2**], for N large enough.
Conjecture 2 is replaced by: For $N \geq 20$, the admissible sets of maximal size and largest element N are either stratified, or one of the sets made of odd elements described in conjecture 3 of [**4**] (whenever N is of the form $m^2 - 1$ or $m^2 + m - 1$), or a quasistratified set described above (whenever N is of the form $m^2 + m - 1$ or $m^2 + m - 2$).
Conjecture 4 of [**4**] then becomes an easy consequence of our new conjecture 2.
This new conjecture 2 fits the table of [4] for $20 \leq N \leq 50$. This table has been extended up to $N = 60$, and the conjecture is verified for $20 \leq N \leq 60$.

4. How many stratified sets are there ?

Theorem 2. — *The set of stratified sets with largest element N and the set of partitions of N have same cardinal. Moreover an explicit one to one correspondence between these two sets is given.*

Proof. — Let m be an integer, and, as above, let us denote $S_k(N)$ the number of stratified sets with largest N and with k elements. From (13) and (14) the generating function of $S_{2m-1}(N) + S_{2m}(N)$ is:

$$\begin{aligned}\sum_{N=0}^{\infty} (S_{2m-1}(N) + S_{2m}(N))\, x^N &= \frac{x^{m^2}}{1-x^m} \prod_{i=1}^{m-1} \frac{1}{(1-x^i)^2} + x^{m^2+m} \prod_{i=1}^{m} \frac{1}{(1-x_i)^2} \\ &= x^{m^2} \prod_{i=1}^{m} \frac{1}{(1-x_i)^2}\end{aligned}$$

Now, the generating function of $S(N)$, the total number of stratified sets with largest elements N will be:

$$\sum_{N=0}^{\infty} S(N)x^N = \sum_{m=1}^{\infty} x^{m^2} \prod_{i=1}^{m} \frac{1}{(1-x_i)^2}.$$

But, by an identity due to Euler (cf. [5], p. 280):

$$\sum_{m=1}^{\infty} x^{m^2} \prod_{i=1}^{m} \frac{1}{(1-x^i)^2} = \prod_{i=1}^{\infty} \frac{1}{1-x^i} = \sum_{N=0}^{\infty} p(N)x^N, \tag{18}$$

and so, $S(N) = p(N)$.

To the partition $n = n_1 + n_2 + \cdots + n_k$, with $n_1 \geq n_2 \geq \cdots \geq n_k$, let us associate the so-called Ferrers diagram, that is the array of dots made with n_1 dots on the first line, n_2 dots on the second line, ..., and so on n_k dots on the k^{th} line. For instance, to $10 = 4 + 3 + 1 + 1 + 1$ corresponds the array:

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• • •
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This graphical representation contains a square in the upper left corner, and the largest such square is called "Durfee square" in [5, p. 281].

In the combinatorial proof of Euler's identity (18), it is observed in [5] that

$$x^{m^2} \prod_{i=1}^{m} \frac{1}{(1-x^i)^2}$$

is the generating function of the number of partitions such that the Durfee squares have an edge of length exactly m. To find the wanted one to one correspondence we just have to use the combinatorial proof of (18) in [5, p. 281].

From the above proof, one can see that $S_{2m}(N)$ is equal to the number of partitions of N such that the Durfee square has an edge of length m, and moreover such that the corresponding array contains the rectangle of length $m+1$ and height m. Similarly, $S_{2m-1}(N)$ is equal to the number of partitions of N such that the Durfee square has an edge of length m, but such that the array does not contain the above mentioned rectangle.

Let us suppose first that the Ferrers diagram does not contain the rectangle $(m+1) \times m$ (that means that $n_m = m$) and choose $k = 2m - 1$. The Ferrers diagram consists of three parts: The Durfee square and two tails. Let us denote by $\mathcal{U}$ the upper right tail and by $\mathcal{V}$ the lower left tail, so that

$$n = m^2 + |\mathcal{U}| + |\mathcal{V}|.$$

Now, $\mathcal{U}$ can be interpreted as the Ferrers diagram (in column) of a partition of $|\mathcal{U}|$, the parts of which are $\leq m-1$. Let us denote by x_i the number of columns of $\mathcal{U}$ with height i ; then this partition writes

$$x_1 + 2x_2 + \cdots + (m-1)x_{m-1} = |\mathcal{U}|. \tag{19}$$

Similarly $\mathcal{V}$ can be interpreted as the Ferrers diagram (in row) of a partition of $|\mathcal{V}|$, the parts of which are $\leq m$. Let us denote by y_i the number of rows of $\mathcal{V}$ with length i ; then this partition writes

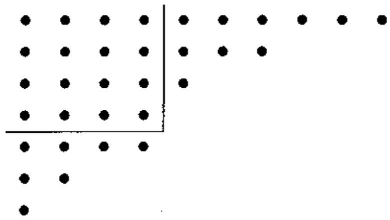
$$y_1 + 2y_2 + \cdots + my_m = |\mathcal{V}|. \tag{20}$$

By introducing a new variable x_k, let us transform the inequality (8) in the equality:

$$(x_1 + x_k) + 2(x_2 + x_{k-1}) + \cdots + (m-1)(x_{m-1} + x_{m+1}) + mx_m = N - m^2 = |\mathcal{U}| + |\mathcal{V}|.$$

The values of $x_1, \ldots, x_{m-1}$ are given by (19), the values of $x_k = y_1, x_{k-1} = y_2, \ldots, x_m = y_m$ are given by (20), and the stratified set $(a_1, a_2, \ldots, a_k = N)$ can be obtained by $a_k = N$, and $a_i = a_{i+1} - 1 - x_i$.

Example $N = 33 = 10 + 7 + 5 + 4 + 4 + 2 + 1$



$m = 4, k = 7$
$|\mathcal{U}| = 10, x_3 = 1, x_2 = 2, x_1 = 3.$
$|\mathcal{V}| = 7, y_4 = 1, y_3 = 0, y_2 = 1, y_1 = 1.$
The stratified set corresponding to this partition is (19, 23, 26, 28, 30, 31, 33).

Whenever the Ferrers diagram does contain the rectangle $(m+1) \times m$ (that means that $n_{m-1} \geq m+1$) one chooses $k = 2m$, and before defining the tails $\mathcal{U}$ and $\mathcal{V}$ we have to take off the rectangle, so that

$$n = m(m+1) + |\mathcal{U}| + |\mathcal{V}|.$$

The parts of the partition represented by $\mathcal{U}$ are allowed to be equal to m, so that (19) becomes

$$x_1 + 2x_2 + \cdots + mx_m = |\mathcal{U}|,$$

while (20) does not change. (7) becomes:

$$(x_1 + x_k) + 2(x_2 + x_{k-1}) + \cdots + m(x_m + x_{m+1}) = N - m(m+1) = |\mathcal{U}| + |\mathcal{V}|$$

and the end of the calculation of the $a_i's$ is similar.

For instance the stratified set associated to the partition of $10 = 4 + 3 + 1 + 1 + 1$, the array of which is displayed above, is $(6, 8, 9, 10)$.

Corollary 3. — *The number $a(N)$ of admissible sets with largest element N is greater than $p(N)$, the number of partitions of N.*

Proof. — It follows immediately from Theorem 2, since any stratified set is admissible.
From the result of Hardy and Ramanujan, it is known that (cf. [**6**], formula 1.41):

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

So the above corollary improves the lower bound given in [**4**]:

$$a(N) \geq 2^{2\sqrt{N-2}} - 1.$$

For the moment, I am not able to improve the upper bound of [4]:

$$a(N) \leq \exp(c\sqrt{N}\log N),$$

but I conjecture that $a(N)$ is not much greater than $p(N)$ and satisfies

$$a(N) = \exp\left((\pi\sqrt{2/3} + o(1))\sqrt{N}\right),$$

and, may be (see the table) that $a(N) \sim p(N)$.

N	$p(N)$	$a(N)$	$a(N)/p(N)$	N	$p(N)$	$a(N)$	$a(N)/p(N)$
1	1	1	1.00	41	44583	235189	5.28
2	2	2	1.00	42	53174	273087	5.14
3	3	3	1.00	43	63261	335262	5.30
4	5	6	1.20	44	75175	394565	5.25
5	7	9	1.29	45	89134	465548	5.22
6	11	15	1.36	46	105558	551586	5.23
7	15	23	1.53	47	124754	659344	5.29
8	22	39	1.77	48	147273	750256	5.09
9	30	54	1.80	49	173525	912459	5.26
10	42	87	2.07	50	204226	1051209	5.15
11	56	121	2.16	51	239943	1230129	5.13
12	77	178	2.31	52	281589	1433643	5.09
13	101	249	2.47	53	329931	1705477	5.17
14	135	362	2.68	54	386155	1900438	4.92
15	176	484	2.75	55	451276	2308752	5.12
16	231	708	3.06	56	526823	2604726	4.94
17	297	928	3.12	57	614154	3041041	4.95
18	385	1265	3.29	58	715220	3483815	4.87
19	490	1685	3.44	59	831820	4132473	4.97
20	627	2306	3.68	60	966467	4527898	4.69
21	792	2886	3.64	61	1121505	5491786	4.90
22	1002	3918	3.91	62	1300156	6101289	4.69
23	1255	4987	3.97	63	1505499	7090459	4.71
24	1575	6418	4.07	64	1741630	8019859	4.60
25	1958	8265	4.22	65	2012558	9504818	4.72
26	2436	10601	4.35	66	2323520	10230396	4.40
27	3010	13104	4.35	67	2679689	12413471	4.63
28	3718	16947	4.56	68	3087735	13595124	4.40
29	4565	21069	4.62	69	3554345	15791911	4.44
30	5604	26088	4.66	70	4087968	17584116	4.30
31	6842	32804	4.79	71	4697205	20860378	4.44
32	8349	40935	4.90	72	5392783	22095088	4.10
33	10143	49360	4.87	73	6185689	26904818	4.35
34	12310	61712	5.01	74	7089500	29025643	4.09
35	14883	75338	5.06	75	8118264	33687817	4.15
36	17977	90456	5.03	76	9289091	37071664	3.99
37	21637	111771	5.17	77	10619863	44046119	4.15
38	26015	134685	5.18	78	12132164	45918783	3.78
39	31185	160353	5.14	79	13848650	56109976	4.05
40	37338	195993	5.25	80	15796476	59689468	3.78

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