Corrigé de l'exam. finale MathIV-analyse

Note Title 6/15/2010

Exo I

$$\int_0^1 \left(\int_0^{\sqrt{x}} \frac{2xy}{1-y^4}\, dy \right) dx = \int_0^1 \left(\int_{y^2}^1 \frac{2xy}{1-y^4}\, dx \right) dy$$

$$0 \le y \le \sqrt{x}, \quad 0 \le x \le 1$$

$$\Longleftrightarrow \quad 0 \le y \le 1, \quad y^2 \le x \le 1,$$

$$= \int_0^1 \frac{y}{1-y^4} \cdot \left[x^2 \right]_{y^2}^{1} dy = \int_0^1 \frac{y - y^5}{1-y^4}\, dy$$

$$= \int_0^1 y\, dy = \frac{1}{2}$$

Exo II

$$x = y^{3/2}, \quad 0 \le y \le 1.$$

$$\begin{cases} x = t^{3/2} \\ y = t \end{cases}, \quad t \in [0,1] \qquad \begin{cases} x' = \frac{3}{2} t^{1/2} \\ y' = 1 \end{cases}$$

$$|\gamma'(t)| = \sqrt{1 + \frac{9}{4} t}$$

$$\mathcal{L}(\gamma) = \int_0^1 \sqrt{1 + \frac{9}{4} t}\, dt = \left[\frac{4}{9} \cdot \frac{2}{3} \cdot \left(1 + \frac{9}{4} t\right)^{3/2} \right]_0^1$$

$$= \frac{2}{3} \cdot \left[\frac{13}{4} \right]^{3/2} - \frac{2}{3} = \frac{2}{3} \left(\frac{13\sqrt{13}}{8} - 1 \right)$$

Exo III

$$(x,y) \longmapsto (u,v)$$

$$(x,y) \longmapsto (8x^3 + y,\ 2x)$$

$$J(g) = \begin{pmatrix} 24x^2 & 1 \\ 2 & 0 \end{pmatrix}$$

$(u,v) \longmapsto (w,z)$

$(u,v) \longmapsto (2u - v^3, 3v)$ $\qquad J(h) = \begin{pmatrix} 2 & -3v^2 \\ 0 & 3 \end{pmatrix}$

$v = 2x \Rightarrow$

$$J(h\circ g) = J(h)\circ J(g) = \begin{pmatrix} 2 & -12x^2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 24x^2 & 1 \\ 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 48x^2 - 24x^2 & 2 \\ 6 & 0 \end{pmatrix} = \begin{pmatrix} 24x^3 & 2 \\ 6 & 0 \end{pmatrix}$$

Ou bien $\quad h\circ g(x,y) = (2\cdot 8x^3 + 2y - (2x)^3, 6x)$

$= 8x^3 + 2y, 6x$

$$J(h\circ g) = \begin{pmatrix} 24x^2 & 2 \\ 6 & 0 \end{pmatrix}$$

Exo IV

1. $$\iint_{D_n} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^n e^{-r^2} r\,dr\,d\theta$$

$dx\,dy = r\,dr\,d\theta$

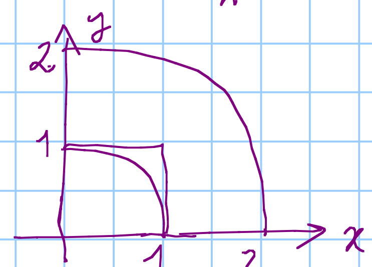
$r^2 = x^2 + y^2$

$$= \left[-\frac{e^{-r^2}}{2}\right]_0^n \cdot \pi/2 = \frac{\pi}{4}\left(1 - e^{-n^2}\right)$$

2. $K_n = \int_0^n e^{-x^2}dx \int_0^n e^{-y^2}dy$ par Fubini

$= I_n^2$

3.



4. Les integrales de fonction positive :

$$D_n \subset Q_n \subset D_{2n} \Rightarrow J_n \leq K_n \leq J_{2n}$$

$\underset{\text{positivité}}{\Rightarrow}$ $\sqrt{J_n} \leq I_n \leq \sqrt{J_{2n}}$

5. Finalement $J_n \underset{n \to +\infty}{\longrightarrow} \pi/4$, J_{2n} aussi donc par le lemme de gend'armes K_n aussi. Donc $I_n = \frac{\sqrt{\pi}}{2}$

Exo V

1. $(\cos t)^2 + \left(2 \cdot \frac{1}{2} \sin t\right)^2 = 1$

$\Rightarrow x^2 + 4y^2 = 1$

$x^2 + 4y^2 = 1$ – c'est une ellipse

2. Tangente : $\begin{pmatrix} \cos t \\ \frac{1}{2}\sin t \end{pmatrix}' = \begin{pmatrix} -\sin t \\ \frac{1}{2}\cos t \end{pmatrix} = \begin{pmatrix} -2y \\ \frac{x}{2} \end{pmatrix}$

Au pt. $t = \pi/4$ $\cos t = \sin t = \frac{\sqrt{2}}{2}$

$x_0 = \frac{\sqrt{2}}{2}$, $y_0 = \frac{\sqrt{2}}{4}$

Donc la droite avec vecteur-directeur $\begin{pmatrix} -2 \cdot \frac{\sqrt{2}}{4} \\ \sqrt{2}/2 \cdot \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \sqrt{2}/4 \end{pmatrix}$ passant par le pt. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}\right)$ a pour l'équation

$$\left(x - \frac{\sqrt{2}}{2}\right) \Big/ \left(-\frac{\sqrt{2}}{2}\right) = \left(y - \sqrt{2}/4\right) \Big/ \frac{\sqrt{2}}{4}$$

$$y = -\frac{1}{2}x + \frac{\sqrt{2}}{4}$$

3. La distance au carré de M_0 au pt $M(x,y)$
$d(M_0,M)^2 = x^2 + (y-1)^2$

Contrainte : $(x,y) \in \mathcal{E}$ i.e. avec multipl. de Lagrange on a

$$\overrightarrow{\text{grad}}\left(x^2+(y-1)^2+\lambda(x^2+4y^2-1)\right)=0$$

$$\begin{cases} \frac{\partial}{\partial x}\left(x^2+(y-1)^2+\lambda(x^2+4y^2-1)\right)=0 \\ \frac{\partial}{\partial y}\left(x^2+(y-1)^2+\lambda(x^2+4y^2-1)\right)=0 \\ \frac{\partial}{\partial \lambda}\left(\text{—} \quad \text{''} \quad \text{—}\right)=0 \end{cases}$$

$$\begin{cases} 2x+2\lambda x=0 \\ 2(y-1)+8\lambda y=0 \\ x^2+4y^2-1=0 \end{cases} \iff \begin{cases} (\lambda+1)x=0 & x=0 \text{ ou } \lambda=-1 \\ (4\lambda+1)y=1 \\ x^2+4y^2=1 \end{cases}$$

Soit $\begin{cases} x=0 \\ y=\frac{1}{2} \end{cases}$ ou $\begin{cases} x=\pm\sqrt{1-4\cdot\frac{1}{9}}=\pm\frac{\sqrt{5}}{3} \\ y=-\frac{1}{3} \end{cases}$

Valeur de $d\left(M_0,\left(0,\frac{1}{2}\right)\right)=\sqrt{\left(\frac{1}{2}-1\right)^2}=\frac{1}{2}$

$d\left(M_0,\left(\pm\frac{\sqrt{5}}{3},-\frac{1}{3}\right)\right)=\sqrt{\frac{5}{9}+\frac{1}{9}}=\frac{\sqrt{6}}{3}$

$\frac{1}{2}<\frac{\sqrt{6}}{3} \Rightarrow M_{min}=\left(0,\frac{1}{2}\right)$, $M_{max}=\left(\pm\frac{\sqrt{5}}{3},-\frac{1}{3}\right)$

4. $$\iint_{\Sigma} dx\,dy = \int_0^{2\pi}\int_0^1 \frac{1}{2} r\,dr\,dt = \frac{2\pi}{4} = \frac{\pi}{2}$$

Car $x = r\cos t$, $y = \frac{1}{2} r\sin t$

$$J\left(\frac{(x,y)}{(r,t)}\right) = \begin{vmatrix} \cos t & \frac{1}{2}\sin t \\ -r\sin t & \frac{1}{2} r\cos t \end{vmatrix} = \frac{1}{2} r$$

$$dx\,dy = \frac{1}{2} r\,dr\,dt$$

5. Green - Riemann :

$$\iint_{\Omega} d\alpha = \int_{\partial\Omega = \Sigma} \alpha,$$

Par exemple on a

$$\iint_{\Omega} dx\,dy = \int_{\Sigma} x\,dy$$

$x = \cos t$, $dy = \frac{1}{2}\cos t\,dt$

$$\int_{\Sigma} x\,dy = \int_0^{2\pi} \frac{1}{2}\cos^2 t\,dt = \frac{1}{4}\int_0^{2\pi} (\cos 2t + 1)\,dt = \pi/2$$