From the motion group of the trivial link to the homology of the hypertree poset

Bérénice Oger

Institut Camille Jordan (Lyon)

Friday January, 10th 2014
Seminar of the ANR HOGT project

From the motion group of ~~the~~ a n-component trivial link to the homology of ~~the~~ a hypertree poset on n vertices

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Motivation : $P\Sigma_n$

- F_n generated by $(x_i)_{i=1}^n$
- $P\Sigma_n$, pure symmetric automorphism group
  - group of automorphisms of F_n which send each x_i to a conjugate of itself,
  - group of motions of a collection of n coloured unknotted, unlinked circles in 3-space.
- It seems that their cohomology groups are not Koszul (A. Conner and P. Goetz).

- Use of the hypertree poset for the computation of the l^2-Betti numbers of $P\Sigma_n$ by C. Jensen, J. McCammond and J. Meier.
- Action of $P\Sigma_n$ on a contractible complex MM_n defined by McCullough and Miller in 1996 in terms of marking of hypertrees, whose fundamental domain is the hypertree poset on n vertices,
- $P\Sigma_n \triangleright Inn(F_n) => O P\Sigma_n = P\Sigma_n / Inn(F_n)$
- $OP\Sigma_n$ acts cocompactly on MM_n
- Use of a theorem by Davis, Januszkiewicz and Leary to obtain the expression of l^2-cohomology of the group in term of the cohomology of the fundamental domain of the complex.

Summary

1. The hypertree poset
   - Hypertrees
   - Hypertree poset
   - Homology of the hypertree poset
2. Computation of the homology of the hypertree poset
   - Species
   - Counting strict chains using large chains
   - Pointed hypertrees
   - Relations between chains of hypertrees
   - Dimension of the homology
3. From the hypertree poset to rooted trees
   - PreLie species
   - Character for the action of the symmetric group on the homology of the poset

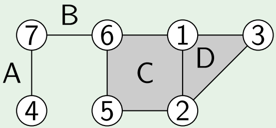
Hypergraphs and hypertrees

Definition ([Ber89])

A *hypergraph* *(on a set* V*) is an ordered pair* (V, E) *where:*

- V *is a finite set (vertices)*
- E *is a collection of subsets of cardinality at least two of elements of* V *(edges).*

Example of a hypergraph on $[1;7]$



Walk on a hypergraph

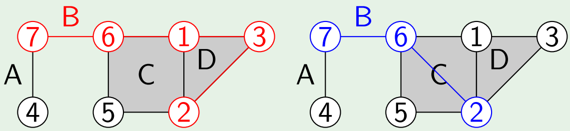
Definition

Let $H = (V, E)$ be a hypergraph.
A walk *from a vertex or an edge d to a vertex or an edge f in H is an alternating sequence of vertices and edges beginning by d and ending by f:*

$$(d, \dots, e_i, v_i, e_{i+1}, \dots, f)$$

where for all i, $v_i \in V$, $e_i \in E$ and $\{v_i, v_{i+1}\} \subseteq e_i$.
The length *of a walk is the number of edges and vertices in the walk.*

Examples of walks



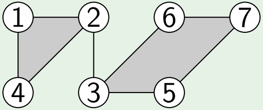
Hypertrees

Definition

A hypertree is a non-empty hypergraph H such that, given any distinct vertices v and w in H,

- *there exists a walk from v to w in H with distinct edges e_i, (H is connected),*
- *and this walk is unique, (H has no cycles).*

Example of a hypertree



The hypertree poset

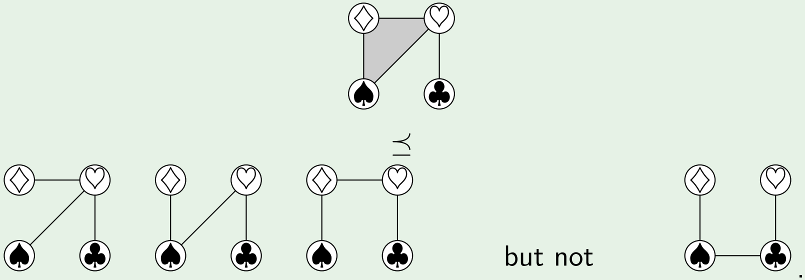
Definition

Let I be a finite set of cardinality n, S and T be two hypertrees on I.

$$S \preceq T \iff \text{Each edge of } S \text{ is the union of edges of } T$$

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.

Example with hypertrees on four vertices



- Graded poset by the number of edges [McCullough and Miller 1996],
- There is a unique minimum $\hat{0}$,
- HT(I) $=$ hypertree poset on I,
- $\mathrm{HT_n}$ $=$ hypertree poset on n vertices.
- Möbius number : $(n-1)^{n-2}$ [McCammond and Meier 2004]

Goal:

- New computation of the homology dimension
- Computation of the action of the symmetric group on the homology (Conjecture of Chapoton 2007)

Homology of the poset

To every poset P, one can associate a simplicial complex (nerve of the poset seen as a category) whose

- vertices are elements of P,
- faces are the chains of P.

Definition

A strict k-chain of hypertrees on I is a k-tuple $(a_1, \ldots, a_k)$, where a_i are hypertrees on I different from the minimum $\hat{0}$ and $a_i \prec a_{i+1}$.

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Let C_k be the vector space generated by strict k-chains. We define $C_{-1} = \mathbb{C}.e$. We define the following linear map on the set $(C_k)_{k \geq -1}$:

$$\partial_k(a_1 \prec \ldots \prec a_{k+1}) = \sum_{i=1}^{k} (-1)^i (a_1 \prec \ldots \prec \hat{a}_i \prec \ldots \prec a_k),$$

$(a_1 \prec \ldots \prec a_{k+1}) \in C_k$.

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$(a_1 \prec \dots \prec a_{k+1}) \in C_k$.

The homology is defined by:

$$\tilde{H}_j = ker\partial_j / im\partial_{j+1}.$$

Theorem ([MM04])

The homology of $\widehat{\mathrm{HT}_\mathrm{n}}$ *is concentrated in maximal degree* $(n-3)$.

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Corollary

The character for the action of the symmetric group on $\tilde{H}_{n-3}$ is given in terms of characters for the action of the symmetric group on C_k by:

$$\chi_{\tilde{H}_{n-3}} = (-1)^{n-3} \sum_{k=-1}^{n-3} (-1)^k \chi_{C_k}, \text{ where } n = \#I.$$

What are species?

Definition

A species F *is a functor from the category of finite sets and bijections to itself. To a finite set* I*, the species* F *associates a finite set* $\mathrm{F}(I)$ *independent from the nature of* I*.*

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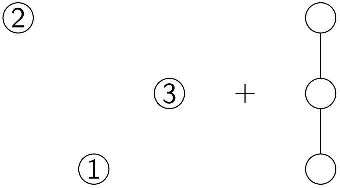

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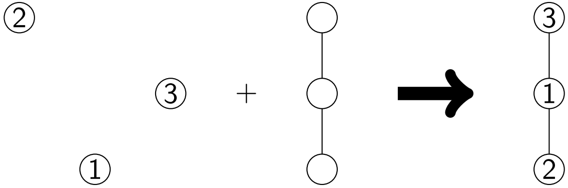

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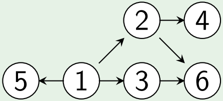
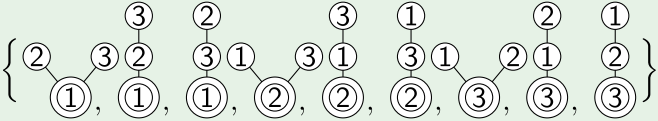
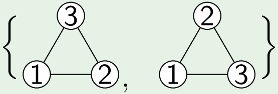
Counterexamples

The following sets are not obtained using species:

- $\{(1,\mathbf{3},2),(2,1,\mathbf{3}),(2,\mathbf{3},1)(3,1,\mathbf{2})\}$(set of permutations of $\{1,2,3\}$ with exactly 1 descent)
- (graph of divisibility of $\{1,2,3,4,5,6\}$)

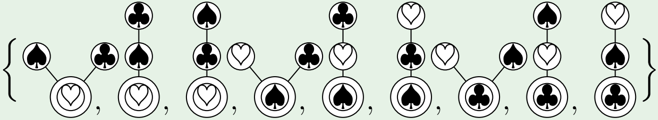
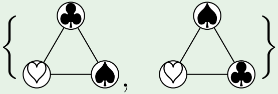


Examples of species

- $\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ (Species of lists Assoc on $\{1,2,3\}$)
- $\{\{1,2,3\}\}$ (Species of non-empty sets Comm)
- $\{\{1\},\{2\},\{3\}\}$ (Species of pointed sets Perm)
-  (Species of rooted trees PreLie)
-  (Species of cycles)

These sets are the image by species of the set $\{1,2,3\}$.

Examples of species

- $\{(\heartsuit,\spadesuit,\clubsuit),(\heartsuit,\clubsuit,\spadesuit),(\spadesuit,\heartsuit,\clubsuit),(\spadesuit,\clubsuit,\heartsuit),(\clubsuit,\heartsuit,\spadesuit),(\clubsuit,\spadesuit,\heartsuit)\}$ (Species of lists Assoc on $\{\clubsuit,\heartsuit,\spadesuit\}$)
- $\{\{\heartsuit,\spadesuit,\clubsuit\}\}$ (Species of non-empty sets Comm)
- $\{\{\heartsuit\},\{\spadesuit\},\{\clubsuit\}\}$ (Species of pointed sets Perm)
-  (Species of rooted trees PreLie)
-  (Species of cycles)

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Operations on species and associated series

Proposition

Let F and G be two species. The following operations can be defined on them:

- $F'(I) = F(I \sqcup \{\bullet\})$, (derivative)

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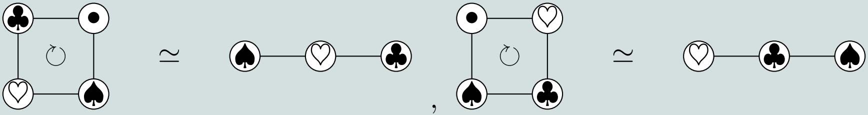
Example: Derivative of the species of cycles on $I = \{\heartsuit, \spadesuit, \clubsuit\}$



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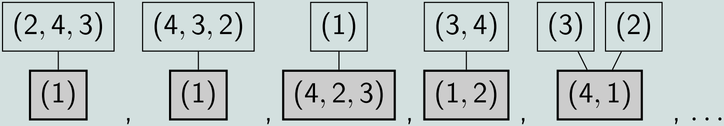
Example of substitution: Rooted trees of lists on $I = \{1, 2, 3, 4\}$



Definition

To a species F, we associate its generating series:

$$C_F(x) = \sum_{n\geq 0} \#F(\{1,\ldots,n\})\frac{x^n}{n!}.$$

Examples of generating series:

- The generating series of the species of lists is $C_{\text{Assoc}} = \frac{1}{1-x}$.
- The generating series of the species of non-empty sets is $C_{\text{Comm}} = \exp(x) - 1$.
- The generating series of the species of pointed sets is $C_{\text{Perm}} = x \cdot \exp(x)$.
- The generating series of the species of rooted trees is $C_{\text{PreLie}} = \sum_{n\geq 0} n^{n-1}\frac{x^n}{n!}$.
- The generating series of the species of cycles is $C_{Cycles} = -\ln(1-x)$.

Definition

The cycle index series of a species F is the formal power series in an infinite number of variables $\mathfrak{p} = (p_1, p_2, p_3, \ldots)$ defined by:

$$Z_F(\mathfrak{p}) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} F^\sigma p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3} \ldots \right),$$

- *with F^σ, is the set of F-structures fixed under the action of σ,*
- *and σ_i, the number of cycles of length i in the decomposition of σ into disjoint cycles.*

Examples

- The cycle index series of the species of lists is $Z_{\text{Assoc}} = \frac{1}{1-p_1}$.
- The cycle index series of the species of non empty sets is $Z_{\text{Comm}} = \exp(\sum_{k \geq 1} \frac{p_k}{k}) - 1$.

Operations on cycle index series

Operations on species give operations on their cycle index series:

Proposition

Let F and G be two species. Their cycle index series satisfy:

$$\begin{aligned} Z_{F+G} &= Z_F + Z_G, & Z_{F\times G} &= Z_F \times Z_G, \\ Z_{F\circ G} &= Z_F \circ Z_G, & Z_{F'} &= \frac{\partial Z_F}{\partial p_1}. \end{aligned}$$

Definition

The suspension Σ of a cycle index series $f(p_1, p_2, p_3, \ldots)$ is defined by:

$$\Sigma f = -f(-p_1, -p_2, -p_3, \ldots).$$

Counting strict chains using large chains

Let I be a finite set of cardinality n.

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Let $M_{k,s}$ be the set of words on $\{0, 1\}$ of length k, with s letters "1". The species $\mathcal{M}_{k,s}$ is defined by:

$$\begin{cases} \emptyset & \mapsto & M_{k,s}, \\ V \neq \emptyset & \mapsto & \emptyset. \end{cases}$$

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Proposition

The species $\mathcal{H}_k$ of large k-chains and $\mathcal{HS}_i$ of strict i-chains are related by:

$$\mathcal{H}_k \cong \sum_{i \geq 0} \mathcal{HS}_i \times \mathcal{M}_{k,i}.$$

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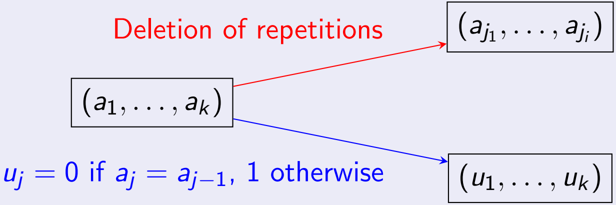
Proof.



with $a_0 = \hat{0}$. □

The previous proposition gives, for all integer $k > 0$:

$$\chi_k = \sum_{i=0}^{n-2} \binom{k}{i} \chi_i^s.$$

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χ_k is a polynomial $P(k)$ in k which gives, once evaluated in -1, the character:

Corollary

$$\chi_{\tilde{H}_{n-3}} = (-1)^n P(-1) =: (-1)^n \chi_{-1}$$

The hypertrees will now be on n vertices.

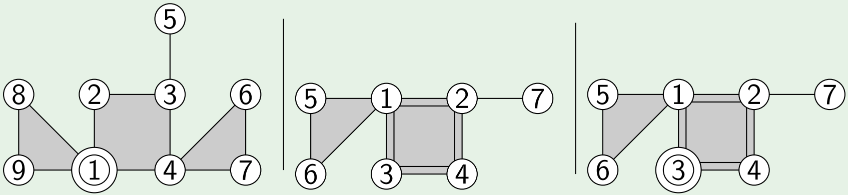
Pointed hypertrees

Definition

Let H be a hypertree on I. H is:

- *rooted in a vertex s if the vertex s of H is distinguished,*
- *edge-pointed in an edge a if the edge a of H is distinguished,*
- *rooted edge-pointed in a vertex s in an edge a if the edge a of H and a vertex s of a are distinguished.*

Example of pointed hypertrees



Proposition: Dissymmetry principle

The species of hypertrees and of rooted hypertrees are related by:

$$\mathcal{H} + \mathcal{H}^{pa} = \mathcal{H}^{p} + \mathcal{H}^{a}.$$

We write:

- $\mathcal{H}_k$, the species of large k-chains of hypertrees,
- $\mathcal{H}_k^{pa}$, the species of large k-chains of hypertrees whose minimum is rooted edge-pointed,
- $\mathcal{H}_k^{p}$, the species of large k-chains of hypertrees whose minimum is rooted,
- $\mathcal{H}_k^{a}$, the species of large k-chains of hypertrees whose minimum is edge-pointed.

Corollary ([Oge13])

The species of large k-chains of hypertrees are related by:

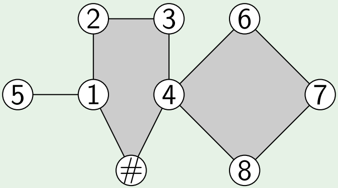
$$\mathcal{H}_k + \mathcal{H}_k^{pa} = \mathcal{H}_k^{p} + \mathcal{H}_k^{a}.$$

Last but not least type of hypertrees

Definition

A hollow hypertree *on n vertices ($n \geq 2$) is a hypertree on the set $\{\#, 1, \ldots, n\}$, such that the vertex labelled by $\#$, called the gap, belongs to one and only one edge.*

Example of a hollow hypertree



We denote by $\mathcal{H}_k^c$, the species of large k-chains of hypertrees whose minimum is a hollow hypertree.

Relations between species of hypertrees

Theorem

The species $\mathcal{H}_k$, $\mathcal{H}_k^p$ and $\mathcal{H}_k^c$ satisfy:

$$\mathcal{H}_k^p = X \times \mathcal{H}_k' \tag{1}$$

$$\mathcal{H}_k^p = X \times \mathrm{Comm} \circ \mathcal{H}_k^c + X, \tag{2}$$

$$\mathcal{H}_k^c = \mathrm{Comm} \circ \mathcal{H}_{k-1}^c \circ \mathcal{H}_k^p, \tag{3}$$

$$\mathcal{H}_k^a = (\mathcal{H}_{k-1} - x) \circ \mathcal{H}_k^p, \tag{4}$$

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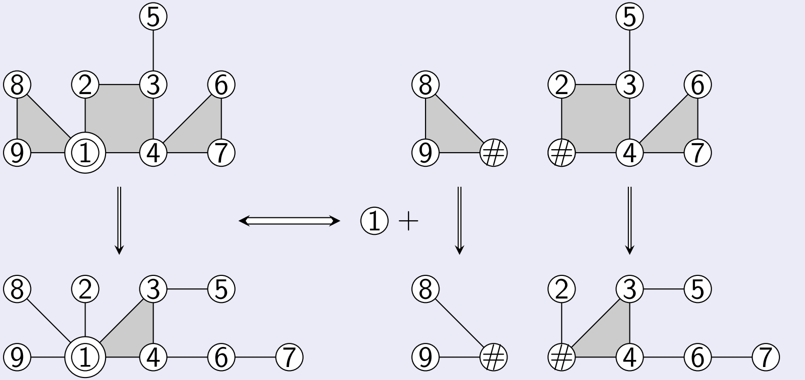
Proof.

1. Rooting a species F is the same as multiplying the singleton species X by the derivative of F,

Second part of the proof.

We separate the root and every edge containing it, putting gaps where the root was,

$$\mathcal{H}_k^p = X \times \mathrm{Comm} \circ \mathcal{H}_k^c + X,$$

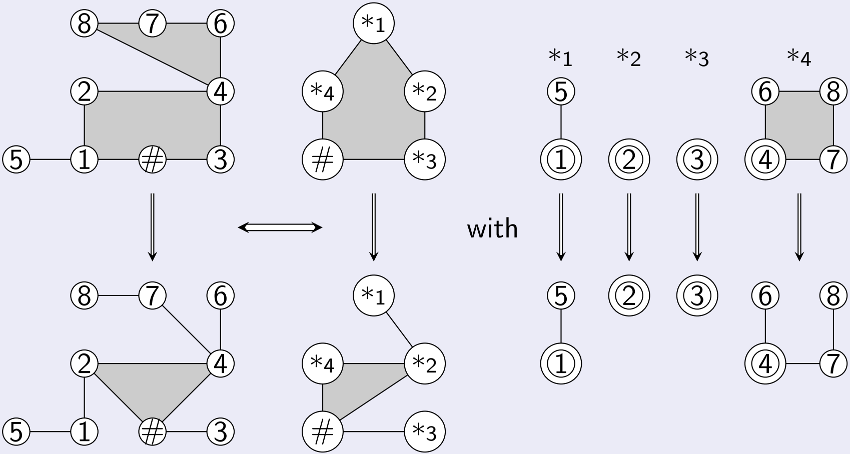


and End!

3. Hollow case:

$$\mathcal{H}_k^c = \mathcal{H}_k^{cm} \circ \mathcal{H}_k^p, \qquad (6)$$

$$\mathcal{H}_k^{cm} = \mathrm{Comm} \circ \mathcal{H}_{k-1}^c. \qquad (7)$$



Dimension of the homology

Proposition

The generating series of the species $\mathcal{H}_k$, $\mathcal{H}_k^p$ and $\mathcal{H}_k^c$ satisfy:

$$\mathcal{C}_k^p = x \cdot \exp\left(\frac{\mathcal{C}_{k-1}^p \circ \mathcal{C}_k^p}{\mathcal{C}_k^p} - 1\right), \qquad (8)$$

$$\mathcal{C}_k^a = (\mathcal{C}_{k-1} - x)(\mathcal{C}_k^p), \qquad (9)$$

$$\mathcal{C}_k^{pa} = (\mathcal{C}_{k-1}^p - x)(\mathcal{C}_k^p), \qquad (10)$$

$$x \cdot \mathcal{C}_k' = \mathcal{C}_k^p, \qquad (11)$$

$$\mathcal{C}_k + \mathcal{C}_k^{pa} = \mathcal{C}_k^p + \mathcal{C}_k^a. \qquad (12)$$

Lemma

The generating series of $\mathcal{H}_0$ and $\mathcal{H}_0^p$ are given by:

$$\mathcal{C}_0 = \sum_{n\geq 1} \frac{x^n}{n!} = e^x - 1,$$

$$\mathcal{C}_0^p = xe^x.$$

Lemma

The generating series of $\mathcal{H}_0$ and $\mathcal{H}_0^p$ are given by:

$$\mathcal{C}_0 = \sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1,$$

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This implies with the previous theorem:

Theorem ([MM04])

The dimension of the only homology group of the hypertree poset is $(n-1)^{n-2}$.

This dimension is the dimension of the vector space PreLie(n-1) whose basis is the set of rooted trees on $n-1$ vertices.

From the hypertree poset to rooted trees

1. This dimension is the dimension of the vector space PreLie(n-1) whose basis is the set of rooted trees on $n-1$ vertices.
   The operad (a species with more properties on substitution) whose vector space are PreLie(n) is PreLie.
2. This operad is anticyclic ([Cha05]): There is an action of the symmetric group $\mathfrak{S}_n$ on PreLie$(n-1)$ which extends the one of $\mathfrak{S}_{n-1}$.
3. Moreover, there is an action of $\mathfrak{S}_n$ on the homology of the poset of hypertrees on n vertices.

Question

Is the action of $\mathfrak{S}_n$ on PreLie(n-1) the same as the action on the homology of the poset of hypertrees on n vertices?

Character for the action of the symmetric group on the homology of the poset

Using relations on species established previously, we obtain:

Proposition

The series Z_k, Z_k^p, Z_k^a and Z_k^{pa} satisfy the following relations:

$$Z_k + Z_k^{pa} = Z_k^p + Z_k^a, \tag{13}$$

$$Z_k^p = p_1 + p_1 \times \mathrm{Comm} \circ \left(\frac{Z_{k-1}^p \circ Z_k^p - Z_k^p}{Z_k^p} \right), \tag{14}$$

$$Z_k^a + Z_k^p = Z_{k-1} \circ Z_k^p, \tag{15}$$

$$Z_k^{pa} + Z_k^p = Z_{k-1}^p \circ Z_k^p, \text{ and } p_1 \frac{\partial Z_k}{\partial p_1} = Z_k^p. \tag{16}$$

Theorem ([Oge13], conjecture of [Cha07])

The cycle index series Z_{-1}, which gives the character for the action of $\mathfrak{S}_n$ on $\tilde{H}_{n-3}$, is linked with the cycle index series M associated with the anticyclic structure of PreLie *by:*

$$Z_{-1} = p_1 - \Sigma M = \mathrm{Comm} \circ \Sigma\, \mathrm{PreLie} + p_1 \left(\Sigma\, \mathrm{PreLie} + 1\right). \tag{17}$$

The cycle index series Z^p_{-1} is given by:

$$Z^p_{-1} = p_1 \left(\Sigma\, \mathrm{PreLie} + 1\right). \tag{18}$$

Theorem ([Oge13], conjecture of [Cha07])

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Proof.

Sketch of the proof

1. Computation of $Z_0 = \mathrm{Comm}$ and $Z_0^p = \mathrm{Perm} = p_1 + p_1 \times \mathrm{Comm}$
2. Replaced in the formula giving Z_0^p in terms of itself and Z^p_{-1}

$$Z_0^p = p_1 + p_1 \times \mathrm{Comm} \circ \left(\frac{Z^p_{-1} \circ Z_0^p - Z_0^p}{Z_0^p} \right),$$

Second part of the proof.

3. As $\Sigma\,\mathrm{PreLie} \circ \mathrm{Perm} = \mathrm{Perm} \circ \Sigma\,\mathrm{PreLie} = p_1$, according to [Cha07], we get:

$$Z^{p}_{-1} = p_1 \left(\Sigma\,\mathrm{PreLie} + 1\right).$$

4. The dissymetry principle associated with the expressions gives:

$$\mathrm{Comm} + Z^{p}_{-1} \circ \mathrm{Perm} - \mathrm{Perm} = \mathrm{Perm} + Z_{-1} \circ \mathrm{Perm} - \mathrm{Perm}.$$

5. Thanks to equation [Cha05, equation 50], we conclude:

$$\Sigma M - 1 = -p_1\left(-1 + \Sigma\,\mathrm{PreLie} + \frac{1}{\Sigma\,\mathrm{PreLie}}\right).$$

□

Thank you for your attention !

[Oge13] Bérénice Oger Action of the symmetric groups on the homology of the hypertree posets. Journal of Algebraic Combinatorics, february 2013.

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Eccentricity

Definition

The eccentricity of a vertex or an edge is the maximal number of vertices on a walk without repetition to another vertex.
The center of a hypertree is the vertex or the edge with minimal eccentricity.

Example of eccentricity