THE RAMANUJAN JOURNAL, 8, 147–175, 2004

Popularity of Sets Represented by the Partitions of *n*

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Received July 24, 2001; Accepted August 9, 2002

Abstract. Let us say that a partition of the positive integer n represents a, $0 \leq a \leq n$, if there is a submultiset of the multiset of the parts whose sum is a. Erdős and Szalay have proved that almost all partitions of n represent all integers a, $0 \leq a \leq n$. If $\mathcal{A}$ is a finite set of positive integers, let us denote by $\tilde{p}(n, \mathcal{A})$ the number of partitions of n which represent all integers a, $0 \leq a \leq n$, $a \notin \mathcal{A}$, $n - a \notin \mathcal{A}$ but do not represent a for $a \in \mathcal{A}$. For instance, $\tilde{p}(n, \emptyset)$ is the number of partitions of n which represent all integers between 0 and n; the result of Erdős and Szalay can be reformulated as $\tilde{p}(n, \emptyset) \sim p(n)$, where $p(n)$ is the total number of partitions of n. The aim of this paper is the study of $\tilde{p}(n, \mathcal{A})$: we shall compare the values of $\tilde{p}(n, \mathcal{A})$ for small sets $\mathcal{A}$ and we shall give a close formula for $\tilde{p}(n, \mathcal{A})$ when $\mathcal{A}$ is the set of the first k integers.

Key words: partitions, generating functions, asymptotic estimate

2000 Mathematics Subject Classification: Primary—11P81, 11P82

1. Introduction

A partition of an integer n is a sum of positive integers in descending order which add up to n. We shall write:

$$n = n_1 + n_2 + \cdots + n_t, \quad n_1 \geq n_2 \geq \cdots \geq n_t. \tag{1.1}$$

We shall denote by $p(n)$ the number of partitions of n. The generating function is well known:

$$\sum_{n=0}^{\infty} p(n)X^n = \prod_{m=1}^{\infty} \frac{1}{1 - X^m} \cdot \tag{1.2}$$

*Research partially supported by French-Hungarian APAPE-OMFB exchange programs F-5/1997, Balaton 98009, F-18/00, Balaton 02798NC, by CNRS, Institut Girard Desargues, UMR 5028 and by Hungarian National Foundation for Scientific Research, Grant No. T029759.

If a satisfies $1 \leq a \leq n-1$, we shall say that the partition (1.1), or, equivalently, the multiset $\{n_1, n_2, \ldots, n_t\}$ represents a if a can be written as a subsum $n_{i_1} + n_{i_2} + \cdots + n_{i_r}$ (with $1 \leq i_1 < i_2 < \cdots < i_r \leq t$) of (1.1). A partition which represents all integers between 0 and n is *practical*. It has been proved in [9] that the number $\tilde{p}(n)$ of practical partitions of n satisfies

$$\tilde{p}(n) \sim p(n), \quad n \to \infty \tag{1.3}$$

(the notation $\tilde{p}(n) \sim p(n)$ means that $\tilde{p}(n) = (1 + o(1))\, p(n)$ or $\tilde{p}(n)/p(n) \to 1$).

If a satisfies $1 \leq a \leq n - 1$, we shall denote by $R(n, a)$ the number of partitions of n which do not represent a. Clearly, $R(n, a) \leq p(n) - \tilde{p}(n)$ and from (1.3), it follows that $R(n, a) = o(p(n))$ as $n \to \infty$. The function $R(n, a)$ has been studied in [2–5, 7, 8]. We shall use the result of [3]: for a fixed and $n \to \infty$, we have

$$R(n, a) \sim p(n) \left(\frac{\pi}{\sqrt{6n}} \right)^{\psi(a)} u(a) \tag{1.4}$$

where

$$\psi(a) = \lfloor a/2 \rfloor + 1, \tag{1.5}$$

($\lfloor x \rfloor$ is the integral part of x) and $u(a)$ is a constant depending on a. We have $u(1) = 1$, $u(2) = 4$, $u(3) = 3$, $u(4) = 16$. A table of the values of $u(a)$ up to $a = 20$ and some information on the asymptotic behaviour of $u(a)$ as $a \to \infty$ are given in [3].

Let $\mathcal{A}$ be a finite set of positive integers and $|\mathcal{A}|$ be the number of its elements. We shall denote by $r(n, \mathcal{A})$ the number of partitions of n without any parts in $\mathcal{A}$ so that the generating function is

$$\sum_{n=0}^{\infty} r(n, \mathcal{A}) X^n = \prod_{\substack{m=1 \\ m \notin \mathcal{A}}}^{\infty} \frac{1}{1 - X^m} \cdot \tag{1.6}$$

As usual, we shall set $p(0) = r(0, \mathcal{A}) = 1$, and for $n < 0$, $p(n) = r(n, \mathcal{A}) = 0$. We shall denote by $r(n, m)$ the number of partitions of n whose parts are at least m, in other terms,

$$r(n, m) = r\left(n, \{1, 2, \ldots, m - 1\}\right). \tag{1.7}$$

An asymptotic estimation of $r(n, \mathcal{A})$ is given in [7], which, for a finite set $\mathcal{A}$ gives, as $n \to \infty$:

$$r(n, \mathcal{A}) = p(n) \left(\frac{\pi}{\sqrt{6n}} \right)^{|\mathcal{A}|} \left(\prod_{a \in \mathcal{A}} a \right) \left(1 + \frac{O(1)}{\sqrt{n}} \right). \tag{1.8}$$

From (1.7), (1.8) yields, for m fixed and as $n \to \infty$:

$$r(n, m) = p(n) \left(\frac{\pi}{\sqrt{6n}} \right)^{m-1} (m - 1)! \left(1 + \frac{O(1)}{\sqrt{n}} \right). \tag{1.9}$$

Two partitions of n are equivalent if they both represent the same integers. The classes of equivalence will be characterized by the sets of integers represented by equivalent partitions. If $\mathcal{A}$ is a finite set of positive integers, a partition of n is *$\mathcal{A}$-practical* if it represents all a's, $1 \le a \le n$, $a \notin \mathcal{A}$, $n - a \notin \mathcal{A}$ but does not represent a for $a \in \mathcal{A}$; we shall denote by $\tilde{\mathcal{P}}(n, \mathcal{A})$ the set of $\mathcal{A}$-practical partitions of n; we shall define $\tilde{p}(n, \mathcal{A}) = |\tilde{\mathcal{P}}(n, \mathcal{A})|$. In some sense, $\tilde{p}(n, \mathcal{A})$ will measure the popularity of the set $\mathcal{A}$.

The aim of this paper is the study of $\tilde{p}(n, \mathcal{A})$. First, we shall classify the most popular sets by proving:

Theorem 1. *For n large enough, we have*

$$\begin{aligned} \tilde{p}(n) = \tilde{p}(n, \emptyset) &> \tilde{p}(n, \{1\}) > \tilde{p}(n, \{1,3\}) > \tilde{p}(n, \{2\}) > \tilde{p}(n, \{1,2\}) > \\ &> \tilde{p}(n, \{1,3,5\}) > \tilde{p}(n, \{2,5\}) > \tilde{p}(n, \{1,2,5\}) > \tilde{p}(n, \{1,4\}) > \\ &> \tilde{p}(n, \{1,2,4\}) > \tilde{p}(n, \{2,3\}) > \tilde{p}(n, \{3\}) > \tilde{p}(n, \{1,2,3\}) > \tilde{p}(n, \mathcal{B}) \end{aligned}$$

for any finite set $\mathcal{B}$ in $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$, different from the sets already mentioned.

M. Deléglise has built a table of the values of $\tilde{p}(n, \mathcal{A})$ for n up to 115 and all possible $\mathcal{A}$'s; we are pleased to thank him strongly for this work which has been very useful. According to the values of $\tilde{p}(100, \mathcal{A})$, the order of the sets is slightly different:

$$\begin{aligned} &\emptyset, \{1\}, \{1,3\}, \{2\}, \{1,2\}, \{1,3,5\}, \{1,4\}, \{3\}, \{1,2,5\}, \{2,3\}, \\ &\quad \{1,3,5,7\}, \{4\}, \{2,5\}, \{1,2,4\}, \{1,2,4,7\}, \{1,3,5,7,9\}, \ldots . \end{aligned}$$

This is due to the fact that the coefficients of the asymptotic expansion of $\tilde{p}(n, \mathcal{A})/p(n)$ according to the powers of $\frac{\pi}{\sqrt{6n}}$ are sometimes rather large and, for $n = 100$, $\frac{\pi}{\sqrt{6n}}$ is not that small.

The proof of Theorem 1 will be given in Section 2. Of course, the method of proof can be extended to compare the values of $\tilde{p}(n, \mathcal{A})$ for a longer list. However, we have not yet succeeded in stating a general theorem comparing $\tilde{p}(n, \mathcal{A}_1)$ and $\tilde{p}(n, \mathcal{A}_2)$ for two given sets $\mathcal{A}_1$ and $\mathcal{A}_2$ when $n \to \infty$.

The result (1.3) has been precised in [6] where an asymptotic expansion of $\tilde{p}(n)/p(n)$ has been given. The proof follows from the formula

$$\tilde{p}(n) = p(n) - \sum_{1 \le a \le n/2} \tilde{p}(a-1) r(n-a+1, a+1). \tag{1.10}$$

In Section 4, we shall prove Theorem 3, which generalizes formula (1.10) to $\tilde{p}(n, \mathcal{A})$ where $\mathcal{A}$ is the set of the first k integers. Unfortunately, the proof is much more complicated than the proof of (1.10) in [6]. Let us introduce the notation:

Definition 1. Let $k \ge 1$ be fixed. For $k + 1 \le n_1 \le n_2 \le \cdots \le n_t$, we define $a(i)$ (for $i = 1, \ldots, t$) with

$$a(i) \text{ is the smallest integer } \ge k+1 \text{ not representable by } \{n_1, \ldots, n_i\}. \tag{1.11}$$

In other words:

$$a(i) \text{ is not representable by } \{n_1, \ldots, n_i\}; \tag{1.12}$$

$$k+1, k+2, \ldots, a(i)-1 \text{ are representable by } \{n_1, \ldots, n_i\}. \tag{1.13}$$

Further, for $1 \le i \le t$, let

$$S(i) = \sum_{\substack{j=1 \\ n_j \le a(i)-k-1}}^{i} n_j. \tag{1.14}$$

Let us observe from the above definition that

$$a(i) \text{ and } S(i) \text{ are not decreasing.} \tag{1.15}$$

The behaviour of $a(i)$ and $S(i)$ is precised in Theorem 2 which will be proved in Section 3.

Theorem 2. *Let us use the notation of Definition* 1. *If* i, $1 \le i \le t$, *satisfies*

$$a(i) \ge 2k+3, \tag{1.16}$$

we have

$$a(i) \ge (3k+2)\frac{k+1}{2} + 1 \ge 4k+2 \ge 3k+3, \tag{1.17}$$

$$a(i)+1 \le S(i), \tag{1.18}$$

and

$$S(i) \le a(i)+k; \tag{1.19}$$

moreover, for $1 \le i \le t-1$,

$$S(i+1) - a(i+1) = \begin{cases} S(i)-a(i)-1 & n_{i+1} = a(i),\ a(i)-k \notin \{n_1, \ldots, n_i\}, \\ & \quad a(i)+1 < S(i) \le a(i)+k; \\ k & n_{i+1} = a(i),\ a(i)-k \in \{n_1, \ldots, n_i\}, \\ & \quad a(i)+1 < S(i) \le a(i)+k; \\ k & n_{i+1} = a(i) \quad \textit{and} \quad S(i) = a(i)+1; \\ S(i)-a(i) & \textit{otherwise}. \end{cases} \tag{1.20}$$

Theorem 2 will be used to prove Theorem 3. (For $u > v$, the sum from u to v is to be considered 0.)

Theorem 3. *Let k be a positive integer, and $\mathcal{A} = \{1, 2, \ldots, k\}$. For $n \geq (3k+4)\frac{k+1}{2} + 1$, we have*:

$$\begin{aligned}\tilde{p}(n, \mathcal{A}) &= r\left(n - (3k+2)\frac{k+1}{2}, k+1\right)\\ &\quad - r\left(n - (3k+2)\frac{k+1}{2}, \{1, 2, \ldots, k, k+1, 2k+2\}\right)\\ &\quad - \sum_{a=(3k+2)\frac{k+1}{2}+1}^{\lfloor n/2 \rfloor} \left\{\sum_{j=0}^{k-1} \tilde{p}(a+k-j, \mathcal{A})\right.\\ &\quad \left. \times r\left(n-k-1-a(j+1) + \frac{(j+1)(j+2)}{2}, \{1, 2, \ldots, a-k-1, a\}\right)\right\}. \end{aligned} \tag{1.21}$$

Theorem 3 can be used to calculate recursively $\tilde{p}(n, \mathcal{A})$, since it is not difficult to compute $r(n, \mathcal{A})$ (use, for instance, formula (2.11) below). Unfortunately, we have not succeeded in extending Theorem 3 to any finite set $\mathcal{A}$. However, after the proof of Theorem 3, we shall give similar formulas for $\tilde{p}(n, \{2\})$, $\tilde{p}(n, \{1, 3\})$ and $\tilde{p}(n, \{1, 3, 5\})$.

We thank the referee for several valuable suggestions.

2. Proof of Theorem 1

We shall start by proving:

Lemma 1. *Let $R(n, a)$ be the number of partitions of n which do not represent a, and let us set*

$$\bar{R}(n, a) = R(n, a) + R(n, a+1) + \cdots + R(n, \lfloor n/2 \rfloor) = \sum_{b=a}^{\lfloor n/2 \rfloor} R(n, b). \tag{2.1}$$

Then for a fixed and $n \to \infty$, we have

$$\bar{R}(n, a) = O\left(p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^{\psi(a)}\right) \tag{2.2}$$

where $\psi(a)$ is defined by (1.5). *More precisely*,

$$\bar{R}(n, a) \sim R(n, a) + R(n, a+1).$$

For odd a, $\bar{R}(n, a) \sim R(n, a)$.

Proof: We shall use a result mainly due to J. Dixmier (cf. [5]) in the form given in [11], Theorem 3: for n large enough and

$$0.18\sqrt{n} \le a \le n - 0.18\sqrt{n} \tag{2.3}$$

the following inequality holds

$$\log(R(n,a)) \le 2.431\sqrt{n}. \tag{2.4}$$

We shall also use the result of [2], p. 44: if $\lambda = a/\sqrt{n}$, then

$$\log(R(n,a)) \le \left(c + \frac{\pi^2}{6c} + \frac{\lambda}{2}\log(1 - e^{-c\lambda})\right)\sqrt{n}, \tag{2.5}$$

where c is any positive real number.

By using (2.5) with $c = \frac{\pi}{\sqrt{6}}$, and observing that, for x real, $1 - e^{-x} \le x$, we get

$$\log R(n,b) \le \pi\sqrt{\frac{2n}{3}} + \frac{b}{2}\log\left(\frac{\pi b}{\sqrt{6n}}\right)$$

or, in other terms

$$R(n,b) \le e^{\pi\sqrt{\frac{2n}{3}}}v(b) \tag{2.6}$$

with

$$v(b) = \left(\frac{\pi}{\sqrt{6n}}\right)^{b/2} b^{b/2}. \tag{2.7}$$

We have

$$\frac{v(b+1)}{v(b)} = \left(\frac{\pi(b+1)}{\sqrt{6n}}\right)^{1/2}\left(1 + \frac{1}{b}\right)^{b/2} \le \left(\frac{e\pi(b+1)}{\sqrt{6n}}\right)^{1/2}$$

so that, for $b + 1 \le 0.18\sqrt{n}$,

$$\frac{v(b+1)}{v(b)} \le \left(\frac{0.18e\pi}{\sqrt{6}}\right)^{1/2} \le 0.8. \tag{2.8}$$

Let us write

$$\bar{R}(n,a) = \sum_{b=a}^{a+7} + \sum_{b=a+8}^{\lfloor 0.18\sqrt{n}\rfloor} + \sum_{0.18\sqrt{n} < b \le n/2} R(n,b) \stackrel{\text{def}}{=} S_1 + S_2 + S_3.$$

From (2.4), we get

$$S_3 \leq \frac{n}{2}\exp(2.431\sqrt{n}) = O\left(p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^{\psi(a+1)+\frac{1}{2}}\right),$$

since a is fixed, and from [10],

$$p(n) = \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}n}\left(1+\frac{O(1)}{\sqrt{n}}\right), \quad \pi\sqrt{2/3} = 2.56\ldots. \tag{2.9}$$

From (2.6) and (2.8) it follows

$$S_2 \leq e^{\pi\sqrt{\frac{2n}{3}}}v(a+8)(1+(.8)+(.8)^2+\cdots) = 5e^{\pi\sqrt{\frac{2n}{3}}}v(a+8). \tag{2.10}$$

The definition (1.5) implies $\psi(a) \leq 1 + a/2$; since a is fixed, by (2.7) and (2.9), (2.10) yields

$$S_2 = O(np(n)v(a+8)) = O\left(np(n)\left(\frac{\pi}{\sqrt{6n}}\right)^{\frac{a+8}{2}}\right) = O\left(p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^{\psi(a+1)+\frac{1}{2}}\right).$$

Finally, by (1.4), it is easily seen that

$$S_1 \sim R(n,a) + R(n,a+1) = O\left(p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^{\psi(a)}\right),$$

which completes the proof of Lemma 1. □

Remark. The constant in (2.4) can be improved for two reasons: (2.5) is slightly better than the upper bound of $R(n, \lambda\sqrt{n})$ used in the proof of Theorem 3 of [11], and J.-C. Aval (cf. [1]) has improved a key lemma of [5]. Unfortunately, these two improvements do not allow to decrease very much the constant in (2.4). The couple of numbers (0.18, 2.431) in (2.3) and (2.4) can, for instance, be replaced by (0.18, 2.422), (0.2, 2.415), (0.3, 2.391) or (0.4, 2.378).

To prove Theorem 1, we shall give an asymptotic equivalent of $\tilde{p}(n, \mathcal{A})$ for all the sets $\mathcal{A}$ considered in the statement. If $\tilde{p}(n, \mathcal{A}) \sim \tilde{p}(n, \mathcal{A}')$, we shall study the difference $\tilde{p}(n, \mathcal{A}) - \tilde{p}(n, \mathcal{A}')$.

For any finite set $\mathcal{A}$, it is possible to find an asymptotic expansion of any order of $r(n, \mathcal{A})/p(n)$. Indeed, from the generating functions (1.2) and (1.6), it follows that

$$\sum_{n=0}^{\infty} r(n, \mathcal{A})X^n = \left(\sum_{n=0}^{\infty} p(n)X^n\right)\prod_{a\in\mathcal{A}}(1-X^a).$$

So, if we expand the polynomial

$$\prod_{a\in\mathcal{A}}(1-X^a)=\sum_m w_m X^m,$$

we can write $r(n,\mathcal{A})$ as a linear combination of the $p(n-m)$'s:

$$r(n,\mathcal{A})=\sum_m w_m p(n-m). \tag{2.11}$$

But, from the famous formula of Hardy and Ramanujan for $p(n)$ (cf. [10]), for m fixed and $n\to\infty$, it is possible to expand $p(n-m)/p(n)$ according to the powers of $1/\sqrt{n}$, as explained in [6]. However the method is a bit technical and needs a computer. Here, we shall prove Theorem 1 by using only the asymptotic estimations (1.8) and (1.9), and the following formula, which follows from (1.6): if $\ell\notin\mathcal{A}$,

$$r(n,\mathcal{A})-r(n-\ell,\mathcal{A})=r(n,\mathcal{A}\cup\{\ell\}). \tag{2.12}$$

Formulas (2.12) and (1.8) imply for fixed ℓ:

$$r(n,\mathcal{A})-r(n-\ell,\mathcal{A})=p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^{|\mathcal{A}|+1}\left(\ell\prod_{a\in\mathcal{A}}a\right)\left(1+\frac{O(1)}{\sqrt{n}}\right). \tag{2.13}$$

In fact, the method used in [7] to prove (1.8) shows that the relation (2.13) above still holds, even if $\ell\in\mathcal{A}$.

It follows from (2.9) that, for m fixed and $n\to\infty$,

$$p(n-m)=p(n)\left(1+\frac{O(1)}{\sqrt{n}}\right), \tag{2.14}$$

and, from (1.8) and (2.14), that, for any finite set $\mathcal{A}$, m fixed and $n\to\infty$,

$$r(n-m,\mathcal{A})=r(n,\mathcal{A})\left(1+\frac{O(1)}{\sqrt{n}}\right). \tag{2.15}$$

We shall use the obvious relations:

$$\tilde{p}(n,\mathcal{A})\le r(n,\mathcal{A}) \tag{2.16}$$

and

$$\tilde{p}(n,\mathcal{A})\le R(n,\max(\mathcal{A})), \tag{2.17}$$

which, together with (1.8) or (1.4), will give an upper bound for $\tilde{p}(n,\mathcal{A})$.

We are now ready to estimate $\tilde{p}(n,\mathcal{A})$, as $n\to\infty$, for the different sets $\mathcal{A}$ considered in Theorem 1.

- $\mathcal{A} = \emptyset$. We know from [9] (cf. (1.3)) that

$$\tilde{p}(n, \emptyset) = \tilde{p}(n) \sim p(n). \tag{2.18}$$

- $\mathcal{A} = \{1\}$. From (2.16) we have

$$\tilde{p}(n, \{1\}) \le r(n, \{1\}) = r(n, 2). \tag{2.19}$$

Moreover, to get $\tilde{\mathcal{P}}(n, \{1\})$ from the set of partitions without any part equal to 1, we have to take off all the partitions which do not represent any of the integers $2, 3, \dots, \lfloor n/2 \rfloor$. Therefore, with the notation of Lemma 1

$$\tilde{p}(n, \{1\}) \ge r(n, 2) - \bar{R}(n, 2). \tag{2.20}$$

Then, it follows from (2.19), (2.20), (1.9) and Lemma 1:

$$\tilde{p}(n, \{1\}) = r(n, 2) + O(\bar{R}(n, 2)) = r(n, 2) + O\left(\frac{p(n)}{n}\right) \sim p(n)\frac{\pi}{\sqrt{6n}}. \tag{2.21}$$

- $\mathcal{A} = \{1, 3\}$. A partition belonging to $\tilde{\mathcal{P}}(n, \{1, 3\})$ should not contain any part equal to 1 or 3, but it should contain at least one part equal to 2 to represent 2. Thus

$$\tilde{p}(n, \{1, 3\}) \le r(n - 2, \{1, 3\}). \tag{2.22}$$

Moreover, to get $\tilde{\mathcal{P}}(n, \{1, 3\})$, we have to take off the partitions which do not represent any of the numbers between 4 and $\lfloor n/2 \rfloor$. Therefore,

$$\tilde{p}(n, \{1, 3\}) \ge r(n - 2, \{1, 3\}) - \bar{R}(n, 4). \tag{2.23}$$

Then, it follows from (2.22), (2.23), (2.15), (1.8) and Lemma 1:

$$\tilde{p}(n, \{1, 3\}) = r(n - 2, \{1, 3\}) + O(p(n)n^{-3/2}) \sim 3p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^2. \tag{2.24}$$

- $\mathcal{A} = \{2\}$. To represent 1, 3, 4 and 5 but not 2, a partition of n should contain one and only one part equal to 1, should not contain any part equal to 2, should contain at least one part equal to 3 and either one part equal to 4 or one part equal to 5 . Thus

$$\tilde{p}(n, \{2\}) \le r(n - 4, 3) - r(n - 4, \{1, 2, 4, 5\}). \tag{2.25}$$

Moreover, to get $\tilde{\mathcal{P}}(n, \{2\})$, we have to take off the partitions which do not represent any of the numbers between 6 and $\lfloor n/2 \rfloor$. Therefore,

$$\tilde{p}(n, \{2\}) \ge r(n - 4, 3) - r(n - 4, \{1, 2, 4, 5\}) - \bar{R}(n, 6). \tag{2.26}$$

Then, it follows from (2.25), (2.26), (2.15), (1.8), (1.9) and Lemma 1:

$$\tilde{p}(n, \{2\}) = r(n - 4, 3) + O(p(n)n^{-2}) \sim 2p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^2. \tag{2.27}$$

- $\mathcal{A} = \{1, 2\}$. A partition belonging to $\tilde{\mathcal{P}}(n, \{1, 2\})$ should not contain any part equal to 1 or 2, but it should contain at least one part equal to 3, 4, 5 to represent 3, 4 and 5. Thus

$$\tilde{p}(n, \{1, 2\}) \le r(n - 12, 3). \tag{2.28}$$

Moreover, to get $\tilde{\mathcal{P}}(n, \{1, 2\})$, we have to take off the partitions which do not represent any of the numbers between 6 and $\lfloor n/2 \rfloor$. Therefore,

$$\tilde{p}(n, \{1, 2\}) \ge r(n - 12, 3) - \bar{R}(n, 6). \tag{2.29}$$

Then, it follows from (2.28), (2.29), (2.15), (1.9) and Lemma 1:

$$\tilde{p}(n, \{1, 2\}) = r(n - 12, 3) + O(p(n)n^{-2}) \sim 2p(n) \left(\frac{\pi}{\sqrt{6n}} \right)^2. \tag{2.30}$$

Since, from (2.27) and (2.30), $\tilde{p}(n, \{2\}) \sim \tilde{p}(n, \{1, 2\})$, we evaluate their difference by (2.13) and (2.14):

$$\begin{aligned} \tilde{p}(n, \{2\}) - \tilde{p}(n, \{1, 2\}) &= r(n - 4, 3) - r(n - 12, 3) + O(p(n)n^{-2}) \\ &\sim 16p(n) \left(\frac{\pi}{\sqrt{6n}} \right)^3 > 0. \end{aligned}$$

- $\mathcal{A} = \{1, 3, 5\}$. A partition belonging to $\tilde{\mathcal{P}}(n, \{1, 3, 5\})$ should not contain any part equal to 1, 3 or 5, but to represent 2 and 4, it should contain either at least two parts equal to 2 or one (and only one) part equal to 2 and at least one part equal to 4. Thus

$$\tilde{p}(n, \{1, 3, 5\}) = r(n - 4, \{1, 3, 5\}) + r(n - 6, \{1, 2, 3, 5\}) - \theta \bar{R}(n, 6), \tag{2.31}$$

where, from now on, θ will denote a real number satisfying $0 \le \theta \le 1$. From (2.15), (1.8) and Lemma 1, (2.31) implies

$$\tilde{p}(n, \{1, 3, 5\}) \sim 15p(n) \left(\frac{\pi}{\sqrt{6n}} \right)^3. \tag{2.32}$$

- $\mathcal{A} = \{2, 5\}$. A partition belonging to $\tilde{\mathcal{P}}(n, \{2, 5\})$ should contain one (and only one) part equal to 1, should not contain any part equal to 2 or 5, should contain at least one part equal to 3; to represent 6 it should contain either at least two parts equal to 3 or one (and only one) part equal to 3 and at least one part equal to 6. Note that such a partition represents also $4 = 1 + 3$ and $7 = 1 + 6$. Thus

$$\begin{aligned} \tilde{p}(n, \{2, 5\}) &= r(n - 7, \{1, 2, 5\}) + r(n - 10, \{1, 2, 3, 5\}) - \theta \bar{R}(n, 8) \\ &\sim 10p(n) \left(\frac{\pi}{\sqrt{6n}} \right)^3. \end{aligned} \tag{2.33}$$

- $\mathcal{A} = \{1, 2, 5\}$. A partition belonging to $\tilde{\mathcal{P}}(n, \{1, 2, 5\})$ should not contain any part equal to 1, 2 or 5, but to represent 3, 4 and 7, it should contain at least one part equal to 3 and one part equal to 4; to represent 6 it should contain either at least two parts equal to 3 or one (and only one) part equal to 3 and at least one part equal to 6. Thus

$$\begin{aligned}\tilde{p}(n, \{1, 2, 5\}) &= r(n-10, \{1, 2, 5\}) + r(n-13, \{1, 2, 3, 5\}) - \theta \bar{R}(n, 8)\\ &\sim 10p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^3. \end{aligned} \tag{2.34}$$

Further, from (2.34) and (2.33), we have by using (2.13), (2.14), (1.8) and Lemma 1

$$\begin{aligned}\tilde{p}(n, \{2, 5\}) - \tilde{p}(n, \{1, 2, 5\}) &= r(n-7, \{1, 2, 5\}) + r(n-10, \{1, 2, 3, 5\})\\ &-r(n-10, \{1, 2, 5\}) - r(n-13, \{1, 2, 3, 5\}) + (2\theta - 1)\bar{R}(n, 8)\\ &\sim r(n-7, \{1, 2, 3, 5\}) \sim 30p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^4 > 0.\end{aligned}$$

- $\mathcal{A} = \{1, 4\}$. A partition belonging to $\tilde{\mathcal{P}}(n, \{1, 4\})$ should contain one (and only one) part equal to 2, should not contain any part equal to 1 or 4, should contain at least one part equal to 3; to represent 6 it should contain either at least two parts equal to 3 or one (and only one) part equal to 3 and at least one part equal to 6. Such a partition will represent 7, if it contains one part equal to 5 or 7. Thus

$$\begin{aligned}\tilde{p}(n, \{1, 4\}) &= r(n-8, \{1, 2, 4\}) + r(n-11, 5)\\ &\quad - r(n-8, \{1, 2, 4, 5, 7\}) - r(n-11, \{1, 2, 3, 4, 5, 7\}) - \theta \bar{R}(n, 8)\\ &\sim 8p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^3.\end{aligned} \tag{2.35}$$

- $\mathcal{A} = \{1, 2, 4\}$. Similarly,

$$\begin{aligned}\tilde{p}(n, \{1, 2, 4\}) &= r(n-18, \{1, 2, 4\}) + r(n-21, 5) - \theta \bar{R}(n, 8)\\ &\sim 8p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^3.\end{aligned} \tag{2.36}$$

We have to compare $\tilde{p}(n, \{1, 4\})$ and $\tilde{p}(n, \{1, 2, 4\})$: from (2.35) and (2.36), it follows

$$\begin{aligned}\tilde{p}(n, \{1, 4\}) - \tilde{p}(n, \{1, 2, 4\}) &= r(n-8, \{1, 2, 4\}) + r(n-11, 5)\\ &-r(n-8, \{1, 2, 4, 5, 7\}) - r(n-11, \{1, 2, 3, 4, 5, 7\})\\ &-r(n-18, \{1, 2, 4\}) - r(n-21, 5) + (2\theta - 1)\bar{R}(n, 8)\\ &\sim 80p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^4 > 0.\end{aligned}$$

- $\mathcal{A} = \{2, 3\}$. To represent all integers between 1 and 7 except 2 and 3, a partition of n should have one (and only one) part equal to 1, no part equal to 2 or 3, at least one part equal to 4, and either one part equal to 6 or parts equal to 5 and 7. Therefore

$$\tilde{p}(n, \{2, 3\}) = r(n - 11, 4) + r(n - 17, \{1, 2, 3, 6\}) - \theta \bar{R}(n, 8) \sim 6p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^3. \tag{2.37}$$

- $\mathcal{A} = \{3\}$. To represent all integers between 1 and 7 except 3, a partition of n should have two (and only two) parts equal to 1, no part equal to 2 or 3, at least one part equal to 4, and at least one part equal to 5, 6 or 7. Therefore

$$\tilde{p}(n, \{3\}) = r(n - 6, 4) - r(n - 6, \{1, 2, 3, 5, 6, 7\}) - \theta \bar{R}(n, 8) \sim 6p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^3. \tag{2.38}$$

We estimate:

$$\begin{aligned}\tilde{p}(n, \{2, 3\}) - \tilde{p}(n, \{3\}) &= r(n - 11, 4) - r(n - 6, 4) + r(n - 17, \{1, 2, 3, 6\}) \\ &\quad + r(n - 6, \{1, 2, 3, 5, 6, 7\}) + (2\theta - 1)\bar{R}(n, 8) \\ &\sim (-30 + 36)p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^4 = 6p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^4 > 0.\end{aligned}$$

- $\mathcal{A} = \{1, 2, 3\}$. A partition belonging to $\tilde{\mathcal{P}}(n, \{1, 2, 3\})$ should not contain any part equal to 1, 2 or 3, but to represent 4, 5, 6 and 7, it should contain at least one part equal to 4, 5, 6 and 7. Therefore

$$\tilde{p}(n, \{1, 2, 3\}) = r(n - 22, 4) - \theta \bar{R}(n, 8) \sim 6p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^3. \tag{2.39}$$

We estimate:

$$\begin{aligned}\tilde{p}(n, \{3\}) - \tilde{p}(n, \{1, 2, 3\}) &= r(n - 6, 4) - r(n - 6, \{1, 2, 3, 5, 6, 7\}) \\ &\quad - r(n - 22, 4) + (2\theta - 1)\bar{R}(n, 8) \\ &\sim 96p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^4 > 0.\end{aligned}$$

- $\mathcal{A} = \mathcal{B}$. It suffices to show that, for any other finite set $\mathcal{B}$ in $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$, the following upper bound holds:

$$\tilde{p}(n, \mathcal{B}) = O(p(n)n^{-2}). \tag{2.40}$$

If $\max(\mathcal{B}) \geq 6$, then (2.40) is satisfied from (2.17) and Lemma 1. If $\max(\mathcal{B}) \leq 5$ and $|\mathcal{B}| \geq 4$ then (2.40) is satisfied from (2.16) and (1.8). For the remaining sets, (2.40) will follow from (1.9) and from the upper bounds given in the array below:

$\mathcal{A} =$	$\tilde{p}(n, \mathcal{A}) \le$	$\mathcal{A} =$	$\tilde{p}(n, \mathcal{A}) \le$
$\{4\}$	$2r(n-3,5)$	$\{1,3,4\}$	$r(n-2,5)$
$\{2,4\}$	0	$\{3,4,5\}$	$r(n-2,6)$
$\{3,4\}$	$r(n-2,5)$	$\{2,4,5\}$	0
$\{5\}$	$2r(n-4,6)$	$\{2,3,4\}$	$r(n-1,5)$
$\{1,5\}$	0	$\{1,4,5\}$	0
$\{3,5\}$	0	$\{2,3,5\}$	0
$\{4,5\}$	$2r(n-3,6)$		

So, the proof of Theorem 1 is completed.

3. Proof of Theorem 2

3.1. Proof of Theorem 2 (1.17)

We start by proving the following lemma:

Lemma 2. *For an arbitrary positive integer $n \ge 2$,*
(a) *the numbers represented by the set $\mathcal{A} = \{n, n+1, \ldots, 2n\}$ are all integers from n to $\frac{3n+1}{2}n$ and $\frac{3n+1}{2}n + n$;*
(b) *the multiset*

$$\mathcal{A}' = \{n, \underbrace{n+1, \ldots, n+1}_{m_{n+1}\,(\ge 1)}, \ldots, \underbrace{2n-1, \ldots, 2n-1}_{m_{2n-1}\,(\ge 1)}, 2n\}$$

represents all integers from n to

$$M = \frac{3n+1}{2}n + (m_{n+1}-1)(n+1) + \cdots + (m_{2n-1}-1)(2n-1)$$

but does not represent $M+1$;
(c) *the multiset $\mathcal{B} = \{n, n, n+1, \ldots, 2n-1\}$ represents all integers between n and $\frac{3n-1}{2}n$;*
(d) *the multiset*

$$\mathcal{B}' = \{\underbrace{n, \ldots, n}_{m_n\,(\ge 2)}, \underbrace{n+1, \ldots, n+1}_{m_{n+1}\,(\ge 1)}, \ldots, \underbrace{2n-2, \ldots, 2n-2}_{m_{2n-2}\,(\ge 1)}, 2n-1\}$$

represents all integers from n to

$$M = \frac{3n-1}{2}n + (m_n-2)n + (m_{n+1}-1)(n+1) + \cdots + (m_{2n-2}-1)(2n-2)$$

but does not represent $M+1$.

Proof of (a): Let k be an integer, $1 \le k \le n$, and $\mathcal{C}$ be a subset of $\mathcal{A}$ with k elements. Let us define $\sigma(\mathcal{C}) = \sum_{m \in \mathcal{C}} m$. One has

$$f(k) \le \sigma(\mathcal{C}) \le g(k)$$

where

$$f(k) = n + n + 1 + \cdots + n + k - 1 = k\left(n + \frac{k-1}{2}\right)$$

and

$$g(k) = 2n - k + 1 + \cdots + 2n = k\left(2n - \frac{k-1}{2}\right).$$

Moreover, for any integer a, $f(k) \le a \le g(k)$, there is a $\mathcal{C}$, $|\mathcal{C}| = k$, such that $\sigma(\mathcal{C}) = a$; this can be seen by the walk of the caterpillar: let us start with $\mathcal{C}_1 = \{n, \dots, n + k - 1\}$, then increase successively each element from the right to the left by 1. We get $\mathcal{C}_2 = \{n, \dots, n + k - 2, n + k\}$, $\mathcal{C}_3 = \{n, \dots, n + k - 3, n + k - 1, n + k\}$, ..., $\mathcal{C}_k = \{n + 1, n + 2 \dots, n + k\}$; and start again: $\mathcal{C}_{k+1} = \{n + 1, n + 2, \dots, n + k - 1, n + k + 1\}$ up to $\mathcal{C}_{(n-k+1)k} = \{2n - k + 1, \dots, 2n\}$. Clearly, $\sigma(\mathcal{C}_i)$ takes all values between $f(k)$ and $g(k)$.

In order to see that $\mathcal{A}$ represents all integers from $f(1) = n$ to $g(n) = \frac{3n+1}{2}n$, it suffices to check that

$$f(k + 1) \le g(k) + 1, \qquad \text{for } k = 1, 2, \dots, n - 1$$

which follows from $g(k) + 1 - f(k + 1) = (k - 1)(n - 1 - k)$.

Finally, since $g(n) = \frac{3n+1}{2}n$ is the sum of the $|\mathcal{A}| - 1$ largest elements of $\mathcal{A}$, the only subsum of $\mathcal{A}$ which is larger than $g(n)$ is $\sigma(\mathcal{A}) = \frac{3n+1}{2}n + n$, which completes the proof of (a).

Proof of (b): Let us write $\mathcal{A}'' = \mathcal{A}' \setminus \mathcal{A} = \{a_1, \dots, a_s\}$, with $s = (m_{n+1} - 1) + \cdots + (m_{2n-1} - 1)$. Since $\mathcal{A}'$ contains $\mathcal{A}$, it follows from (a) that $\mathcal{A}'$ represents all integers from n to $\frac{3n+1}{2}n \ge 3n + 1$. Further, since a_i, $1 \le i \le s$ satisfies $a_i \le 2n - 1 \le (3n + 1) - n$, the multiset $\mathcal{A} \cup \{a_1\}$ represents all integers from n to $\frac{3n+1}{2}n + a_1$, and so on, the multiset $\mathcal{A}' = \{a_1, \dots, a_s\} \cup \mathcal{A}$ represents all integers from n to M. By the same argument as the one at the end of the proof of (a), $\mathcal{A}'$ does not represent $M + 1, \dots, M + n - 1$ but represents $M + n$.

Proof of (c) and (d): The proof of (c) is similar to the one of (a) with, for $1 \le k \le n$, $f(k) = kn + \frac{(k-1)(k-2)}{2}$ and $g(k) = 2kn - \frac{k(k+1)}{2}$, while the proof of (d) is the same as the one of (b). □

To prove (1.17), we can suppose that there exists a *minimal* i_0 satisfying $a(i_0) \ge 2k + 3$. Then, from (1.11), $n_1, \dots, n_{i_0}$ represents $k + 1, \dots, 2k + 2$. Therefore, $k + 1, \dots, 2k + 1$ should belong to $\{n_1 = k + 1, n_2, , \dots, n_{i_0}\}$; and to represent $2k + 2$, there are two possibilities: either $n_2 \ne k + 1$ and $2k + 2 \in \{n_1, \dots, n_{i_0}\}$ or $n_2 = k + 1$.

(A) For $n_2 \neq k+1$, then we should have

$$\{k+1, \dots, 2k+2\} \subset \left\{n_1, \dots, n_{i_0}\right\}, \tag{3.1}$$

and the first elements of the multiset $\{n_1, \dots, n_{i_0}\}$ are

$$k+1, \underbrace{k+2, \dots, k+2}_{m_{k+2}\,(\geq 1)}, \dots, \underbrace{2k+1, \dots, 2k+1}_{m_{2k+1}\,(\geq 1)}, 2k+2, \dots .$$

Let us set $i_1 = 1 + m_{k+2} + \cdots + m_{2k+1} + 1$. We have $i_1 \leq i_0$ and $n_{i_1} = 2k+2$. From Lemma 2(b), $\{n_1 = k+1, n_2, \dots, n_{i_1}\}$ represents all integers from $k+1$ to

$$M = \frac{3(k+1)^2}{2} + \frac{k+1}{2} + (k+2)(m_{k+2} - 1) + \cdots + (2k+1)(m_{2k+1} - 1),$$

but does not represent $M+1$. So, from (1.11),

$$a(i_1) = M + 1 \geq \frac{3(k+1)^2}{2} + \frac{k+1}{2} + 1 = (3k+4)\frac{k+1}{2} + 1 > 3k+3,$$

so that, from the minimality of i_0, we have $i_1 = i_0$, and, from (1.14),

$$S(i_0) = S(i_1) = k + 1 + m_{k+2}(k+2) + \cdots + m_{2k+1}(2k+1) + 2k + 2 = a(i_0) + k.$$

(B) $n_2 = k+1$. Now, the first elements of the multiset $\{n_1, \dots, n_{i_0}\}$ are

$$\underbrace{k+1, \dots, k+1}_{m_{k+1}\,(\geq 2)}, \underbrace{k+2, \dots, k+2}_{m_{k+2}\,(\geq 1)}, \dots, \underbrace{2k, \dots, 2k}_{m_{2k}\,(\geq 1)}, 2k+1, \dots .$$

Let us set $i_2 = m_{k+1} + m_{k+2} + \cdots + m_{2k} + 1$. We have $i_2 \leq i_0$ and $n_{i_2} = 2k+1$. From Lemma 2(d), $\{n_1, n_2, \dots, n_{i_2}\}$ represents all integers from $k+1$ to

$$M = \frac{3(k+1)^2}{2} - \frac{k+1}{2} + (k+1)(m_{k+1} - 2) + (k+2)(m_{k+2} - 1) + \cdots + 2k(m_{2k} - 1),$$

but does not represent $M+1$. So, from (1.11),

$$a(i_2) = M + 1 \geq \frac{3(k+1)^2}{2} - \frac{k+1}{2} + 1 = (3k+2)\frac{k+1}{2} + 1 > 3k+2,$$

so that, from the minimality of i_0, we have $i_2 = i_0$, and, from (1.14),

$$S(i_0) = S(i_2) = m_{k+1}(k+1) + m_{k+2}(k+2) + \cdots + m_{2k}(2k) + 2k + 1 = a(i_0) + k.$$

In both cases (A) and (B), we have proved

$$a(i_0) \geq (3k+2)\frac{k+1}{2} + 1 \geq 4k + 2 \geq 3k + 3 \tag{3.2}$$

which, from (1.15), implies (1.17) and

$$S(i_0) = a(i_0) + k. \tag{3.3}$$

3.2. Proof of Theorem 2 (1.18)

From (1.16) and the definition (1.11), $a(i) - k - 1$ is represented by $\{n_1, \dots, n_i\}$, and more precisely by the elements of $\{n_1, \dots, n_i\}$ smaller than $a(i) - k$. Therefore, from (1.14), $a(i) - k - 1$ is a subsum of $S(i)$, so that

$$S(i) \geq a(i) - k - 1. \tag{3.4}$$

For the same reasons, $a(i)-k-2$ is a subsum of $S(i)$. So, $S(i) = a(i)-k-1$ is impossible, otherwise $S(i) - (a(i) - k - 2) = 1$ would be represented by $\{n_1, \dots, n_i\}$. So, from (3.4), we have

$$S(i) > a(i) - k - 1. \tag{3.5}$$

But, since $a(i)-k-1$ is a subsum of $S(i)$, $S(i)-(a(i)-k-1)$ is also a subsum of $S(i)$ and is represented by $\{n_1, \dots, n_i\}$. It follows from (3.5) that $S(i)-(a(i)-k-1) \geq n_1 = k+1$, in other terms, $S(i) \geq a(i)$. Finally, $S(i) \neq a(i)$ (otherwise $a(i)$ would be represented by $\{n_1, \dots, n_i\}$), so that $S(i) \geq a(i) + 1$, which completes the proof of (1.18).

3.3. Proof of Theorem 2 (1.19) and (1.20)

We shall prove together (1.19) and (1.20) by induction on $i \geq i_0$. From (3.3), (1.19) is true for $i = i_0$. Let us suppose that

$$i \geq i_0 \quad \text{and} \quad S(i) \leq a(i) + k. \tag{3.6}$$

We shall give the values of $S(i + 1)$ and $a(i + 1)$; there are different cases:

I. $a(i) - k \leq n_{i+1} \leq a(i) - 1$.

From (1.11), $a(i)$ is not representable by $\{n_1, \dots, n_i\}$. So a representation of $a(i)$ by $\{n_1, \dots, n_{i+1}\}$ should use n_{i+1}. But $1 \leq a(i)-n_{i+1} \leq k$, so that $a(i)$ cannot be represented by $\{n_1, \dots, n_{i+1}\}$. Consequently, from (1.11) and (1.14), $a(i + 1) = a(i)$, $S(i + 1) = S(i)$ and $S(i + 1) \leq a(i + 1) + k$ follows from (3.6).

II. $n_{i+1} \leq a(i) - k - 1$.

We have $n_1 \leq n_2 \leq \dots \leq n_i \leq n_{i+1} \leq a(i) - k - 1$; so it follows from (1.14) that

$$S(i) = n_1 + n_2 + \dots + n_i. \tag{3.7}$$

We shall prove that

$$a(i+1) = a(i) + n_{i+1}. \tag{3.8}$$

Indeed, from (1.11), $k+1, \dots, a(i)-1$ are represented by $\{n_1, \dots, n_i\}$ and, for $0 \le j < n_{i+1}$, $a(i) + j = (a(i) + j - n_{i+1}) + n_{i+1}$ is represented by $\{n_1, \dots, n_{i+1}\}$.

To show (3.8), it remains to prove that $a(i) + n_{i+1}$ is not represented by $\{n_1, \dots, n_{i+1}\}$. If such a representation did exist, it should not contain n_{i+1} (otherwise $a(i)$ would be represented by $\{n_1, \dots, n_i\}$) so that, from (3.7), $a(i) + n_{i+1}$ is a subsum of $S(i)$ and thus, by our induction hypothesis (3.6)

$$a(i) + n_{i+1} \le S(i) \le a(i) + k < a(i) + n_1 \le a(i) + n_{i+1}$$

which is impossible and so, (3.8) is settled. From (1.14) we have $S(i+1) = n_1 + n_2 + \cdots + n_{i+1}$, which implies $S(i+1) - a(i+1) = S(i) - a(i) \le k$.

III. $n_{i+1} > a(i)$.

This case is easy: clearly, we have $a(i+1) = a(i)$ and $S(i+1) = S(i)$.

IV. $n_{i+1} = a(i)$.

We have $n_i \ne a(i)$ (otherwise $a(i)$ would be represented by $\{n_1, \dots, n_i\}$); so, since $n_i \le n_{i+1} = a(i)$, we have

$$n_i \le a(i) - 1. \tag{3.9}$$

IV/1. $n_{i+1} = a(i)$, $a(i) - k \notin \{n_1, \dots, n_i\}$, $a(i) + 1 < S(i) \le a(i) + k$ $(k \ge 2)$.

We want to show

$$a(i+1) = a(i) + 1. \tag{3.10}$$

From the definition (1.11), $k+1, \dots, a(i)-1$ are represented by $\{n_1, \dots, n_i\}$; $a(i) = n_{i+1}$ is represented by $\{n_1, \dots, n_{i+1}\}$; so, to prove (3.10), we must show that $a(i)+1$ is not represented by $\{n_1, \dots, n_{i+1}\}$. If it was represented by $\{n_1, \dots, n_{i+1}\}$, such a representation could not use n_{i+1}, otherwise, $a(i) + 1 - n_{i+1} = 1$ would be represented by $\{n_1, \dots, n_i\}$. Let us assume that $a(i)+1$ is represented by $\{n_1, \dots, n_i\}$:

$$a(i) + 1 = n_{i_1} + \cdots + n_{i_s}, \quad 1 \le i_1 < i_2 < \cdots < i_s \le i. \tag{3.11}$$

From (3.9), we have $s \ge 2$.

- If $n_{i_1} = k+1$, we have $n_{i_2} + \cdots + n_{i_s} = a(i) - k$ so that $s \geq 3$ (if $s = 2$, $a(i) - k$ would belong to $\{n_1, \ldots, n_i\}$); therefore $n_{i_s} \leq a(i) - k - 1$.
- If $n_{i_1} \geq k+2$, we have $n_{i_2} + \cdots + n_{i_s} = a(i) + 1 - n_{i_1} \leq a(i) - k - 1$.

So, in all cases, the representation (3.11) would imply that $n_{i_s} \leq a(i) - k - 1$, in other terms, from (1.14), $a(i) + 1$ would be a subsum of $S(i)$. But, this would imply that $S(i) - (a(i) + 1)$ is also a subsum of $S(i)$ and thus either vanishes or is at least $k + 1$. But, this is impossible, since it follows from our hypothesis that $0 < S(i) - (a(i) + 1) \leq k - 1$ and the proof of (3.10) is completed.

Finally, from (1.14), $S(i+1) = S(i)$ and $S(i+1) - a(i+1) = S(i) - a(i) - 1 < k$ hold.

IV/2. $n_{i+1} = a(i)$, $a(i) - k \in \{n_1, \ldots, n_i\}$, $a(i) + 1 < S(i) \leq a(i) + k$ $(k \geq 2)$.

Now, the multiset $\{n_1, \ldots, n_{i+1}\}$ writes

$$\{n_1, \ldots, \underbrace{a(i)-k, \ldots, a(i)-k}_{m_k\,(\geq 1)}, \underbrace{a(i)-(k-1), \ldots, a(i)-(k-1)}_{m_{k-1}\,(\geq 0)}, \ldots, \underbrace{a(i)-1, \ldots, a(i)-1}_{m_1\,(\geq 0)}, a(i)(= n_{i+1})\} \tag{3.12}$$

and we have

$$S(i) = a(i) + j_0, \quad 2 \leq j_0 \leq k. \tag{3.13}$$

From (1.11) and (1.14), $k+1, \ldots, a(i) - k - 1$ are subsums of $S(i)$, and by (1.17), we have $a(i) \geq 3k + 2$, so that $k \leq a(i) - 2k - 2$, and, from (3.6), $\frac{1}{2}S(i) \leq \frac{1}{2}(a(i) + k) \leq \frac{1}{2}(a(i) + (a(i) - 2k - 2)) = a(i) - k - 1$. Therefore, each integer from $k+1$ to $S(i) - (k+1)$ can be written u or $S(i) - u$, where $k + 1 \leq u \leq a(i) - k - 1$, and thus is a subsum of $S(i)$, i.e., from (3.13),

$$k+1, \ldots, a(i) - k - 2 + j_0, a(i) - k - 1 + j_0 \text{ are subsums of } S(i). \tag{3.14}$$

For $j_0 < k$, $a(i) - u$, $1 \leq u \leq k - j_0$, is not a subsum of $S(i)$ (otherwise $S(i) - (a(i) - u) = j_0 + u \leq k$ would be a subsum of $S(i)$); as it is, from (1.11), represented by $\{n_1, \ldots, n_i\}$ its representation needs a part larger than $a(i) - k - 1$, but this part is the only one, since $a(i) - u - (a(i) - k) = k - u \leq k$, thus

$$m_{k-j_0} \geq 1, \ldots, \; m_1 \geq 1. \tag{3.15}$$

We shall now prove the following assertion

Assertion 1. With the notation of (3.12) and (3.13), we have

$$a(i+1) = 2a(i) + j_0 - k + \sum_{j=1}^{k} m_j (a(i) - j). \tag{3.16}$$

Proof of Assertion 1: In order to prove (3.16), from (1.11), we first have to show that each N satisfying

$$k+1 \le N < 2a(i) + j_0 - k + \sum_{j=1}^{k} m_j(a(i)-j) \tag{3.17}$$

can be represented by $\{n_1, \dots, n_{i+1}\}$. For such an N, there exist $u_1, \dots, u_k$ and a *minimal* V such that

$$N = V + \sum_{j=1}^{k} u_j(a(i)-j) \tag{3.18}$$

with

$$V \ge 0; \quad 0 \le u_j \le m_j, \quad j = 1, \dots, k-1; \quad 0 \le u_k < m_k.$$

We shall consider several cases:

(a) $V = 0$. Here, from (3.12), (3.18) is a representation of N by $\{n_1, \dots, n_i\}$.

(b) $1 \le V \le k$. Since from (3.17), $N \ge k+1$, there exists a *minimal* j_1 such that, in (3.18), $u_{j_1} \ge 1$.

(b_1) $1 \le j_1 < V$. We have $j_1 < k$, so we can write

$$N = V + k - j_1 + \big(u_{j_1} - 1\big)(a(i) - j_1) + \sum_{j=j_1+1}^{k-1} u_j(a(i)-j) + (u_k+1)(a(i)-k). \tag{3.19}$$

The first term on the right hand side of (3.19) satisfies from (1.17)

$$k+1 \le V + k - j_1 \le 2k - 1 \le a(i) - k - 1$$

and so, is a subsum of $S(i)$; since, in (3.18), u_k has been chosen smaller than m_k, (3.19) is a representation of N by $\{n_1, \dots, n_i\}$.

(b_2) $j_1 = V$. We write

$$N = a(i) + \big(u_{j_1} - 1\big)(a(i) - j_1) + \sum_{j=j_1+1}^{k} u_j(a(i)-j)$$

and, since $a(i) = n_{i+1}$, N is represented by $\{n_1, \dots, n_{i+1}\}$.

(b_3) $V < j_1 \le k$. We write

$$N = V + a(i) - j_1 + \big(u_{j_1} - 1\big)(a(i) - j_1) + \sum_{j=j_1+1}^{k} u_j(a(i)-j). \tag{3.20}$$

If we set $j_1 - V = j'$, we have $1 \le j' < j_1 \le k$ and (3.20) becomes

$$N = a(i) - j' + \big(u_{j_1} - 1\big)(a(i) - j_1) + \sum_{j=j_1+1}^{k} u_j(a(i) - j). \tag{3.21}$$

But, from (1.11), $a(i) - j'$ is represented by $\{n_1, \ldots, n_i\}$. Since, for $j_1 \le j \le k$, we have $1 \le a(i) - j' - (a(i) - j) = j - j' \le k - 1$, such a representation cannot use any part $a(i) - j$, $j_1 \le j \le k$, and (3.21) is a representation of N by $\{n_1, \ldots, n_i\}$.

(c) $k + 1 \le V \le a(i) - k - 1$. From (1.11) and (1.14), V is a subsum of $S(i)$, and so, (3.18) is a representation of N by $\{n_1, \ldots, n_i\}$.

(d) $a(i) - k \le V \le a(i) - 1$.

(d_1) If $u_1 = u_2 = \cdots = u_k = 0$, from (1.11), $N = V$ is represented by $\{n_1, \ldots, n_i\}$.

(d_2) If $u_1 = u_2 = \cdots = u_k = 0$ does not hold, there exists a *maximal* j_2, $1 \le j_2 \le k$, such that $u_{j_2} \ne 0$. We write

$$\begin{aligned} N &= V + \sum_{j=1}^{j_2} u_j(a(i) - j) \\ &= n_{i+1} + (V - j_2) + \sum_{j=1}^{j_2-1} u_j(a(i) - j) + \big(u_{j_2} - 1\big)(a(i) - j_2). \end{aligned} \tag{3.22}$$

We have from (1.17)

$$k + 1 < a(i) - 2k \le V - j_2 \le a(i) - (j_2 + 1),$$

so, from (1.11), $V - j_2$ is represented by $\{n_1, \ldots, n_i\}$ without using any part $a(i) - j$, $j \le j_2$ and (3.22) is a representation of N by $\{n_1, \ldots, n_{i+1}\}$.

(e) $V = a(i)$. Since $a(i) = n_{i+1}$, (3.18) is a representation of N by $\{n_1, \ldots, n_{i+1}\}$.

(f) $a(i) + 1 \le V \le 2a(i) - 2k - 1$. Since u_k has been chosen smaller than m_k in (3.18), we write

$$N = V - (a(i) - k) + \sum_{j=1}^{k-1} u_j(a(i) - j) + (u_k + 1)(a(i) - k). \tag{3.23}$$

Here we have $k + 1 \le V - (a(i) - k) \le a(i) - k - 1$, and (3.23) is a representation of N by $\{n_1, \ldots, n_i\}$.

(g) $2a(i) - 2k \le V \le 2a(i) - k - 1 + j_0$. In (3.18) we write $V = n_{i+1} + V - a(i)$. From (1.17), we have $k + 1 < a(i) - 2k \le V - a(i) \le a(i) - k - 1 + j_0$, so that, from (3.14), $V - a(i)$ is a subsum of $S(i)$ and (3.18) is a representation of N by $\{n_1, \ldots, n_{i+1}\}$.

(h) $V \ge 2a(i) - k + j_0$. Here, from (3.13) and (1.17), we have $V \ge a(i) + 2k + 6 > a(i) - 1$, and since we have chosen V *minimal*, we have $u_1 = m_1, \ldots, u_{k-1} = m_{k-1}$ and $u_k = m_k - 1$. Then (3.18) can be written

$$N = V - (a(i) - k) + \sum_{j=1}^{k} m_j(a(i) - j). \tag{3.24}$$

Let us set $V' = V - (a(i) - k)$ so that $V' \geq a(i) + j_0$. From (3.17) and (3.24), it follows

$$a(i) + j_0 \leq V' < 2a(i) + j_0 - k. \tag{3.25}$$

We distinguish three cases:

(h_1) $V' = a(i) + j_0$. From (3.13) and (3.24), we have $N = S(i) + \sum_{j=1}^{k} m_j(a(i) - j)$ and so, N is represented by $\{n_1, \dots, n_i\}$.

(h_2) $a(i) + j_0 + 1 \leq V' \leq a(i) + k$ (and $j_0 < k$). We write $V' = a(i) + j_0 + \ell$, $1 \leq \ell \leq k - j_0$, so that, from (3.15), $m_\ell \geq 1$. We have

$$N = V' + a(i) - \ell + \cdots + (m_\ell - 1)(a(i) - \ell) + \cdots$$

and since, from (3.13), $V' + a(i) - \ell = n_{i+1} + S(i)$, N is represented by $\{n_1, \dots, n_{i+1}\}$.

(h_3) $a(i) + k + 1 \leq V' \leq 2a(i) + j_0 - k - 1$. Here we have

$$k + 1 \leq V' - n_{i+1} \leq a(i) - k - 1 + j_0;$$

so, from (3.14), $V' - n_{i+1}$ is a subsum of $S(i)$ and N is represented by $\{n_1, \dots, n_{i+1}\}$.

So, we have proved that each N satisfying (3.17) is represented by $\{n_1, \dots, n_{i+1}\}$; to prove (3.16), it remains to show that

$$2a(i) + j_0 - k + \sum_{j=1}^{k} m_j(a(i) - j) \tag{3.26}$$

cannot be represented by $\{n_1, \dots, n_{i+1}\}$. But, from (3.12), (1.14) and (3.13), we get

$$\begin{aligned} n_1 + \cdots + n_{i+1} &= S(i) + \sum_{j=1}^{k} m_j(a(i) - j) + n_{i+1} \\ &= 2a(i) + j_0 + \sum_{j=1}^{k} m_j(a(i) - j) \\ &= k + \left(2a(i) + j_0 - k + \sum_{j=1}^{k} m_j(a(i) - j)\right) \end{aligned} \tag{3.27}$$

so that (3.26) cannot be represented by $\{n_1, \dots, n_{i+1}\}$, and the proof of Assertion 1 is completed. □

Since, from (3.16), (3.13) and (1.17),

$$a(i + 1) - k - 1 \geq 2a(i) + j_0 - 2k - 1 > 2a(i) - 2k - 1 > a(i) = n_{i+1},$$

it follows from (1.14) that $S(i + 1) = n_1 + \cdots + n_{i+1}$ and thus, from (3.16) and (3.27), that $S(i + 1) = a(i + 1) + k$.

IV/3. $n_{i+1} = a(i)$, $S(i) = a(i) + 1$ $(k \geq 1)$. In this case,

$$k+1, \ldots, a(i)-k-1, a(i)-k = S(i)-(k+1) \text{ are subsums of } S(i); \tag{3.28}$$

but, for $k \geq 2$, $a(i)-(k-1) = S(i)-k, \ldots, a(i)-1 = S(i)-2$ cannot be subsums of $S(i)$. So, if $a(i)-u$, $1 \leq u \leq k-1$, is represented by $\{n_1, \ldots, n_i\}$, such a representation needs a part $a(i)-j$, $u \leq j \leq k$. But, since $a(i)-u-(a(i)-j) \leq k$, this is the only one. Consequently,

$$a(i)-(k-1), a(i)-(k-2), \ldots, a(i)-1 \in \{n_1, \ldots, n_i\} \tag{3.29}$$

and from (3.9), the multiset $\{n_1, \ldots, n_{i+1}\}$ can be written:

$$\begin{aligned}\{n_1, \ldots, &\underbrace{a(i)-k, \ldots, a(i)-k}_{m_k\,(\geq 0)}, \underbrace{a(i)-(k-1), \ldots, a(i)-(k-1)}_{m_{k-1}\,(\geq 1)},\\ &\ldots, \underbrace{a(i)-1, \ldots, a(i)-1}_{m_1\,(\geq 1)}, a(i)(= n_{i+1})\}\end{aligned} \tag{3.30}$$

with $m_1(\geq 0)$ for $k = 1$. We shall now prove the following assertion

Assertion 2. With the notation of (3.30), we have

$$a(i+1) = 2a(i) - k + 1 + \sum_{j=1}^{k} m_j(a(i)-j). \tag{3.31}$$

Proof of Assertion 2: The proof looks like the proof of Assertion 1; that is why we shall omit some details.

For $k \geq 2$, and

$$k+1 \leq N < 2a(i) + 1 - k + \sum_{j=1}^{k} m_j(a(i)-j), \tag{3.32}$$

there exist $u_1, \ldots, u_k$ and a *minimal* V such that

$$N = V + \sum_{j=1}^{k} u_j(a(i)-j) \tag{3.33}$$

with

$$V \geq 0; \quad 0 \leq u_{k-1} < m_{k-1}; \quad 0 \leq u_j \leq m_j, \quad j = 1, \ldots, k-2, k.$$

We shall consider several cases:

(a) $V = 0$. Here, from (3.30), (3.33) is a representation of N by $\{n_1, \ldots, n_i\}$.

(b) $1 \leq V \leq k$. There exists a *minimal* j_1 such that $u_{j_1} \geq 1$.

(b$_1$) $1 \le j_1 < V - 1$ ($\le k - 1$). We write

$$N = V + (k-1) - j_1 + \big(u_{j_1} - 1\big)(a(i) - j_1) + \cdots \\ + (u_{k-1} + 1)(a(i) - (k-1)) + \cdots . \tag{3.34}$$

The first term satisfies from (1.17):

$$k + 1 \le V + (k-1) - j_1 \le 2k - 2 \le a(i) - k - 1$$

and (3.34) is a representation of N by $\{n_1, \dots, n_i\}$.

(b$_2$) $j_1 = V - 1$. We write from (3.33)

$$N = \underbrace{(a(i) + 1)}_{S(i)} + \cdots + (u_{V-1} - 1)(a(i) - (V-1)) + \cdots,$$

and N is represented by $\{n_1, \dots, n_i\}$.

(b$_3$) $j_1 = V$. We write from (3.33)

$$N = n_{i+1} + \cdots + (u_V - 1)(a(i) - V) + \cdots,$$

and N is represented by $\{n_1, \dots, n_{i+1}\}$.

(b$_4$) $V < j_1 \le k$. We write from (3.33)

$$N = (a(i) - (j_1 - V)) + \big(u_{j_1} - 1\big)(a(i) - j_1) + \cdots . \tag{3.35}$$

We have $1 \le j_1 - V < j_1 \le k$, and, from (3.30), $m_{j_1 - V} \ge 1$; $u_{j_1 - V} = 0$ follows from the minimality of j_1, so that (3.35) is a representation of N by $\{n_1, \dots, n_i\}$.

(c) $k+1 \le V \le a(i) - k - 1$. Here, V is a subsum of $S(i)$, and so, (3.33) is a representation of N by $\{n_1, \dots, n_i\}$.

(d) $a(i) - k \le V \le a(i) - 1$.

(d$_1$) If $u_1 = u_2 = \cdots = u_k = 0$, $N = V$ is represented by $\{n_1, \dots, n_i\}$.

(d$_2$) If $u_1 = u_2 = \cdots = u_k = 0$ does not hold, there exists a *maximal* j_2, $1 \le j_2 \le k$, such that $u_{j_2} \ne 0$. We write

$$N = V + \sum_{j=1}^{j_2} u_j (a(i) - j) \\ = n_{i+1} + (V - j_2) + \cdots + \big(u_{j_2} - 1\big)(a(i) - j_2). \tag{3.36}$$

We have from (1.17)

$$k + 1 < a(i) - 2k \le V - j_2 \le a(i) - (j_2 + 1),$$

so, as in the proof of Assertion 1 (d_2), (3.36) is a representation of N by $\{n_1, \dots, n_{i+1}\}$.

(e) $V = a(i) = n_{i+1}$. (3.33) is a representation of N by $\{n_1, \dots, n_{i+1}\}$.

(f) $V = a(i) + 1 = S(i)$. (3.33) is a representation of N by $\{n_1, \dots, n_i\}$.

(g) $a(i)+2 \le V \le a(i)+k$. We write

$$N = (V - a(i) + k - 1) + \cdots + (u_{k-1}+1)(a(i)-(k-1)) + \cdots .$$

Here we have $k+1 \le V - a(i) + k - 1 \le 2k-1 < a(i) - k - 1$, and N is represented by $\{n_1, \ldots, n_i\}$.

(h) $a(i)+k+1 \le V \le 2a(i) - k$. In (3.33) we write $V = n_{i+1} + V - a(i)$. We have $k+1 \le V - a(i) \le a(i) - k$, so that, from (3.28), $V - a(i)$ is a subsum of $S(i)$ and N is represented by $\{n_1, \ldots, n_{i+1}\}$.

(i) $V = 2a(i) - k + 1$. We write

$$N = n_{i+1} + \cdots + (u_{k-1}+1)(a(i)-(k-1)) + \cdots$$

which shows that N is represented by $\{n_1, \ldots, n_{i+1}\}$.

(j) $V \ge 2a(i) - k + 2$. Here, we have $V > a(i) - 1$, and since we have chosen V *minimal*, we have $u_1 = m_1, \ldots, u_{k-2} = m_{k-2}, u_{k-1} = m_{k-1} - 1$ and $u_k = m_k$. Then (3.33) can be written

$$N = V' + \sum_{j=1}^{k} m_j(a(i) - j), \tag{3.37}$$

with $V' = V - (a(i) - (k-1)) \ge a(i) + 1$. From (3.32) and (3.37), it follows

$$a(i) + 1 \le V' < 2a(i) - k + 1. \tag{3.38}$$

We distinguish three cases:

(j_1) $V' = a(i) + 1 = S(i)$. Here (3.37) is a representation of N by $\{n_1, \ldots, n_i\}$.

(j_2) $a(i) + 2 \le V' \le a(i) + k$. We write $V' = a(i) + 1 + \ell$, $1 \le \ell \le k - 1$, so that, from (3.30), $m_\ell \ge 1$. We have

$$N = n_{i+1} + S(i) + \cdots + (m_\ell - 1)(a(i) - \ell) + \cdots$$

and N is represented by $\{n_1, \ldots, n_{i+1}\}$.

(j_3) $a(i) + k + 1 \le V' \le 2a(i) - k$. Here we have

$$k + 1 \le V' - n_{i+1} \le a(i) - k;$$

so, from (3.28), $V' - n_{i+1}$ is a subsum of $S(i)$ and N is represented by $\{n_1, \ldots, n_{i+1}\}$.

So, we have proved that each N satisfying (3.32) is represented by $\{n_1, \ldots, n_{i+1}\}$; to prove (3.31), it remains to show that

$$2a(i) - k + 1 + \sum_{j=1}^{k} m_j(a(i) - j) \tag{3.39}$$

cannot be represented by $\{n_1, \ldots, n_{i+1}\}$. But, from (3.30),

$$\begin{aligned} n_1 + \cdots + n_{i+1} &= S(i) + \sum_{j=1}^{k} m_j(a(i) - j) + n_{i+1} \\ &= 2a(i) + 1 + \sum_{j=1}^{k} m_j(a(i) - j) \\ &= k + \left(2a(i) - k + 1 + \sum_{j=1}^{k} m_j(a(i) - j)\right) \end{aligned} \tag{3.40}$$

so that (3.39) cannot be represented by $\{n_1, \ldots, n_{i+1}\}$, and the proof of Assertion 2 is completed for $k \geq 2$.

The case $k = 1$ can be settled in a similar way with $0 \leq u_1 \leq m_1$ and $0 \leq V < 2a(i)$ considering (a), (b_3), (c), (d_1), (d_2), (e), (f) and (h). □

Like in the case **IV/2**, it is easy to show from (1.14), (3.31) and (3.40) that $S(i + 1) = a(i + 1) + k$, and the proof of Theorem 2 is completed.

4. Proof of Theorem 3

Before starting the proof of Theorem 3, let us observe that, for $\mathcal{A} = \{1, 2, \ldots, k\}$, $\tilde{p}(n, \mathcal{A})$ is easy to compute for $2k + 2 \leq n \leq (3k + 4)\frac{k+1}{2}$. We have

$$\tilde{p}(2k + 2, \mathcal{A}) = \tilde{p}(2k + 3, \mathcal{A}) = \tilde{p}\left((3k + 4)\frac{k + 1}{2}, \mathcal{A}\right) = 1,$$

and $\tilde{p}(n, \mathcal{A}) = 0$ for $2k + 4 \leq n \leq (3k + 4)\frac{k+1}{2} - 1$.

For $2k + 2 \leq a < n$, let us define $\mathcal{X}(n, a)$ as the set of partitions of n not containing $1, 2, \ldots, k$ but representing $k + 1, k + 2, \ldots, a - 1$, further not representing a, and $X(n, a) = |\mathcal{X}(n, a)|$.

A generic partition (1.1) of n, belonging to $\tilde{\mathcal{P}}(n, \mathcal{A})$ should contain no part up to k and, for $n \geq 3k + 2$, should contain parts equal to $k + 1, k + 2, \ldots, 2k + 1$ in order to represent $k + 1, k + 2, \ldots, 2k + 1$. The number of such partitions is $r(n - (3k + 2)\frac{k+1}{2}, k + 1)$. Thus, from the definition of $\mathcal{X}(n, a)$, we have

$$\tilde{p}(n, \mathcal{A}) = r\left(n - (3k + 2)\frac{k + 1}{2}, k + 1\right) - \sum_{a=2k+2}^{\lfloor n/2 \rfloor} X(n, a). \tag{4.1}$$

For $n \geq (3k + 4)\frac{k+1}{2} + 1$, we have

$$X(n, 2k + 2) = r\left(n - (3k + 2)\frac{k + 1}{2},\ \{1, 2, \ldots, k, k + 1, 2k + 2\}\right) \tag{4.2}$$

since a partition of $\mathcal{X}(n, 2k+2)$ should contain $k+1$ exactly once, should contain $k+2, k+3, \dots, 2k+1$ at least once and should not contain $2k+2$.

Further, for $a \ge 2k+3$, if (1.1) is a partition of $\mathcal{X}(n, a)$, $a(t)$ defined by (1.11) satisfies $a(t) = a \ge 2k+3$, and so, from Theorem 2, it satisfies also $a(t) \ge (3k+2)\frac{k+1}{2} + 1$, so that

$$X(n, a) = 0 \quad \text{for } 2k+3 \le a \le (3k+2)\frac{k+1}{2}\cdot \tag{4.3}$$

In view of applying (4.1), it remains to calculate $X(n, a)$ when

$$3k+3 \le 4k+2 \le (3k+2)\frac{k+1}{2} + 1 \le a \le \frac{n}{2}\cdot \tag{4.4}$$

From now on, we shall assume that (4.4) holds; if the partition (1.1) belongs to $\mathcal{X}(n, a)$, let us define ℓ, $1 \le \ell \le t$, by

$$n_\ell \le a - k - 1 < n_{\ell+1}. \tag{4.5}$$

Note that $\ell = t$ is impossible; indeed, if $n_t \le a - k - 1$, we would have from (1.11), (1.14), (1.19) and (4.4)

$$n = S(t) \le a(t) + k = a + k \le n/2 + k,$$

which does not hold since n is supposed to satisfy $n \ge (3k+4)\frac{k+1}{2} > 2k$. So, we have:

$$1 \le \ell < t. \tag{4.6}$$

From the definitions (1.11) and (1.14), we have $a(t) = a$ and with (4.5),

$$S(t) = n_1 + n_2 + \cdots + n_\ell. \tag{4.7}$$

Since our partition belongs to $\mathcal{X}(n, a)$, the integers $k+1, \dots, a-k-1$ are represented by $\{n_1, \dots, n_t\}$ and, from (4.5), are represented by $\{n_1, \dots, n_\ell\}$. This implies that $a - k \le a(\ell)$ and we get from (4.4) and (1.15)

$$2k+3 \le a - k \le a(\ell) \le a(t) = a. \tag{4.8}$$

By applying Theorem 2 and (1.15), it follows

$$a - k + 1 \le a(\ell) + 1 \le S(\ell) \le S(t) \le a(t) + k = a + k$$

and

$$0 \le S(t) - S(\ell) \le 2k - 1. \tag{4.9}$$

Comparing (4.7) and

$$S(\ell) = \sum_{\substack{j=1 \\ n_j \le a(\ell)-k-1}}^{\ell} n_j$$

gives, from (4.5)

$$S(t) - S(\ell) = \sum_{\substack{j=1 \\ a(\ell)-k \le n_j}}^{\ell} n_j$$

so that, if $S(t) - S(\ell) \neq 0$, we would have from (4.8) and (4.4)

$$S(t) - S(\ell) \ge a(\ell) - k \ge a - 2k > 4k + 1 - 2k > 2k - 1$$

which contradicts (4.9). Consequently, $S(t) - S(\ell) = 0$ and $n_\ell \le a(\ell) - k - 1$. Therefore, it follows from Theorem 2 that

$$a + 1 = a(t) + 1 \le S(t) = S(\ell) = n_1 + n_2 + \cdots + n_\ell \le a(\ell) + k \tag{4.10}$$

and

$$a(\ell) \ge a - (k - 1). \tag{4.11}$$

From (1.11), the multiset $\{n_1, \ldots, n_\ell\}$ represents $a(\ell) - 1$ and thus, by (4.10), it also represents $n_1 + \cdots + n_\ell - (a(\ell) - 1) = S(\ell) - (a(\ell) - 1)$. But, from (4.8) and (4.10) we have

$$a(\ell) - 1 < a(\ell) + 1 \le a + 1 \le S(\ell)$$

and therefore, $S(\ell) - (a(\ell) - 1) > 0$. Since $S(\ell) - (a(\ell) - 1)$ is represented by $\{n_1, \ldots, n_\ell\}$, we have $S(\ell) - (a(\ell) - 1) \ge k + 1$; in other terms, $S(\ell) \ge a(\ell) + k$ which, together with (4.10) gives

$$n_1 + n_2 + \cdots + n_\ell = a(\ell) + k. \tag{4.12}$$

We introduce the set $\mathcal{X}(n, a, j)$ (for $0 \le j \le k - 1$) which is the subset of $\mathcal{X}(n, a)$ such that

$$a(\ell) = a - j, \quad 0 \le j \le k - 1 \tag{4.13}$$

where ℓ is defined by (4.5). From (4.8) and (4.11), it follows that

$$\mathcal{X}(n, a) = \bigcup_{0 \le j \le k-1} \mathcal{X}(n, a, j).$$

We shall assume that our partition belongs to $\mathcal{X}(n, a, j)$. It follows from (4.12) and (1.14) that

$$S(\ell) = n_1 + \cdots + n_\ell = a + k - j, \quad n_\ell \le a - k - 1 - j. \tag{4.14}$$

From (1.11) and (4.13),

$$\text{the multiset } \{n_1, \ldots, n_\ell\} \text{ represents } k + 1, \ldots, a - j - 1. \tag{4.15}$$

Let us assume that it represents $a-u$, $0<u\le j$. Then, it would also represent $n_1+\cdots+n_\ell-(a-u)=k-j+u$ by (4.14). But, $1\le k-j+u\le k<n_1$, so that

$$\{n_1,\dots,n_\ell\} \text{ does not represent } 1,2,\dots,k,a-j,a-j+1,\dots,a. \tag{4.16}$$

If $j\ge 1$, $a-j,a-j+1,\dots,a-1$ are represented by $\{n_1,\dots,n_t\}$. But, from (4.16), a representation of $a-u$, $1\le u\le j$, needs at least one part n_r, $\ell+1\le r\le t$. From (4.5), $n_r\ge a-k$, and $a-u-n_r\le a-1-(a-k)=k-1$. Thus, $a-u-n_r=0$, and

$$a-j,a-j+1,\dots,a-1\in\{n_{\ell+1},\dots,n_t\}. \tag{4.17}$$

From (4.14), (4.15) and (4.16), $n_1+n_2+\cdots+n_\ell$ is a partition of $\tilde{\mathcal{P}}(a+k-j,\mathcal{A})$. From (4.17) and (4.5), $n_{\ell+1}+\cdots+n_t$ is a partition of $n-(a+k-j)$ which contains $a-j,\dots,a-1$ and does not contain $1,2,\dots,a-k-1,a$. The number of such partitions is

$$\begin{aligned} r&\Big(n-(a+k-j)-(a-j)-\cdots-(a-1),\{1,2,\dots,a-k-1,a\}\Big)\\ &=r\left(n-k-1-(j+1)a+\frac{(j+1)(j+2)}{2},\{1,2,\dots,a-k-1,a\}\right).\end{aligned}$$

Conversely, if a and n satisfy (4.4) then any partition (1.1) of n such that, for some ℓ, $n_1+n_2+\cdots+n_\ell\in\tilde{\mathcal{P}}(a+k-j,\mathcal{A})$ for some j, $0\le j\le k-1$, and $n_{\ell+1}+\cdots+n_t$ is a partition of $n-(a+k-j)$ which contains $a-j,\dots,a-1$ (if $j\ge 1$) and does not contain $1,2,\dots,a-k-1,a$ (consequently, ℓ satisfies (4.5)) is a partition of $\mathcal{X}(n,a,j)$ and therefore,

$$\begin{aligned} X(n,a)=\sum_{j=0}^{k-1}\tilde{p}(a+k-j,\mathcal{A})r&\Bigg(n-k-1-(j+1)a\\ &+\frac{(j+1)(j+2)}{2},\{1,2,\dots,a-k-1,a\}\Bigg).\end{aligned} \tag{4.18}$$

Replacing $X(n,a)$ in (4.1) by its value given in (4.2), (4.3) and (4.18) completes the proof of Theorem 3.

We end this paper by writing three formulas looking like (1.21) and giving the value of $\tilde{p}(n,\mathcal{A})$ for $\mathcal{A}=\{2\},\{1,3\},\{1,3,5\}$:

For $n\ge 8$, we have:

$$\begin{aligned}\tilde{p}(n,\{2\})&=r(n-4,3)-r(n-4,\{1,2,4,5\})\\ &\quad-\sum_{a=6}^{\lfloor n/2\rfloor}\tilde{p}(a+2,\{2\})r(n-a-2,\{1,2,\dots,a-3,a-1,a\}).\end{aligned} \tag{4.19}$$

For $n\ge 10$, we have:

$$\begin{aligned}\tilde{p}(n,\{1,3\})&=r(n,\{1,3\})-r(n,4)-r(n-2,5)\\ &\quad-(r(n-2,\{1,3,5\})-r(n-2,6))-r(n-9,\{1,2,3,4,6\})\\ &\quad-\sum_{a=8}^{\lfloor n/2\rfloor}\tilde{p}(a+3,\{1,3\})r(n-a-3,\{1,2,\dots,a-4,a-2,a\}).\end{aligned} \tag{4.20}$$

For $n \geq 17$, we have:

$$\begin{aligned}
&\tilde{p}(n, \{1, 3, 5\}) \\
&\quad = r(n, \{1, 3, 5\}) - r(n, \{1, 2, 3, 5\}) - r(n-2, 6) - r(n-4, 7) \\
&\qquad - (r(n-2, \{1, 3, 5, 7\}) - r(n-2, \{1, 2, 3, 4, 5, 7\}) - r(n-4, 8)) \\
&\qquad - 2r(n-13, \{1, 2, 3, 4, 5, 6, 8\}) - 2r(n-15, \{1, 2, 3, 4, 5, 6, 8, 10\}) \\
&\qquad - \sum_{a=12}^{\lfloor n/2 \rfloor} \tilde{p}(a+5, \{1, 3, 5\}) r(n-a-5, \{1, 2, \ldots, a-6, a-4, a-2, a\}). \qquad (4.21)
\end{aligned}$$

References

1. J.-C. Aval, "On sets represented by partitions," *Europ. J. Combinatorics* **20** (1999), 317–320.
2. M. Deléglise, P. Erdős, and J.-L. Nicolas, "Sur les ensembles représentés par les partitions d'un entier n," *Discrete Math.* **200** (1999), 27–48.
3. J. Dixmier, "Sur les sous–sommes d'une partition," *Mémoire Soc. Math. France* **35** (1988).
4. J. Dixmier, "Sur les sous–sommes d'une partition, III," *Bull. Sci. Math.* **113** (1989), 125–149.
5. J. Dixmier, "Partitions avec sous–sommes interdites," *Bull. Soc. Math. Belgique* **42** (1990), 477–500.
6. J. Dixmier and J.-L. Nicolas, "Partitions without small parts," *Colloquia Mathematica Societatis János Bolyai* **51** (1987), 9–33. Number Theory, Budapest, Hungary.
7. P. Erdős, J.-L. Nicolas, and A. Sárközy, "On the number of partitions of n without a given subsum I," *Discrete Math.* **75** (1989), 155–166.
8. P. Erdős, J.-L.Nicolas, and A. Sárközy, "On the number of partitions of n without a given subsum II," in *Analytic Number Theory* (B. Berndt, H. Diamond, H. Halberstam, and A. Hildebrand, eds.), Birkhäuser, 1990, pp. 205–234.
9. P. Erdős and M. Szalay, "On some problems of J. Dénes and P. Turán," *Studies in Pure Mathematics to the memory of Paul Turán*, Budapest, 1983, pp. 187–212.
10. G. H. Hardy and S. Ramanujan, "Asymptotic formulae in combinatory analysis," *Proc. London Math. Soc.* (2), **17** (1918), 75–115, and Collected Papers of S. Ramanujan, pp. 276–309.
11. J.-L. Nicolas and A. Sárközy, "On two partitions problems," *Acta Math. Hungar.* **77** (1997), 95–121.