RIGIDITY THEOREM, FREENESS OF ALGEBRAS AND APPLICATIONS

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Abstract. A generalised bialgebra is an algebra which is also a coalgebra, with a relation λ linking coproducts of products of elements to products of coproducts of them. The rigidity theorem, first proven by Loday in 2008, states that, under certain conditions on λ, any associated generalised bialgebra is free and cofree over its primitive elements. We give here an easy-to-check criterion to build by duality such a theorem and apply it to prove the freeness of some combinatorial algebras.

Une bigèbre généralisée est une algèbre munie d'une structure de cogèbre et d'une famille de relations λ permettant la réécriture de coproduits de produits d'éléments en produits de coproduits de ces mêmes éléments. Le théorème de rigidité, prouvé en 2008 par Loday, permet d'établir, sous certaines conditions sur λ, la liberté et la coliberté de toute bigèbre généralisée associée. Nous donnons ici un critère facilement vérifiable pour l'obtention par dualité d'un tel théorème avant de l'appliquer pour montrer la liberté de certaines algèbres combinatoires.

Introduction

In 1969, Beck introduced in [1] the notion of distributive laws relating products of different types in an algebras, such as, for instance, the one linking the commutative product and the Lie bracket in a Poisson algebra. In 1997, Markl introduced in [12] this notion for operads and sketched the notion of generalised bialgebra. In 2008, Loday proved in [9] what is called the rigidity theorem: under certain conditions on the relations λ linking products and coproducts, a generalised bialgebra is free and cofree over its primitive elements. In 2015, Livernet, Mesablishvili and Wisbauer generalised in [8] this result to monads and the authors loosened the conditions on λ in [3], showing that for any type of algebra and coalgebra, there exists relations such that the rigidity theorem holds. We present here a way to explicitly build a rigidity theorem, in the first section, and apply this in the other sections to preLie algebras (section 2), dendriform algebras (section 3) and tridendriform algebras (section 4) to show the freeness of some combinatorial algebras (section 2 and 5), such as WQSym-algebras or hypertree algebras. We recover in the last section results of Vong in [17], giving an operadic frame to his proof, and of Burgunder-Curien-Ronco in [2], with a simpler description of the generators of the algebra. The notion of operad is a keypoint in the proof of these results, but the results themselves can be understood without knowing it.

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1. General results

An *operad* is a functor $\mathcal{F}$ from the category of finite dimensional $\mathfrak{S}$-module to itself, *i.e.* a *linear species*, endowed with a linear map $\pi : \mathcal{F} \circ \mathcal{F} \to \mathcal{F}$. It is used to encode a family of operations graded by arity and satisfying some given relations. For instance, Operad Comm associates to any vector space the trivial representation of the symmetric group and encodes commutative algebras. We consider two connected algebraic operads $\mathcal{A}$ and $\mathcal{C}$, or, in other words, a type of algebra $\mathcal{A}$ and a type of coalgebra $\mathcal{C}^c$, such that there is no operations or cooperations of arity 0, and the spaces of operations and cooperations of arity 1 is generated by the identity.

Let us first recall that a coalgebra is *conilpotent* if and only if for any x in C, there exists n such that $\delta(x) = 0$, for any $\delta \in \mathcal{C}(k)$, $k > n$, that *primitives* are elements x such that $\delta(x) = 0$, for any cooperation δ (the coproduct is reduced) and define the cofiltration on any conilpotent coalgebra C, $\mathcal{F}_n C = \{x \in C | \delta(x) = 0,$ for any $\delta \in \mathcal{C}(k), k > n\}$. We moreover denote by $\mathbb{K}$ a field of characteristic 0.

Definition 1.0.1. A *generalised bialgebra* of type $\mathcal{C}^c -_\lambda \mathcal{A}$ is a vector space C which is an $\mathcal{A}$-algebra, a conilpotent $\mathcal{C}^c$-coalgebra and such that any operation μ of arity n and any cooperation δ of coarity m satisfy the rewriting rule

$$\lambda : \delta \circ \mu \mapsto \sum_{\mu_i, \delta_j, \sigma} c_{\mu_i, \delta_j} \left(\mu_1 \otimes \ldots \otimes \mu_m\right) \circ \sigma \circ \left(\delta_1 \otimes \ldots \otimes \delta_n\right),$$

where μ_i are operations, δ_i are cooperations, σ is a permutation and c_{μ_i, δ_j} is a function from C^n to $\mathbb{K}$, constant on the sets $\{(x_1, \ldots, x_n) | x_j \in \mathcal{F}_{i_j} C - \mathcal{F}_{i_j - 1} C\}$, which enables to rewrite any coproducts of products of elements in terms of products of coproducts of the same elements. The rewriting rule λ is moreover asked to be compatible with the structure of operads of $\mathcal{C}$ and $\mathcal{A}$, *i.e.* that the rewriting of $(\delta_1 \otimes \ldots \otimes \delta_m) \circ \delta \circ \mu \circ (\mu_1 \otimes \ldots \otimes \mu_n)$ does not depend on the order used to apply the different rules.

Remark 1.0.2. It is then sufficient to give the rewriting rules on the generating operations and cooperations.

Example 1.0.3. A commutative cocommutative Hopf bialgebra, *i.e.* such that its cocommutative coproduct is a morphism of algebra, is a $\mathrm{Comm}^c -_{Hopf}$ Comm-generalised bialgebra. A $\mathrm{As} -_{\mathrm{n.u.i.}}$ As generalised bialgebra is defined as an associative coassociative bialgebra (B, μ, Δ) with rewriting rule given, for any $u, v \in B$ by:

$$\Delta \circ \mu(u, v) = u \otimes v + \Delta(u)_1 \otimes \mu(\Delta(u)_2, v) + \mu(u, \Delta(v)_1) \otimes \Delta(v)_2,$$

with Sweedler's notation of the coproduct.

Let us now consider a basis B of an operad $\mathcal{A}$, or equivalently a family of bases $B_\mathcal{H}$ of every free $\mathcal{A}$-algebra $\mathcal{H}$. Given an element x of $B_\mathcal{H}$ and a product $\mu \in \mathcal{A}(n)$, the coproduct Δ_μ given by duality on $x \in B_\mathcal{H}$ is then defined as:

$$\Delta_\mu(x) = \sum_{x_1, \ldots, x_n \in B_\mathcal{H}} \delta_{x^*(\mu(x_1, \ldots, x_n)) \neq 0} \frac{1}{x^*(\mu(x_1, \ldots, x_n))} x_1 \otimes \ldots \otimes x_n,$$

where x^* is the dual of x with respect to $B_\mathcal{H}$ and $\delta_{x^*(\mu(x_1, \ldots, x_n)) \neq 0}$ is the Kronecker symbol.

To fulfil the requirements of the rigidity theorem, the bases $B_\mathcal{H}$ have to satisfy some conditions:

Definition 1.0.4. A basis $B_{\mathcal{H}}$ is said to be a *compatible basis* if products and dual coproducts expressed in this basis commute with the action of the symmetric group. In other words, for any cooperation $\Delta \in \mathcal{C}(n)$, any operation $\mu \in \mathcal{A}(n)$, any $\sigma \in \mathfrak{S}_n$ and any $x_1, \ldots, x_n \in \operatorname{Prim}\mathcal{H}$, we have:

$$\Delta \circ \mu\,(\sigma \cdot (x_1, \ldots, x_n)) = \sigma \cdot (\Delta \circ \mu(x_1, \ldots, x_n)),$$

where the symmetric group acts by permutation on the $\{x_i\}$.

We then obtain an analogue of the rigidity theorem:

Proposition 1.0.5. *If the considered basis B is a compatible basis, then the rigidity theorem applies: any conilpotent $\mathcal{C}^c - \mathcal{A}$-bialgebra with rewriting rule given by duality (computed on the free algebra) is free and cofree over the vector space of its primitives.*

Remark 1.0.6. An example of bases of operads which are not compatible is given by the Lyndon basis and the comb basis of the operad Lie (see [3]).

Example 1.0.7. Using the duality on usual bases of known operads, we find back the following cases: cocommutative-commutative Hopf algebras by Borel, coassociative-associative bialgebras by Loday and Ronco, coassociative-Zinbiel and comagmatic-magmatic bialgebras by Burgunder, coNAP-preLie bialgebras by Livernet, conilpotent-nilpotent and coduplicial-duplicial bialgebras by Loday [9].

We apply Proposition 1.0.5 in the following sections.

2. The preLie Case

We consider here the preLie operad and the rooted tree basis introduced by Chapoton and Livernet in [5]: the free preLie algebra on a vector space V is spanned by rooted (non planar) trees with vertices indexed by V. We recall that the relation satisfied by a preLie product $\curvearrowleft$ is given by:

$$(x \curvearrowleft y) \curvearrowleft z - x \curvearrowleft (y \curvearrowleft z) = (x \curvearrowleft z) \curvearrowleft y - x \curvearrowleft (z \curvearrowleft y).$$

Combinatorially, the product of two rooted trees T and S, $T \curvearrowleft S$, is the sum over all possible ways to add an edge between a vertex of T and the root of S. The root of the obtained tree is the root of T. The dual coproduct is then given by the sum over all possible ways to delete an edge in the tree:

$$\Delta(T) = \sum_{a \in E(T)} R_a(T) \otimes L_a(T),$$

where $R_a(T)$ is the connected component of $T - \{a\}$ containing the root of T and $L_a(T)$ is the other connected component.

Remark 2.0.8. This coproduct is obtained by taking only connected components in the Connes-Kreimer coproduct.

To apply the rigidity theorem to some algebras, we compute the associated relation:

Proposition 2.0.9. *The preLie product and its dual coproduct satisfy the following relation:*

$$(1)\quad \Delta(T \curvearrowleft S) = \#T \times T \otimes S + (T \curvearrowleft S_1) \otimes S_2 + (T_1 \curvearrowleft S) \otimes T_2 + T_1 \otimes (T_2 \curvearrowleft S),$$

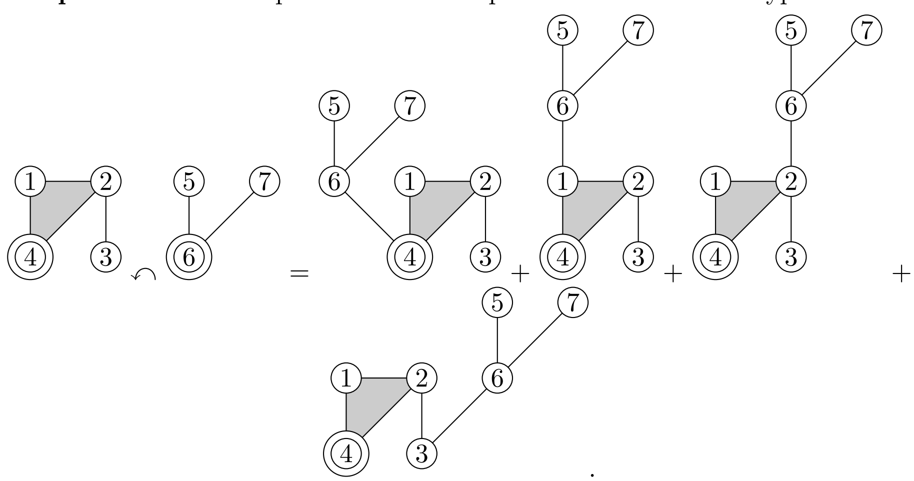
where $\Delta(T) = T_1 \otimes T_2$, $\Delta(S) = S_1 \otimes S_2$ *and* $\#T = \max\{k | \exists \delta \in \mathcal{C}(k) : \delta(T) \neq 0\}$. *Thus any conilpotent preLie-copreLie bialgebra whose product and coproduct are linked by this relation is free and cofree over its primitives.*

Example 2.0.10 (Hypertrees)**.** The bijection between decorated hypertrees and a pair given by some types of box trees and decorated sets in [15] motivated the introduction of the following product on hypertrees introduced by Berge.

Definition 2.0.11 (Berge)**.** A *hypergraph (on a set V)* is an ordered pair (V, E) where V is a finite set and E is a collection of elements of cardinality at least two, belonging to the power set $\mathcal{P}(V)$. The elements of V are called *vertices* and those of E are called *edges*. Defining a walk as a collection of adjacent edges, a *hypertree* is then a connected acyclic hypergraph.

Let us consider a rooted hypertree H, *i.e.* a hypertree with a distinguished vertex. We define a preLie product on rooted hypertrees $H \curvearrowleft G$ as the sum of all the ways to graft the root of G on a vertex v of H, where the grafting is given by adding an edge of cardinality 2 between v and r.

Example 2.0.12. We represent below the product of two rooted hypertrees:



On this algebra, we define the coproduct given by all the ways to delete an edge of cardinality two in the hypertree. This coproduct satisfies the dual relations of Equation 1. Hence the associated algebra is preLie free, with primitive elements given by the hypertrees with no binary edge, *i.e.* edge of cardinality two.

3. The dendriform case

3.1. **Rigidity theorem.**

We discuss in this section the case of coduplicial-dendriform algebras:

Definition 3.1.1. A *dendriform algebra* (see [10]) structure on A is a pair of binary products $\prec: A \otimes A \to A$ and $\succ: A \otimes A \to A$, satisfying, for any $a, b, c \in A$:

- $(a \prec b) \prec c = a \prec (b * c)$, • $(a \succ b) \prec c = a \succ (b \prec c)$,
- $(a * b) \succ c = a \succ (b \succ c)$, with $* = \prec + \succ$.

The free dendriform algebra on a vector space V is spanned by labelled binary planar trees labelled by elements of a basis of V. Consider a tree T and denote T

as the grafting of its left hand-side tree t_l and right hand-side tree t_r as follows: $T = \vee(t_l; t_r)$. The operations $\prec, \succ$ are defined on two trees T and S as:

$$T \prec S = \vee(t_l; t_r * S) \text{ and } T \succ S = \vee(T * s_l; s_r), \text{ with } \ T * \emptyset = \emptyset * T = T.$$

We recall the definition of duplicial algebras.

Definition 3.1.2. A *duplicial algebra* [9] structure on A is a pair of binary products $\triangleright : A \otimes A \to A$ and $\triangleleft : A \otimes A \to A$, satisfying that $\triangleright$ and $\triangleleft$ are associative and

$$(x \triangleright y) \triangleleft z = x \triangleright (y \triangleleft z), \text{ for any } \ x, y, z \text{ in } \ A.$$

The free duplicial structure on the planar binary rooted trees is given by:

$$T \triangleright S = \vee(T \triangleright s_l; s_r) \text{ and } T \triangleleft S = \vee(t_l; t_r \triangleleft S)$$

with $T \triangleright \emptyset = \emptyset \triangleleft T = T$.

We define dual coproducts induced by the duplicial structure for trees as follows.

Definition 3.1.3. Let $T = \vee(t_1, t_2)$ be a tree, define

$$\Delta_\triangleright(T) = \Delta_\triangleright(t_1)_1 \otimes \vee(\Delta_\triangleright(t_1)_2, t_2) \text{ and } \Delta_\triangleleft(T) = \vee(t_1, \Delta_\triangleleft(t_2)_1) \otimes \Delta_\triangleleft(t_2)_1$$

, with $\Delta_\triangleright(\vee(\emptyset, \emptyset)) = 0$ and $\Delta_\triangleleft(\vee(\emptyset, \emptyset)) = 0$, using Sweedler notation for the coproduct.

Proposition 3.1.4 (Rigidity theorem for coduplicial dendriform bialgebras)**.** *Any connected dendriform coduplicial bialgebra satisfying the following relations is free and cofree over its primitive elements:*

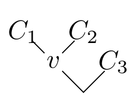
$$\begin{aligned}
&\Delta_\triangleright(T \succ S) = T \otimes S + T * (\Delta_\triangleright(S))_1 \otimes (\Delta_\triangleright(S))_2 + (\Delta_\triangleright(T))_1 \otimes (\Delta_\triangleright(T))_2 \succ S,\\
&\Delta_\triangleright(T \prec S) = (\Delta_\triangleright(T))_1 \otimes (\Delta_\triangleright(T))_2 \prec S,\\
&\Delta_\triangleleft(T \succ S) = T \succ (\Delta_\triangleleft(S))_1 \otimes (\Delta_\triangleleft(S))_2,\\
&\Delta_\triangleleft(T \prec S) = T \otimes S + T \prec (\Delta_\triangleleft(S))_1 \otimes (\Delta_\triangleleft(S))_2 + (\Delta_\triangleleft(T))_1 \otimes (\Delta_\triangleleft(T))_2 * S,
\end{aligned}$$

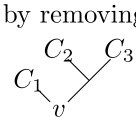
where $* = \succ + \prec$.

3.2. **Link with the Tamari lattice.** Let T, S be two planar binary trees. A right path is a path from an inner edge to the root with only right inner edges. Define the right (resp. left) path r_T (resp. l_T) to be the longest path with only inner right (resp. left) edges, and define R_T (resp. L_T) to be the number of vertices on the right (resp. left) path.

Lemma 3.2.1. *The number of terms in* $T \succ S$ *(resp.* $T \prec S$*) is* $\binom{R_T + L_S - 1}{R_T}$ *(resp.* $\binom{R_T + L_S - 1}{L_S}$*).*

Tamari endowed in 1962 the set of planar binary trees with a structure of poset given by the following covering relation: a planar binary tree T covers a planar binary tree S if there exists a vertex v in S and T such that the three connected components C_1, C_2, C_3 obtained by removing v are the same in S and T,

i.e. $S =$ 

and $T =$ 

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We denote by $\leq$ the partial order obtained by transitive closure of this covering relations: the obtained poset is a lattice called *Tamari lattice*.

The intervals in the Tamari lattice are described as follows.

Proposition 3.2.2 ([4], Prop 3.2)**.** *The dendriform and duplicial products enable the description of the following intervals:*

$$[T \triangleright S; T \triangleleft S] = \{X | X \in T \prec S\},$$

for any planar binary trees T and S.

Recalling from [6, Theorem 2.8] that the intervals in Tamari lattice are in one-to-one correspondance with interval-poset, remark that the interval-poset associated with the interval $[T \triangleright S; T \triangleleft S]$ has two connected components, corresponding respectively to T and S. We can now compute the cardinal of this type of interval, using Lemma 3.2.1.

Proposition 3.2.3. *The number of elements in the intervals described above is given by:*

$$|[T \triangleright S; T \triangleleft S]| = \binom{R_T + L_S - 1}{L_S},$$

for any planar binary trees T and S.

4. The tridendriform case

Let us recall from [11] and [14] the relations governing a tridendriform algebra:

Definition 4.0.4. A *tridendriform algebra* is a vector space A together with three operations $\prec: A \otimes A \to A$, $\cdot : A \otimes A \to A$ and $\succ: A \otimes A \to A$, satisfying the following relations:

$$\begin{array}{ll} (a \prec b) \prec c = a \prec (b \prec c + b \succ c + \ b \cdot c), & (a \succ b) \prec c = a \succ (b \prec c), \\ (a \prec b + a \succ b + \ a \cdot b) \succ c = a \succ (b \succ c), & (a \cdot b) \cdot c = a \cdot (b \cdot c), \\ (a \succ b) \cdot c = a \succ (b \cdot c), & (a \prec b) \cdot c = a \cdot (b \succ c), \\ (a \cdot b) \prec c = a \cdot (b \prec c). & \end{array}$$

Note that the operation $* := \prec + \cdot + \succ$ is associative. Let T_n be the set of all planar rooted trees (also known as Schröder trees) with $n+1$ leaves, $n \geq 1$. Given trees $t^1, \ldots, t^r$, let $\vee(t^1, \ldots, t^r)$ be the tree obtained by joining the roots of $t^1, \ldots, t^r$, ordered from left to right, to a new root. Then, any $t \in T_n$ may be written in a unique way as $t = \vee(t^1, \ldots, t^r)$, with $t^i \in T_{n_i}$ and $\sum_{i=1}^{r} n_i + r - 1 = n$.

Following [11], the free tridendriform algebra on one generator is spanned by planar rooted trees with operations $\succ$, $\cdot$ and $\prec$ recursively given by:

$$\begin{array}{rl} t \prec w & := \vee(t^1, \ldots, t^{r-1}, t^r * w), \\ t \cdot w & := \vee(t^1, \ldots, t^{r-1}, t^r * w^1, w^2, \ldots, w^l), \\ t \succ w & := \vee(t * w^1, w^2, \ldots, w^l), \end{array}$$

for $t = \vee(t^1, \ldots, t^r)$ and $w = \vee(w^1, \ldots, w^l)$.

Lemma 4.0.5. *The number of terms appearing in $T*S$ is $D(R_T, L_S)$, the Delannoy number of n, m ([16, A266213]). The number of elements in $T \prec S$ (resp. $T \cdot S$ and $T \succ S$) is $D(R_T, L_S - 1)$ (resp. $D(R_T - 1, L_S - 1)$ and $D(R_T - 1, L_S)$).*

From the tridendriform operad, we define a new operad called *terplicial*, on which the tridendriform operad is quasi-set (see [3]), by analogy with coduplicial-dendriform bialgebras.

Definition 4.0.6. A *terplicial algebra* is a vector space V endowed with three binary products $\{\blacktriangleleft, \blacktriangledown, \blacktriangleright\}$ satisfying the following relations:

$$\begin{array}{ll} \blacktriangleright \text{ and } \blacktriangledown \text{ are associative,} & (x \blacktriangleleft y) \blacktriangleleft z = x \blacktriangleleft (y \blacktriangleright z), \\ (x \blacktriangleright y) \blacktriangleleft z = x \blacktriangleright (y \blacktriangleleft z), & (x \blacktriangledown y) \blacktriangleleft z = x \blacktriangledown (y \blacktriangleleft z), \\ (x \blacktriangleright y) \blacktriangledown z = x \blacktriangleright (y \blacktriangledown z), & (x \blacktriangleleft y) \blacktriangledown z = x \blacktriangledown (y \blacktriangleright z). \end{array}$$

All the equations but the second and the last coincide with relations satisfied by triduplicial algebra defined by J.-C. Novelli and J.-Y. Thibon in [13]. The products introduced by V. Vong in [17] satisfy these relations.

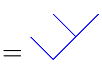
Theorem 4.0.7. *The free terplicial algebra on one generator is spanned by planar trees. Hence, the dimension of the space of operations of arity n in the terplicial operad is given by the Schröder-Hipparchus number.*

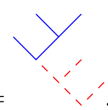
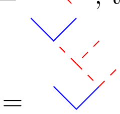
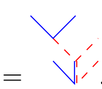
The operations $\blacktriangleleft$, $\blacktriangleright$ and $\blacktriangledown$ on a free terplicial algebra are described recursively as follows, for any tree $T = \vee(t_1, \ldots, t_n)$ and $S = \vee(s_1, \ldots, s_m)$, and denoting by $\emptyset$ the empty tree:

$$\begin{aligned} T \blacktriangleleft S &= \vee(t_1, \ldots, t_{n-1}, t_n \blacktriangleright S), \\ T \blacktriangledown S &= \vee(t_1, \ldots, t_{n-1}, t_n \blacktriangleright s_1, s_2, \ldots, s_m), \\ T \blacktriangleright S &= \vee(T \blacktriangleright s_1, s_2 \ldots, s_m), \end{aligned}$$

given that for any T, $T \blacktriangleright \emptyset = \emptyset \blacktriangleright T = T$.

Remark 4.0.8. Let us denote by RS_T (resp. LS_T) the rightmost (resp. leftmost) subtree of a planar tree T obtained by deleting the root of T. Combinatorially, $T \blacktriangleright S$ (resp. $T \blacktriangleleft S$) is given by the grafting of T on the leftmost path of S (resp. by the grafting of RS_T on the leftmost path of S, itself grafted on the rightmost edge of $T - RS_T$) and $T \blacktriangledown S$ is given by grafting RS_T on the leftmost path of LS_S and grafting it on $T - RS_T$ and $S - LS_S$ glued together, where for any tree T and subtree S of T, $T - S$ denotes the tree obtained by deleting vertices and edges of S in T.

Example 4.0.9. If $T =$  and $S =$ , the products are given by:

$T \blacktriangleright S =$ , $T \blacktriangleleft S =$  and $T \blacktriangledown S =$ .

We introduce the dual coproduct associated with the products $\blacktriangleleft$, $\blacktriangledown$ and $\blacktriangleright$. They are given inductively on $T = \vee(t_1, \ldots, t_n)$ by $\Delta_\blacktriangleright$ (resp. $\Delta_\blacktriangleleft$, resp. $\Delta_\blacktriangledown$)$(T) = 0$, if $t_1 = \emptyset$ (resp. $t_n = \emptyset$, resp. $n = 2$) and, otherwise,

$$\begin{aligned} \Delta_\blacktriangleright(T) &= t_1 \otimes \vee(\emptyset, t_2, \ldots, t_n) + \Delta_\blacktriangleright(t_1)_1 \otimes \vee(\Delta_\blacktriangleright(t_1)_2, t_2, \ldots, t_n), \\ \Delta_\blacktriangleleft(T) &= \vee(t_1, \ldots, t_{n-1}, \emptyset) \otimes t_n + \vee(t_1, \ldots, t_{n-1}, \Delta_\blacktriangleright(t_n)_1) \otimes \Delta_\blacktriangleright(t_n)_2, \\ \Delta_\blacktriangledown(T) &= \sum_{i=2}^{n-1} \vee(t_1, \ldots, t_{i-1}, \emptyset) \otimes \vee(t_i, \ldots, t_n) + \vee(t_1, \ldots, t_i) \otimes \vee(\emptyset, t_{i+1}, \ldots, t_n) \\ &\qquad + \vee(t_1, \ldots, t_{i-1}, \Delta_\blacktriangleright(t_i)_1) \otimes \vee(\Delta_\blacktriangleright(t_i)_2, t_{i+1}, \ldots, t_n), \end{aligned}$$

A combinatorial interpretation of the coproducts is the following: $\Delta_\blacktriangleright(T)$ (resp. $\Delta_\blacktriangleleft(T)$) is given by all the ways to cut edges in the leftmost path of T (resp. of RS_T or the one linking the root to RS_T) . The coproduct $\Delta_\blacktriangledown(T)$ is given by cutting edges in the leftmost path of children of the root, which are not cut by the two other coproducts.

We now use terplicial operad to obtained a new rigidity theorem for tridendriform algebras.

Proposition 4.0.10 (Rigidity theorem for coterplicial tridendriform bialgebras)**.** *Any connected tridendriform coterplicial bialgebra satisfying the following mixed ditributive laws is free and cofree over its primitive elements:*

$$\begin{aligned}
\Delta_{\blacktriangleright}(T \prec S) =&(\Delta_{\blacktriangleright}(T))_1 \otimes (\Delta_{\blacktriangleright}(T))_2 \prec S,\\
\Delta_{\blacktriangleright}(T \cdot S) =&(\Delta_{\blacktriangleright}(T))_1 \otimes (\Delta_{\blacktriangleright}(T))_2 \cdot S,\\
\Delta_{\blacktriangleright}(T \succ S) =&T \otimes S + (\Delta_{\blacktriangleright}(T))_1 \otimes (\Delta_{\blacktriangleright}(T))_2 \succ S + T * (\Delta_{\blacktriangleright}(S))_1 \otimes (\Delta_{\blacktriangleright}(S))_2,\\
\Delta_{\blacktriangledown}(T \prec S) =&(\Delta_{\blacktriangledown}(T))_1 \otimes (\Delta_{\blacktriangledown}(T))_2 \prec S,\\
\Delta_{\blacktriangledown}(T \cdot S) =&T \otimes S + (\Delta_{\blacktriangledown}(T))_1 \otimes (\Delta_{\blacktriangledown}(T))_2 \cdot S + T \cdot (\Delta_{\blacktriangledown}(S))_1 \otimes (\Delta_{\blacktriangledown}(S))_2 +\\
&(\Delta_{\blacktriangleleft}(T))_1 \otimes (\Delta_{\blacktriangleleft}(T))_2 \succ S + T \prec (\Delta_{\blacktriangleright}(S))_1 \otimes (\Delta_{\blacktriangleright}(S))_2,\\
\Delta_{\blacktriangledown}(T \succ S) =&T \succ (\Delta_{\blacktriangledown}(S))_1 \otimes (\Delta_{\blacktriangledown}(S))_2,\\
\Delta_{\blacktriangleleft}(T \prec S) =&T \otimes S + (\Delta_{\blacktriangleleft}(T))_1 \otimes (\Delta_{\blacktriangleleft}(T))_2 * S + T \prec (\Delta_{\blacktriangleright}(S))_1 \otimes (\Delta_{\blacktriangleright}(S))_2\\
\Delta_{\blacktriangleleft}(T \cdot S) =&T \cdot (\Delta_{\blacktriangleleft}(S))_1 \otimes (\Delta_{\blacktriangleleft}(S))_2\\
\Delta_{\blacktriangleleft}(T \succ S) =&T \succ (\Delta_{\blacktriangleleft}(S))_1 \otimes (\Delta_{\blacktriangleleft}(S))_2
\end{aligned}$$

where $* = \succ + \prec + \cdot$.

5. Application: freeness of WQSym algebra

Let us consider the space of surjections: $\mathbf{WQSym}_n^r := \{x : [n] \to [r], x \text{ surjective }\}$ and let $\mathbf{WQSym} = \oplus_{n \geq r \geq 1} \mathbb{K}[\mathbf{WQSym}_n^r]$. For $x \in \mathbf{WQSym}_n^r$, we write $x = x(1) \dots x(n)$. This vector space can be endowed with a dendriform structure $(\prec, \succ)$, see for example [14], as follows: let $x \in \mathbf{WQSym}_n^r$, $y \in \mathbf{WQSym}_m^s$ define

$$x \succ y := \sum_{\substack{a \times b \in \mathbf{WQSym} \\ \max(a) < \max(b) \\ \mathrm{std}(a) = x, \mathrm{std}(b) = y}} a \times b \qquad \text{and} \qquad x \prec y := \sum_{\substack{a \times b \in \mathbf{WQSym} \\ \max(a) \geq \max(b) \\ \mathrm{std}(a) = x, \mathrm{std}(b) = y}} a \times b,$$

where $\times$ is the concatenation of words and std is the identity on words belonging to $\mathbf{WQSym}$ and if a word is not surjective (*i.e.* its image is in $\{1, \dots, j-1\} \cup \{j+1, k\}$ with a gap in j), its standardisation is recursively defined as the standardisation of the word obtained by decreasing by one all letters bigger than a gap.

Let $x \in \mathbf{WQSym}_n^r$, and suppose that $x^{-1}(r) = \{j_1 < \dots < j_{\lambda(x)}\}$. Recursively define the left-to-right maxima as $LR(x) := (LR(x_{j_1}), j_1))$, with $x_{j_1} = x(1) \dots x(j_1 - 1)$ and $LR(\emptyset) = \emptyset$. Similarly define the right-to-left maxima $RL(x) := \big(j_{\lambda(x)}, RL(x^{j_{\lambda(x)}})\big)$, with $x^{j_{\lambda(x)}} = x(j_{\lambda(x)} + 1) \dots x(n)$ and $RL(\emptyset) = \emptyset$.

Example 5.0.11. For $x = 2\ 4\ 1\ 5\ 5\ 3\ 6\ 7\ 3\ 1\ 2\ 5$ the left to right maxima are in bold red: **2** **4** 1 **5** 5 3 **6** **7** 3 1 2 5 and $LR(x) = (1, 2, 4, 7, 8)$. The right to left maxima are in red: 2 4 1 5 5 3 6 **7** 3 1 2 **5** and $RL(x) = (8, 12)$.

We can now define a coduplicial structure on $\mathbf{WQSym}$. For $x \in \mathbf{WQSym}_n^r$, denote $LR(x) = (l_1, \dots, l_x)$ and $RL(x) = (r_1, \dots, r_y)$:

$$\Delta_{\triangleright}(x) = \sum_{k=1}^{x} x(1) \dots x(l_k - 1) \otimes \mathrm{std}(x(l_k) \dots x(n)),$$

$$\Delta_{\triangleleft}(x) = \sum_{k=1}^{y} \mathrm{std}(x(1) \dots x(r_k)) \otimes x(r_k + 1) \dots x(n).$$

Proposition 5.0.12. **WQSym** *endowed with the coproducts* $\Delta_{\triangleright}, \Delta_{\triangleleft}$ *and the dendriform structure* $(\prec, \succ)$ *is a* Dup^c*-Dend bialgebra.*

Hence, we can reprove the known result (see [7, 2, 14, 17]) using the coduplicial-dendriform structure on **WQSym**:

Proposition 5.0.13. **WQSym** *is free as a dendriform algebra on its primitives.*

Remark 5.0.14. The primitives are not the same as in [2], since in dimension 3 the primitives are $(1,3,2)$ and $(2,3,1)$, whereas in [2] it is $(1,2,1)$ and $(2,3,1)$. The number of primitives elements for each dimension is given by the Fubini numbers [16, A00670].

The algebra **WQSym** can be endowed with a tridendriform structure, as follows. Let $x \in \mathbf{WQSym}_n^r, y \in \mathbf{WQSym}_m^s$ and define:

$$x \succ y := \sum_{\substack{a\times b \in \mathbf{WQSym} \\ \max(a)<\max(b) \\ \mathrm{std}(a)=x, \mathrm{std}(b)=y}} a\times b,\ x \prec y := \sum_{\substack{a\times b \in \mathbf{WQSym} \\ \max(a)>\max(b) \\ \mathrm{std}(a)=x, \mathrm{std}(b)=y}} a\times b \text{ and } x\cdot y := \sum_{\substack{a\times b \in \mathbf{WQSym} \\ \max(a)=\max(b) \\ \mathrm{std}(a)=x, \mathrm{std}(b)=y}} a\times b.$$

Suppose that $x^{-1}(r) = \{j_1 < \cdots < j_r\}$ and let $x' \in \mathbf{WQSym}_{n-k}^{r-1}$ be the co-restriction $x' := x|^{\{1,\dots,r-1\}}$. We denote x as $x = \prod_{j_1<\dots<j_{\lambda(x)}} x'$. The work of Vong [17] gives a construction of a terplicial algebraic structure on $\mathbf{WQSym}_m^s$. For $x \in \mathbf{WQSym}_n^r, y \in \mathbf{WQSym}_m^s$ define the operations

$$x \blacktriangleleft y = \prod_{j_1^x,\dots,j_{\lambda(x)}^x} x' \times y,\ x \blacktriangledown y = \prod_{j_1^x,\dots,j_{\lambda(x)}^x, j_1^y,\dots,j_{\lambda(x)}^y} x' \times y' \text{ and } x \blacktriangleright y = x \times y,$$

where $x = \prod_{j_1^x,\dots,j_{\lambda(x)}^x} x'$ and $y = \prod_{j_1^y,\dots,j_{\lambda(x)}^y} y'$. The relations are checked by direct inspection.

The coterplicial structure on **WQSym** is combinatorially constructed as follows: for $x \in \mathbf{WQSym}_n^r$ there is a unique way to describe it as $x_1 \times \ldots \times x_p$ such that every x_i is irreducible that is to say that there does not exist $u, v \in \mathbf{WQSym}$ such that $x_i = u \times v$. Suppose $x = \prod_{j_1,\dots,j_{\lambda(x)}} x' = \prod_{j_1,\dots,j_{\lambda(x)}} u_1 \times \ldots \times u_q$ where $u_1 \times \ldots \times u_q$ is the irreducible decomposition of x', $u_i \in \mathbf{WQSym}_{m_i}^{s_i}$. Denote by $U_1 = u_1 \times \ldots \times u_{p_1}$ the decomposition $x = \prod_{j_1,\dots,j_{\lambda(x)}} U_1 \times u_{s+1} \times \ldots \times u_q$ with $\sum_{i=1} p_1 - 1 m_i + \lambda(x) < j_{\lambda(x)} \le \sum_{i=1} p_1 m_i + \lambda(x)$,

$$\Delta_{\blacktriangleleft}(x) = \sum_i \prod_{j_1,\dots,j_{\lambda(x)}} \big(U_1 \times u_{p_1+1} \times \ldots \times u_{p_1+i}\big) \otimes \mathrm{std}(u_{p_1+i+1} \times \ldots \times u_q),$$

$$\Delta_{\blacktriangleright}(x) = \sum_i x_1 \times \ldots \times x_i \otimes x_{i+1} \times \ldots \times x_p,$$

$$\Delta_{\blacktriangledown}(x) = \sum_{i,l} \prod_{j_1,\dots,j_i} \big(u_1 \times \ldots \times u_l\big) \otimes \prod_{j_{i+1},\dots,j_{\lambda(x)}} \big(u_{l+1} \times \ldots \times u_q\big)$$

where the last sum runs over i, l such that $m_1 + \ldots + m_{l-1} < j_i \le m_1 + \ldots + m_l$.

These cooperations enable us to prove the following proposition:

Proposition 5.0.15 ([2])**.** *The algebra* **WQSym** *is endowed with a terplicial*^{c}*-tridendriform bialgebra structure. Hence, the algebra* **WQSym** *is free as a terplicial algebra and free as a tridendriform algebra.*

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