Variations on a Schwarzian theme

Christian DUVAL
CPT–UM

Congrès international en l'honneur de Claude Roger
Institut Camille Jordan, Lyon
25-28 November 2009

A tribute to Lagrange

If φ is a conformal mapping of $\mathbb{C}$, Lagrange introduces the function

$$S(\varphi) = -2\sqrt{\varphi'}\left(\frac{1}{\sqrt{\varphi'}}\right)''$$

in his treatise on the *cartes géographiques* — Vol IV des œuvres complètes — see [G-R07,O-T09].

This Lagrangian is, today, called the Schwarzian (derivative)

$$\boxed{S(\varphi) = \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2}$$

of φ and is an object of projective geometry.

A tribute to Lagrange

If φ is a conformal mapping of $\mathbb{C}$, Lagrange introduces the function

$$S(\varphi) = -2\sqrt{\varphi'}\left(\frac{1}{\sqrt{\varphi'}}\right)''$$

in his treatise on the *cartes géographiques* — Vol IV des œuvres complètes — see [G-R07,O-T09].

This Lagrangian is, today, called the Schwarzian (derivative)

$$\boxed{S(\varphi) = \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2}$$

of φ and is an object of projective geometry.

- Q: Does it appear/generalize in other geometrical contexts?
- A: Yes! See below ...

The properties of the Schwarzian

The Schwarzian $\mathcal{S}(\varphi)$ measures, at each point x, the shift between a diffeomorphism $\varphi \in \mathrm{Diff}(\mathrm{S}^1)$ and its approximating homography, $h \in \mathrm{PGL}(2,\mathbb{R})$,[1]

$$\mathcal{S}(\varphi)(x) = (\widehat{h}^{-1} \circ \varphi)'''(x)$$

- It is a $\mathrm{PSL}(2,\mathbb{R})$-differential invariant for $\mathrm{Diff}_+(\mathrm{S}^1)$: $\mathcal{S}(\varphi) = \mathcal{S}(\psi)$ iff $\varphi = A \circ \psi$ where $A \in \mathrm{PSL}(2,\mathbb{R})$.
- It is a non trivial 1-cocycle of $\mathrm{Diff}_+(\mathrm{S}^1)$ with coefficients in the module of quadratic differentials $\mathcal{Q}(\mathrm{S}^1)$:

$$\mathcal{S}(\varphi \circ \psi) = \psi^* S(\varphi) + \mathcal{S}(\psi)$$

It has kernel $\mathrm{PSL}(2,\mathbb{R})$.

[1] sth $\widehat{h}^{-1} \circ \varphi$ has the 2-jet of Id at x

The three geometries of the circle

Highlight an important classification result, see [Fuk87, O-T05, G-R07]:

Theorem

The cohomology spaces $H^1(\mathrm{Diff}_+(S^1), \mathcal{F}_\lambda)$ are given by

$$H^1(\mathrm{Diff}_+(S^1), \mathcal{F}_\lambda) = \begin{cases} \mathbb{R} & \text{if } \lambda = 0, 1, 2 \\ \{0\} & \text{otherwise} \end{cases}$$

These 3 cohomology spaces are resp. generated by $\mathcal{E}, \mathcal{A}$, & $\mathcal{S}$:

$$\mathcal{E}(\varphi) = \log(\varphi'), \qquad \mathcal{A}(\varphi) = d\mathcal{E}(\varphi)$$

and Schwarzian cocycle

$$\mathcal{S}(\varphi) = \left(\frac{\varphi'''}{\varphi'} - \frac{3}{2} \left(\frac{\varphi''}{\varphi'} \right)^2 \right) dx^2$$

Invariants of $\mathrm{E}(1) \subset \mathrm{Aff}_+(1) \subset \mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{Diff}_+(\mathrm{S}^1)$

The kernels of these 3 cocycles define resp. the Euclidean, affine, and projective groups, i.e., the **3 geometries** of the circle, whose *discrete* invariants are

- Euclidean invariant (translations): distance

$$[x_1, x_2] = x_2 - x_1$$

- Affine invariant (homotheties, translations): distance-ratio

$$[x_1, x_2, x_3] = \frac{[x_1, x_3]}{[x_1, x_2]}$$

- Projective invariant (homographies): cross-ratio

$$[x_1, x_2, x_3, x_4] = \frac{[x_1, x_3][x_2, x_4]}{[x_2, x_3][x_1, x_4]}$$

Var. 1
The Schwarzian derivative and Lorentz surfaces

Var. 1
The Schwarzian derivative and Lorentz surfaces

- Prolonging to null infinity a “coboundary” of conformal group
- Looking at curvature of timelike Lorentz worldlines

The fourth geometry of Poincaré [K-S87, D-G2k]

Let $H^{1,1} \cong \mathrm{S}^1 \times \mathrm{S}^1 \setminus \Delta$ be hyperboloid of one sheet (of radius 1) in $\mathbb{R}^{2,1}$ (AdS space). Its induced Lorentz metric is

$$\mathrm{g}_1 = \frac{4\, d\theta_1 d\theta_2}{\left|e^{i\theta_1} - e^{i\theta_2}\right|^2}$$

1. $\mathrm{Conf}(H^{1,1}) \cong \mathrm{Diff}(\Delta)$ with $\Delta \cong \mathrm{S}^1$ conformal boundary.
2. If $\varphi \in \mathrm{Conf}_+(H^{1,1})$, then $\varphi^*\mathrm{g}_1 - \mathrm{g}_1$ extends smoothly to $\mathrm{S}^1 \times \mathrm{S}^1$.
3. "Prolongation" to Δ of phoney 1-coboundary

$$\mathcal{S}_1(\varphi) = \tfrac{3}{2}\left(\varphi^*\mathrm{g} - \mathrm{g}\right)|\Delta$$

$\Rightarrow$ non-trivial $\mathrm{Diff}_+(\mathrm{S}^1)$ 1-cocycle $\mathcal{S}_1(\varphi) = \mathrm{S}_1(\varphi)\, d\theta^2 \in \mathcal{Q}(\mathrm{S}^1)$:

$$\mathrm{S}_1(\varphi) = \underbrace{\frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2}_{S(\varphi)} + \frac{1}{2}\left(\varphi'^2 - 1\right)$$

The fourth geometry of Poincaré (cont'd)

Holography: Conformal Lorentzian geometry of bulk $H^{1,1}$ $\Longleftrightarrow$ projective geometry of its conformal boundary, $\Delta \cong \mathrm{S}^1$:

- $\mathrm{Conf}(H^{1,1}) \cong \mathrm{Diff}(\mathrm{S}^1)$.
- $\mathrm{Isom}_+(H^{1,1}) \cong \mathrm{PSL}(2,\mathbb{R})$.
- $(\varphi^* \mathrm{g}_1 - \mathrm{g}_1)|\mathrm{S}^1 \cong \mathcal{S}_1(\varphi)$.
- $\mathrm{Conf}_+(H^{1,1})/\mathrm{Isom}_+(H^{1,1})$ is a $\mathrm{Vir}(\mathrm{S}^1)$ **coadjoint orbit** with central charge $c = 1$, and symplectic 2-form Ω coming from

$$\omega(\delta_1 \mathrm{g}, \delta_2 \mathrm{g}) = \tfrac{3}{2}\int_\Delta i_{\xi_1} L_{\xi_2} \mathrm{g}$$

  where $\mathrm{g} \in [\mathrm{g}_1]$, $\delta_k \mathrm{g} = L_{\xi_k} \mathrm{g}$ & $\xi_k \in \mathrm{Vect}(\mathrm{S}^1)$.
- Etc.

Curvature of worldlines in Lorentz surfaces

Consider a curve $x \mapsto y = \varphi(x)$ and its graph in $\mathbb{R}^{1,1} = (\mathbb{R}^2, \mathrm{g} = dxdy)$.

If velocity $v = \partial/\partial x + \varphi'(x)\partial/\partial y$ is *timelike*, i.e., $\mathrm{g}(v, v) = \varphi'(x) > 0$, the Frenet curvature $\varrho = \sigma(v, a)/\mathrm{g}(v, v)^{\frac{3}{2}}$, with $a = \nabla_v v$ (acceleration) and $\sigma = dx \wedge dy$, reads

$$\varrho(x) = (\varphi'(x))^{-\frac{3}{2}}\varphi''(x).$$

Then

$$\boxed{\varrho'(x)\sqrt{\varphi'(x)} = \frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{2}\left(\frac{\varphi''(x)}{\varphi'(x)}\right)^2}$$

Curvature of worldlines in Lorentz surfaces

Consider a curve $x \mapsto y = \varphi(x)$ and its graph in $\mathbb{R}^{1,1} = (\mathbb{R}^2, \mathrm{g} = dxdy)$.

If velocity $v = \partial/\partial x + \varphi'(x)\partial/\partial y$ is *timelike*, i.e., $\mathrm{g}(v, v) = \varphi'(x) > 0$, the Frenet curvature $\varrho = \sigma(v, a)/\mathrm{g}(v, v)^{\frac{3}{2}}$, with $a = \nabla_v v$ (acceleration) and $\sigma = dx \wedge dy$, reads

$$\varrho(x) = (\varphi'(x))^{-\frac{3}{2}}\varphi''(x).$$

Then

$$\boxed{\varrho'(x)\sqrt{\varphi'(x)} = \frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{2}\left(\frac{\varphi''(x)}{\varphi'(x)}\right)^2}$$

- Q: Does this relationship generalize to curved Lorentz surfaces?
- A: Yes, provided ...

Curvature of worldlines in Lorentz surfaces (cont'd)

Theorem [D-O2k]

Let $\varphi \in \mathrm{Diff}_+(\mathbb{R}P^1)$, and let ϱ be curvature of its graph in $\mathbb{R}P^1 \times \mathbb{R}P^1$ with metric $\mathrm{g} = g(x,y)dxdy$, and t be *proper time*, then

$$d\varrho\, dt = S(\varphi)$$

iff

$$\mathrm{g} = \frac{dxdy}{(axy + bx + cy + d)^2}$$

with $a, b, c, d \in \mathbb{R}$.

Metric of constant curvature $K = 8(ad - bc)$ on $\Sigma = \mathbb{R}P^1 \times \mathbb{R}P^1 \setminus \Gamma$ is projectively equivalent to

$$\mathrm{g} = \begin{cases} dxdy & (K = 0) \\ \dfrac{8}{K}\dfrac{dxdy}{(x-y)^2} & (K \neq 0) \end{cases}$$

Curvature of worldlines in Lorentz surfaces (cont'd)

Ghys' theorem ('95) — "*The Schwarzian derivative* $S(\varphi)$ *of a diffeomorphism* φ *of* $\mathbb{R}P^1$ *has at least* 4 *distinct zeroes*" — hence corresponds to the **4-vertex theorem** for closed timelike curves in $\Sigma \subset \mathbb{R}P^1 \times \mathbb{R}P^1$ with the above metric.

Var. 2
The Schwarzian derivative and Finsler geometry

Var. 2
The Schwarzian derivative and Finsler geometry

- Check out metrics of scalar curvature, e.g., Numata metrics
- Specialize the flag curvature to the . . . 1-dim case

Schwarzian derivative & Finsler scalar curvature (I)

A Finsler structure on a smooth manifold M is defined by a "metric"

$$F : TM \to \mathbb{R}^+$$

whose restriction to $TM \setminus M$ is strictly positive, smooth, and sth $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$; Hessian $\mathrm{g}_{ij}(x, y) = \left(\frac{1}{2}F^2\right)_{y^i y^j}$ is assumed positive definite. The fundamental tensor

$$\mathrm{g} = \mathrm{g}_{ij}(x, y) dx^i \otimes dx^j$$

defines a *sphere's worth of Riemannian metrics* on each T_xM. Also

$$\ell = \ell^i \frac{\partial}{\partial x^i}, \qquad \text{with} \qquad \ell^i(x, y) = \frac{y^i}{F(x, y)}$$

is a distinguished unit section of $\pi^*(TM)$, i.e., $\mathrm{g}_{ij}(x, y)\ell^i \ell^j = 1$, where $\pi : TM \setminus M \to M$.

Schwarzian derivative & Finsler scalar curvature (II)

Unlike Riemannian case, $\nexists$ canonical linear connection on $\pi^*(TM)$.
Example: Chern connection $\omega^i_j = \Gamma^i_{jk}(x,y)dx^k$ uniquely characterized by (i) symmetry: $\Gamma^i_{jk} = \Gamma^i_{kj}$, and (ii) almost "metric transport": $dg_{ij} - \omega^k_i g_{jk} - \omega^k_j g_{ik} = 2C_{ijk}\delta y^k$.[2]

[2] Here, $C_{ijk}(x,y) = \left(\frac{1}{4}F^2\right)_{y^i y^j y^k}$ (Cartan tensor), $\delta y^i = dy^i + N^i_j dx^j$, with $N^i_j(x,y) = \gamma^i_{jk} y^k$ (Ehresmann connection), where γ^i_{jk} formal Christoffel symbols.

Schwarzian derivative & Finsler scalar curvature (II)

Unlike Riemannian case, $\nexists$ canonical linear connection on $\pi^*(TM)$. Example: Chern connection $\omega^i_j = \Gamma^i_{jk}(x,y)dx^k$ uniquely characterized by (i) symmetry: $\Gamma^i_{jk} = \Gamma^i_{kj}$, and (ii) almost "metric transport": $dg_{ij} - \omega^k_i g_{jk} - \omega^k_j g_{ik} = 2C_{ijk}\delta y^k$.[2]

Using "horizontal derivatives" $\delta/\delta x^i = \partial/\partial x^i - N^j_i \partial/\partial y^j$, one gets hh-Chern curvature

$$R_j{}^i{}_{kl} = \frac{\delta}{\delta x^k}\Gamma^i_{jl} + \Gamma^i_{mk}\Gamma^m_{jl} - (k \leftrightarrow l)$$

and **flag curvature** (for the flag $\ell \wedge v$ with $v \in T_xM$) by

$$K(x,y,v) = \frac{R_{ik}v^i v^k}{\mathrm{g}(v,v) - \mathrm{g}(\ell,v)^2}, \qquad \text{where} \qquad R_{ik} = \ell^j R_{jikl}\,\ell^l$$

[2]Here, $C_{ijk}(x,y) = \left(\frac{1}{4}F^2\right)_{y^i y^j y^k}$ (Cartan tensor), $\delta y^i = dy^i + N^i_j dx^j$, with $N^i_j(x,y) = \gamma^i_{jk}y^k$ (Ehresmann connection), where γ^i_{jk} formal Christoffel symbols.

Schwarzian derivative & Finsler scalar curvature (III)

Finsler structure (M, F) **of scalar curvature** $\Longleftrightarrow K(x, y, v)$ is independent of the vector v, i.e.,

$$R_{ik} = K(x, y) h_{ik} \qquad (\star)$$

with $h_{ik} = \mathrm{g}_{ik} - \ell_i \ell_k$ the "angular metric", $\ell_i = \mathrm{g}_{ij} \ell^j$.

Example: The Numata Finsler structure: $F(x, y) = \sqrt{\delta_{ij} y^i y^j} + f_{x^i} y^i$, where $M = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n (f_{x^i})^2 < 1 \right\}$, and $f \in C^\infty(M)$. The flag curvature reads *provocatively*

$$K(x, y) = -\frac{1}{2F^2} \left[\frac{1}{F} f_{x^i x^j x^k} y^i y^j y^k - \frac{3}{2} \frac{1}{F^2} \left(f_{x^i x^j} y^i y^j \right)^2 \right]$$

Schwarzian derivative & Finsler scalar curvature (III)

Finsler structure (M, F) **of scalar curvature** $\Longleftrightarrow$ $K(x, y, v)$ is independent of the vector v, i.e.,

$$R_{ik} = K(x, y) h_{ik} \qquad (\star)$$

with $h_{ik} = \mathrm{g}_{ik} - \ell_i \ell_k$ the "angular metric", $\ell_i = \mathrm{g}_{ij} \ell^j$.

Example: The Numata Finsler structure: $F(x, y) = \sqrt{\delta_{ij} y^i y^j} + f_{x^i} y^i$, where $M = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n (f_{x^i})^2 < 1 \right\}$, and $f \in C^\infty(M)$. The flag curvature reads *provocatively*

$$K(x, y) = -\frac{1}{2F^2} \left[\frac{1}{F} f_{x^i x^j x^k} y^i y^j y^k - \frac{3}{2} \frac{1}{F^2} \left(f_{x^i x^j} y^i y^j \right)^2 \right]$$

Idea: Investigate the case $n = 1$.

Schwarzian derivative & Finsler scalar curvature (IV)

Although Eq. ($\star$) trivially satisfied, the flag curvature admits a nontrivial prolongation to this 1-dim case, where $F(x,y) = |y| + f'(x)y$ with $-1 < f'(x) < +1$ on $M \subset \mathbb{R}P^1$. Its restrictions to $T^\pm M \cong M \times \mathbb{R}^\pm_*$ read $F_\pm(x,y) = \varphi'_\pm(x)y > 0$, where

$$\varphi'_\pm(x) = f'(x) \pm 1 \qquad (*)$$

implying $\varphi_\pm \in \mathrm{Diff}_\pm(\mathbb{R}P^1)$, with $|\varphi'_\pm(x)| < 2$.

Schwarzian derivative & Finsler scalar curvature (IV)

Although Eq. ($\star$) trivially satisfied, the flag curvature admits a nontrivial prolongation to this 1-dim case, where $F(x,y) = |y| + f'(x)y$ with $-1 < f'(x) < +1$ on $M \subset \mathbb{R}P^1$. Its restrictions to $T^\pm M \cong M \times \mathbb{R}^\pm_*$ read $F_\pm(x,y) = \varphi'_\pm(x)y > 0$, where

$$\varphi'_\pm(x) = f'(x) \pm 1 \qquad (*)$$

implying $\varphi_\pm \in \mathrm{Diff}_\pm(\mathbb{R}P^1)$, with $|\varphi'_\pm(x)| < 2$.

Theorem [Duv08]

The 1-dim Numata Finsler structure induces a Riemannian metric, $\mathrm{g}(\varphi) = \varphi^*(dx^2)$, where $\varphi \in \mathrm{Diff}(\mathbb{R}P^1)$ is as in ($*$). The flag curvature is

$$\boxed{K = -\frac{1}{2}\frac{\mathrm{S}(\varphi)}{\mathrm{g}(\varphi)}}$$

with $\mathrm{S}(\varphi)$ the Schwarzian quadratic differential of φ.

Var. 3
The Schwarzian derivative and contact geometry of $S^{1|1}$

Var. 3
The Schwarzian derivative and contact geometry of $S^{1|1}$

- Seek super geometric versions of the Euclidean, affine, and projective invariants of S^1. Super cross-ratio?

- What are then the 1- cocycles associated with super extensions of $\mathrm{Diff}(S^1)$? Super Schwarzian derivative?

Var. 3
The Schwarzian derivative and contact geometry of $S^{1|1}$

- Seek super geometric versions of the Euclidean, affine, and projective invariants of S^1. Super cross-ratio?
- What are then the 1- cocycles associated with super extensions of $\mathrm{Diff}(S^1)$? Super Schwarzian derivative?
- How can one relate theses new geometric objects?
- Classification of the geometries of the supercircle!

The supercircle $S^{1|1}$

The supercircle $S^{1|1}$: the circle $\mathbb{S}^1$, endowed with (a sheaf of associative commutative $\mathbb{Z}/(2\mathbb{Z})$-graded algebras, with sections) the superfunctions $C^\infty(S^{1|1}) = C^\infty(\mathbb{S}^1)[\xi]$ where $\xi^2 = 0$ & $x\xi = \xi x$.

The supercircle $S^{1|1}$

The supercircle $S^{1|1}$: the circle S^1, endowed with (a sheaf of associative commutative $\mathbb{Z}/(2\mathbb{Z})$-graded algebras, with sections) the superfunctions $C^\infty(S^{1|1}) = C^\infty(\mathrm{S}^1)[\xi]$ where $\xi^2 = 0$ & $x\xi = \xi x$.

- If (x, ξ) are local coordinates of (affine) superdomain, every superfunction writes

$$f(x,\xi) = f_0(x) + \xi f_1(x), \qquad \text{where} \qquad f_0, f_1 \in C^\infty(\mathrm{S}^1)$$

- Parity: $p(f_0) = 0$, $p(\xi f_1) = 1$.

- Projection: $\pi : C^\infty(S^{1|1}) \to C^\infty(\mathrm{S}^1)$ where $\ker(\pi)$: ideal generated by nilpotent elements.

The supercircle $S^{1|1}$

The supercircle $S^{1|1}$: the circle S^1, endowed with (a sheaf of associative commutative $\mathbb{Z}/(2\mathbb{Z})$-graded algebras, with sections) the superfunctions $C^\infty(S^{1|1}) = C^\infty(\mathrm{S}^1)[\xi]$ where $\xi^2 = 0$ & $x\xi = \xi x$.

- If (x, ξ) are local coordinates of (affine) superdomain, every superfunction writes

$$f(x,\xi) = f_0(x) + \xi f_1(x), \qquad \text{where} \qquad f_0, f_1 \in C^\infty(\mathrm{S}^1)$$

- Parity: $p(f_0) = 0$, $p(\xi f_1) = 1$.

- Projection: $\pi : C^\infty(S^{1|1}) \to C^\infty(\mathrm{S}^1)$ where $\ker(\pi)$: ideal generated by nilpotent elements.

- Group of diffeomorphisms: $\mathrm{Diff}(S^{1|1}) = \mathrm{Aut}(C^\infty(S^{1|1}))$ consists of pairs $\Phi = (\varphi, \psi)$ of superfunctions sth $(\varphi(x,\xi), \psi(x,\xi))$ are new coordinates on $S^{1|1}$.

Vector fields & 1-forms of the supercircle

Vector fields $\mathrm{Vect}(S^{1|1}) = \mathrm{SuperDer}(C^\infty(S^{1|1}))$ with local expression

$$X = f(x,\xi)\partial_x + g(x,\xi)\partial_\xi \qquad \text{where} \qquad f, g \in C^\infty(S^{1|1})$$

NB: $\mathrm{Vect}(S^{1|1})$ is a $C^\infty(S^{1|1})_L$-module locally generated by $(\partial_x, \partial_\xi)$ where $p(\partial_x) = 0$, $p(\partial_\xi) = 1$. It is also a Lie superalgebra with Lie bracket $[X, Y] = XY - (-1)^{p(X)p(Y)}YX$.

[3] Cohomological degree $|\cdot|$, parity p.

Vector fields & 1-forms of the supercircle

Vector fields $\mathrm{Vect}(S^{1|1}) = \mathrm{SuperDer}(C^\infty(S^{1|1}))$ with local expression

$$X = f(x,\xi)\partial_x + g(x,\xi)\partial_\xi \qquad \text{where} \qquad f, g \in C^\infty(S^{1|1})$$

NB: $\mathrm{Vect}(S^{1|1})$ is a $C^\infty(S^{1|1})_L$-module locally generated by $(\partial_x, \partial_\xi)$ where $p(\partial_x) = 0$, $p(\partial_\xi) = 1$. It is also a Lie superalgebra with Lie bracket $[X, Y] = XY - (-1)^{p(X)p(Y)}YX$.

Differential 1-forms $\Omega^1(S^{1|1}) = C^\infty(S^{1|1})_R$-module locally generated by dual basis $(dx, d\xi)$ where $p(dx) = 0$, $p(d\xi) = 1$. The module $\Omega^\bullet(S^{1|1})$ of differential forms is bigraded: our choice of *Sign Rule* is[3]

$$\alpha \wedge \beta = (-1)^{|\alpha||\beta| + p(\alpha)p(\beta)}\beta \wedge \alpha$$

[3]Cohomological degree $|\cdot|$, parity p.

Contact structure on supercircle $S^{1|1}$

It is given by direction of contact 1-form [Lei80]

$$\alpha = dx + \xi d\xi$$

We have $d\alpha = \beta \wedge \beta$ where $\beta = d\xi$.
- Contact distribution, $\ker(\alpha)$, generated by SUSY vector field

$$D = \partial_\xi + \xi \partial_x$$

- Contactomorphisms:[4]

$$K(1) = \{\Phi \in \mathrm{Diff}(S^{1|1}) \,|\, \Phi^* \alpha = E_\Phi \, \alpha\}$$

[4]One shows that $\Phi = (\varphi, \psi) \in K(1) \iff D\varphi = \psi D\psi$, with **multiplier** $E_\Phi = (D\psi)^2$.

Contact structure on supercircle $S^{1|1}$

It is given by direction of contact 1-form [Lei80]

$$\alpha = dx + \xi d\xi$$

We have $d\alpha = \beta \wedge \beta$ where $\beta = d\xi$.
- Contact distribution, $\ker(\alpha)$, generated by SUSY vector field

$$D = \partial_\xi + \xi \partial_x$$

- Contactomorphisms:[4]

$$K(1) = \{\Phi \in \mathrm{Diff}(S^{1|1}) \,|\, \Phi^*\alpha = E_\Phi\, \alpha\}$$

- Infinitesimal contactomorphisms:
$k(1) = \{X \in \mathrm{Vect}(S^{1|1}) \,|\, L_X\alpha = e_X\alpha\}$. Canonical Lie superalgebra isomorphism $k(1) \to C^\infty(S^{1|1}) : X \mapsto f = \langle X, \alpha \rangle$.

[4]One shows that $\Phi = (\varphi, \psi) \in K(1) \iff D\varphi = \psi D\psi$, with **multiplier** $E_\Phi = (D\psi)^2$.

Densities, 1-forms & quadratic differentials

Let $\mathcal{F}_\lambda(S^{1|1})$ be the $K(1)$-module of λ-densities ($\lambda \in \mathbb{C}$): $C^\infty(S^{1|1})$ endowed with (anti)action $\Phi_\lambda f = (E_\Phi)^\lambda \Phi^* f$. Write $F \in \mathcal{F}_\lambda$ as $F = f\alpha^\lambda$.

Densities, 1-forms & quadratic differentials

Let $\mathcal{F}_\lambda(S^{1|1})$ be the $K(1)$-module of λ-densities ($\lambda \in \mathbb{C}$): $C^\infty(S^{1|1})$ endowed with (anti)action $\Phi_\lambda f = (E_\Phi)^\lambda \Phi^* f$. Write $F \in \mathcal{F}_\lambda$ as $F = f\alpha^\lambda$.

- The $C^\infty(S^{1|1})_R$-module $\Omega^1(S^{1|1})$ of 1-forms is generated by α et β.

Densities, 1-forms & quadratic differentials

Let $\mathcal{F}_\lambda(S^{1|1})$ be the $K(1)$-module of λ-densities ($\lambda \in \mathbb{C}$): $C^\infty(S^{1|1})$ endowed with (anti)action $\Phi_\lambda f = (E_\Phi)^\lambda \Phi^* f$. Write $F \in \mathcal{F}_\lambda$ as $F = f\alpha^\lambda$.

- The $C^\infty(S^{1|1})_R$-module $\Omega^1(S^{1|1})$ of 1-forms is generated by α et β.
- The $C^\infty(S^{1|1})_R$-module $\mathcal{Q}(S^{1|1})$ of quadratic differentials is generated by $\alpha^2 = \alpha \otimes \alpha$ et $\alpha\beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$.

Densities, 1-forms & quadratic differentials

Let $\mathcal{F}_\lambda(S^{1|1})$ be the $K(1)$-module of λ-densities ($\lambda \in \mathbb{C}$): $C^\infty(S^{1|1})$ endowed with (anti)action $\Phi_\lambda f = (E_\Phi)^\lambda \Phi^* f$. Write $F \in \mathcal{F}_\lambda$ as $F = f\alpha^\lambda$.

- The $C^\infty(S^{1|1})_R$-module $\Omega^1(S^{1|1})$ of 1-forms is generated by α et β.
- The $C^\infty(S^{1|1})_R$-module $\mathcal{Q}(S^{1|1})$ of quadratic differentials is generated by $\alpha^2 = \alpha \otimes \alpha$ et $\alpha\beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$.

Proposition

Both $\Omega^1(S^{1|1})$ and $\mathcal{Q}(S^{1|1})$ are $K(1)$-modules ; they admit the decomposition into $K(1)$-submodules:

$$\Omega^1(S^{1|1}) \cong \mathcal{F}_{\frac{1}{2}} \oplus \mathcal{F}_1, \qquad \mathcal{Q}(S^{1|1}) \cong \mathcal{F}_{\frac{3}{2}} \oplus \mathcal{F}_2$$

The projections $\Omega^1(S^{1|1}) \to \mathcal{F}_{\frac{1}{2}}$ (resp. $\mathcal{Q}(S^{1|1}) \to \mathcal{F}_{\frac{3}{2}}$) are given by $\alpha^{\frac{1}{2}}\langle D, \cdot\rangle$, and the corresponding sections by $\alpha^{\frac{1}{2}} L_D$ (resp. $\frac{2}{3}\alpha^{\frac{1}{2}} L_D$).

Orthosymplectic group, Euclidean & affine subgroups

It is the subgroup $\mathrm{SpO}(2|1) \subset \mathrm{GL}(2|1)$ of symplectomorphisms of $(\mathbb{R}^{2|1}, d\varpi)$ where $\varpi = \frac{1}{2}(pdq - qdp + \theta d\theta)$; one has[5]

$$h = \begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix} \in \mathrm{SpO}(2|1)$$

The group $\mathrm{SpO}(2|1)$ preserves 1-forme $\varpi = \frac{1}{2}p^2\alpha$ (where $p \neq 0$); it thus acts by contactomorphisms via the projective action on $S^{1|1}$:

$$\widehat{h}(x, \xi) = \left(\frac{ax + b + \gamma\xi}{cx + d + \delta\xi}, \frac{\alpha x + \beta + e\xi}{cx + d + \delta\xi} \right)$$

The Berezinian is $\mathrm{Ber}(h) = e + \alpha\beta e^{-1}$ and $\mathrm{SpO}_+(2|1) = \mathrm{Ber}^{-1}(1)$, super-extension of $\mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$.

[5] with $ad - bc - \alpha\beta = 1, e^2 + 2\gamma\delta = 1, \alpha e - a\delta + c\gamma = 0, \beta e - b\delta + d\gamma = 0.$

Orthosymplectic group, Euclidean & affine subgroups (cont'd)

One uses the local factorization

$$\mathrm{SpO}_+(2|1) \ni h = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{c} & 1 & \tilde{\delta} \\ \tilde{\delta} & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{a}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \overbrace{\begin{pmatrix} \epsilon & \tilde{b} & -\tilde{\beta} \\ 0 & \epsilon & 0 \\ 0 & \epsilon\tilde{\beta} & 1 \end{pmatrix}}^{\mathrm{E}(1|1)}}_{\mathrm{Aff}(1|1)}$$

where $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{\beta}, \tilde{\delta}) \in \mathbb{R}^{3|2}$, with $\epsilon^2 = 1$, $\tilde{a} > 0$.

Orthosymplectic group, Euclidean & affine subgroups (cont'd)

One uses the local factorization

$$\mathrm{SpO}_+(2|1) \ni h = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{c} & 1 & \tilde{\delta} \\ \tilde{\delta} & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{a}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \overbrace{\begin{pmatrix} \epsilon & \tilde{b} & -\tilde{\beta} \\ 0 & \epsilon & 0 \\ 0 & \epsilon\tilde{\beta} & 1 \end{pmatrix}}^{\mathrm{E}(1|1)}}_{\mathrm{Aff}(1|1)}$$

where $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{\beta}, \tilde{\delta}) \in \mathbb{R}^{3|2}$, with $\epsilon^2 = 1$, $\tilde{a} > 0$.

- Note that $\mathrm{E}(1|1) = \{\Phi \in \mathrm{Diff}(S^{1|1}) \,|\, \Phi^*\alpha = \alpha\}$

- Also $\mathrm{Aff}(1|1) = \{\Phi \in \mathrm{Diff}(S^{1|1}) \,|\, \Phi^*\alpha = a^2\alpha, a \neq 0\}$

Notion of $p|q$-transitivity [M-D08]

Extension of the notion of n-transitivity to supergroup actions.

Consider $E = E_0 \times E_1$ and canonical projections p_0 & p_1. Two n-uplets $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$ of distinct points of E are said $p|q$-equivalent, $s \stackrel{p|q}{=} t$, where $n = \max(p, q)$, if $p_0(s_i) = p_0(t_i)$ $\forall i = 1, \ldots, p$ and $p_1(s_i) = p_1(t_i)$ $\forall i = 1, \ldots, q$.

Notion of $p|q$-transitivity [M-D08]

Extension of the notion of n-transitivity to supergroup actions.

Consider $E = E_0 \times E_1$ and canonical projections p_0 & p_1. Two n-uplets $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$ of distinct points of E are said $p|q$-equivalent, $s \stackrel{p|q}{=} t$, where $n = \max(p, q)$, if $p_0(s_i) = p_0(t_i)$ $\forall i = 1, \ldots, p$ and $p_1(s_i) = p_1(t_i)$ $\forall i = 1, \ldots, q$.

The action $(h \mapsto \hat{h})$ of group G on E is said simply $p|q$-transitive if for all $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$, $\exists!\, g \in G$ sth $\hat{g}(t) \stackrel{p|q}{=} s$.

Notion of $p|q$-transitivity [M-D08]

Extension of the notion of n-transitivity to supergroup actions.

Consider $E = E_0 \times E_1$ and canonical projections p_0 & p_1. Two n-uplets $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$ of distinct points of E are said $p|q$-equivalent, $s \overset{p|q}{=} t$, where $n = \max(p, q)$, if $p_0(s_i) = p_0(t_i)$ $\forall i = 1, \ldots, p$ and $p_1(s_i) = p_1(t_i)$ $\forall i = 1, \ldots, q$.

The action $(h \mapsto \hat{h})$ of group G on E is said simply $p|q$-transitive if for all $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$, $\exists!\, g \in G$ sth $\hat{g}(t) \overset{p|q}{=} s$.

Examples:

- The $\mathrm{PSL}(2, \mathbb{R})$-action on S^1 is simply 3-transitive.
- The $\mathrm{SpO}_+(2|1)$-action on $S^{1|1}$ is $3|2$-transitive.

Construction of the discrete invariants

Theorem

Let G act simply $p|q$-transitively on $E = E_0 \times E_1$, and m be an n-uplet, $n = \max(p, q)$, of distinct points of E. The $(n+1)$-point function $I_{[m]}$ of E with values in E defined by

$$I_{[m]}(t_1, \ldots, t_{n+1}) = \widehat{h}(t_{n+1})$$

where $\widehat{h}(t) \stackrel{p|q}{=} m$, and $t = (t_1, \ldots, t_n) \in E^n \setminus \Gamma$ enjoys the properties:

1. $I_{[m]}$ is G-invariant.
2. If $\Phi \in E!$ preserves $I_{[m]}$, then $\Phi = \widehat{g}$ for some $g \in G$.

Theorem I [M-D08]

- Euclidean invariant $I_e(t_1, t_2) = ([t_1, t_2], \{t_1, t_2\})$:

$$[t_1, t_2] = x_2 - x_1 - \xi_2\xi_1, \qquad \{t_1, t_2\} = \xi_2 - \xi_1$$

- Affine invariant $I_a(t_1, t_2, t_3) = ([t_1, t_2, t_3], \{t_1, t_2, t_3\})$:

$$[t_1, t_2, t_3] = \frac{[t_1, t_3]}{[t_1, t_2]}, \qquad \{t_1, t_2, t_3\} = [t_1, t_2, t_3]^{\frac{1}{2}} \frac{\{t_1, t_3\}}{[t_1, t_3]^{\frac{1}{2}}}$$

- Projective invariant $I_p(t_1, t_2, t_3, t_4) = ([t_1, t_2, t_3, t_4], \pm\{t_1, t_2, t_3, t_4\})$, i.e., super cross-ratio:

$$[t_1, t_2, t_3, t_4] = \frac{[t_1, t_3][t_2, t_4]}{[t_2, t_3][t_1, t_4]},$$

$$\{t_1, t_2, t_3, t_4\} = [t_1, t_2, t_3, t_4]^{\frac{1}{2}} \frac{\{t_2, t_4\}[t_1, t_2] - \{t_1, t_2\}[t_2, t_4]}{([t_1, t_2][t_2, t_4][t_1, t_4])^{\frac{1}{2}}}$$

Theorem I (cont'd)

- If a contactomorphism $\Phi \in K(1)$ preserves the even part of I_e, or I_a, or I_p, then $\Phi = \widehat{h}$ for h in $\mathrm{E}(1|1)$, or $\mathrm{Aff}(1|1)$, or $\mathrm{SpO}_+(2|1)$, respectively.
- If a bijection Φ of $S^{1|1}$ preserves I_e, or I_a, or I_p, then $\Phi = \widehat{h}$ for h in $\mathrm{E}_+(1|1)$, or $\mathrm{Aff}_+(1|1)$, or $\mathrm{SpO}_+(2|1)$, respectively.

See [Aok88, Nel88, U-Y90, Man91, Gidd92] for pioneering introduction of super cross-ratio (super Riemann surfaces, superstrings, . . .)

Theorem I (cont'd)

- If a contactomorphism $\Phi \in K(1)$ preserves the even part of I_e, or I_a, or I_p, then $\Phi = \widehat{h}$ for h in $\mathrm{E}(1|1)$, or $\mathrm{Aff}(1|1)$, or $\mathrm{SpO}_+(2|1)$, respectively.
- If a bijection Φ of $S^{1|1}$ preserves I_e, or I_a, or I_p, then $\Phi = \widehat{h}$ for h in $\mathrm{E}_+(1|1)$, or $\mathrm{Aff}_+(1|1)$, or $\mathrm{SpO}_+(2|1)$, respectively.

See [Aok88, Nel88, U-Y90, Man91, Gidd92] for pioneering introduction of super cross-ratio (super Riemann surfaces, superstrings, ...)

Seek now the corresponding differential invariants ...

The Cartan formula [Car37]

Consider $\Phi \in \mathrm{Diff}(S^1)$, the flow $\phi_\varepsilon = \mathrm{Id} + \varepsilon X + O(\varepsilon^2)$ of a vector field X, and 4 points t_1, $t_2 = \phi_\varepsilon(t_1)$, $t_3 = \phi_{2\varepsilon}(t_1)$, $t_4 = \phi_{3\varepsilon}(t_1)$.

The Schwarzian derivative of Φ is defined, via the cross-ratio, as the quadratic differential $\mathcal{S}(\Phi) \in \mathcal{Q}(S^1)$ appearing in

$$\frac{\Phi^*[t_1, t_2, t_3, t_4]}{[t_1, t_2, t_3, t_4]} - 1 = \langle \varepsilon X \otimes \varepsilon X, \mathcal{S}(\Phi) \rangle + O(\varepsilon^3)$$

This formula (and its avatar for $\mathcal{A}(\Phi)$) admits a prolongation to the case of the supercircle; it leads to the following result:

Theorem II [M-D08]

The even Euclidean, affine and projective invariants $\Rightarrow$ three 1-cocycles of $K(1)$, with kernel $\mathrm{E}(1|1)$, $\mathrm{Aff}(1|1)$ et $\mathrm{SpO}_+(2|1)$ resp:

- the Euclidean cocycle $\mathcal{E} : K(1) \to \mathcal{F}_0(S^{1|1})$:

$$\mathcal{E}(\Phi) = \log E_\Phi$$

- the affine cocycle $\mathcal{A} : K(1) \to \Omega^1(S^{1|1})$:

$$\mathcal{A}(\Phi) = d\mathcal{E}(\Phi)$$

- the projective cocycle (super Schwarzian) $\mathcal{S} : K(1) \to \mathcal{Q}(S^{1|1})$:

$$\boxed{\mathcal{S}(\Phi) = \frac{2}{3}\,\alpha^{\frac{1}{2}} L_D\, \mathrm{S}(\Phi)}$$

where

$$\mathrm{S}(\Phi) = \frac{1}{4}\left(\frac{D^3 E_\Phi}{E_\Phi} - \frac{3}{2}\frac{DE_\Phi\, D^2 E_\Phi}{E_\Phi^2}\right)\alpha^{\frac{3}{2}}$$

Theorem II (cont'd)

Using the projections of $\mathcal{Q}(S^{1|1})$ to summands of densities, one obtains two new affine and projective 1-cocycles:

- the projection of the affine cocycle $\mathrm{A} : K(1) \to \mathcal{F}_{\frac{1}{2}}(S^{1|1})$:

$$\mathrm{A}(\Phi) = \alpha^{\frac{1}{2}} \langle D, \mathcal{A}(\Phi) \rangle = \frac{DE_\Phi}{E_\Phi}\, \alpha^{\frac{1}{2}}$$

- the projection of the Schwarzian cocycle $\mathrm{S} : K(1) \to \mathcal{F}_{\frac{3}{2}}(S^{1|1})$:

$$\mathrm{S}(\Phi) = \alpha^{\frac{1}{2}} \langle D, \mathcal{S}(\Phi) \rangle = \frac{1}{4} \left(\frac{D^3 E_\Phi}{E_\Phi} - \frac{3}{2} \frac{DE_\Phi\, D^2 E_\Phi}{E_\Phi^2} \right) \alpha^{\frac{3}{2}}$$

This expression is due to [Rad86]; see [Fri86, Coh87, G-T93].

Note the super Lagrange formula

$$\mathrm{S}(\Phi) = -\frac{1}{2} E_\Phi^{\frac{1}{2}}\, D^3 (E_\Phi^{-\frac{1}{2}})\, \alpha^{\frac{3}{2}}$$

The three geometries of supercircle $S^{1|1}$

The 1-cocycles of $k(1)$ (Lie superalgebra of hamiltonian vector fields of $(S^{1|1}, [\alpha])$) associated with $\mathcal{E}$, A et S are trivially the $c_i : k(1) \to \mathcal{F}_{i/2}$:

$$c_i(X_f) = (D^{i+2} f)\, \alpha^{i/2} \qquad (i = 0, 1, 3)$$

These are the 3 out of 4 generators of $H^1(k(1), \mathcal{F}_\lambda)$ [A-BF03] the only ones which integrate as (non trivial) $K(1)$-cocycles.

Theorem [M-D08]

The cohomology spaces

$$H^1(K(1), \mathcal{F}_\lambda(S^{1|1})) = \begin{cases} \mathbb{R} & \text{if } \lambda = 0, \frac{1}{2}, \frac{3}{2} \\ \{0\} & \text{otherwise} \end{cases}$$

are resp. generated by $\mathcal{E}$, A et S. The cohomology spaces

$$H^1(K(1), \Omega^1(S^{1|1})) = \mathbb{R} \qquad \text{et} \qquad H^1(K(1), \mathcal{Q}(S^{1|1})) = \mathbb{R}$$

are resp. generated by $\mathcal{A}$ et $\mathcal{S}$.

Var. 4
The super Virasoro group $\mathrm{Vir}(S^{1|1})$

Var. 4
The super Virasoro group $\mathrm{Vir}(S^{1|1})$

- The symplectic structure of $K(1)/\mathrm{SpO}_{+}(2|1)$
- The Bott-Thurston cocycle of $K(1)$

The Bott-Thurston cocycle of $K(1)$

We work out the super Virasoro group, $\mathrm{Vir}(S^{1|1})$, via a distinguished **affine-coadjoint orbit** of $K(1)$.

Consider the 1-form Θ of $K(1)$ defined by the Berezin integral

$$\Theta(\delta_f\Phi) = \frac{1}{2}\int_{S^{1|1}} \mathrm{A}(\Phi)\,\delta_f\mathcal{E}(\Phi)$$

with $A = \alpha^{\frac{1}{2}}\langle D, \mathcal{A}\rangle$ (resp. $\mathcal{E}$) the affine (resp. Euclidean) $K(1)$ cocycle (recall $\mathcal{A} = d\mathcal{E}$); also $\delta_f\Phi = \delta(\Phi\circ\Psi)$ with $\delta\Psi = X_f$ at $\Psi = \mathrm{Id}$, for some contact Hamiltonian $f \in C^\infty(S^{1|1})$.

Easy calculation yield

$$\delta_f\mathcal{E} = \langle X_f, \mathcal{A}\rangle + f' \qquad \& \qquad \delta_f\mathrm{A} = \alpha^{\frac{1}{2}} D\,\delta_f\mathcal{E}$$

The Bott-Thurston cocycle of $K(1)$ (cont'd)

Theorem

The exterior derivative $d\Theta$ of the 1-form Θ of $K(1)$, viz.

$$d\Theta(\delta_f\Phi, \delta_g\Phi) = \frac{1}{2}\int_{S^{1|1}} \delta_f \mathrm{A}(\Phi)\, \delta_g \mathcal{E}(\Phi) - (-1)^{p(f)p(g)} \delta_g \mathrm{A}(\Phi)\, \delta_f \mathcal{E}(\Phi)$$

descends to $\mathrm{S}(K(1)) \cong K(1)/\mathrm{SpO}_+(2|1)$, the *affine-coadjoint* orbit of the origin in $\mathcal{F}_{\frac{3}{2}} \subset k(1)^*$, as a symplectic 2-form Ω sth

$$\begin{aligned}\Omega(X_f, X_g) &= \int_{S^{1|1}} \langle \mathrm{S}(\Phi), [X_f, X_g] \rangle + \int_{S^{1|1}} (D^5 f) g\, \alpha^{\frac{1}{2}} \\ &= \mathsf{S}(\Phi) \cdot [X_f, X_g] + \mathsf{GF}(X_f, X_g)\end{aligned}$$

where the Schwarzian, S, induces the Souriau cocycle, S, and GF is the Gelfand-Fuchs cocycle of $k(1)$.

The super Virasoro group $\mathrm{Vir}(S^{1|1})$

Now Θ fails to be $K(1)$-invariant; introduce hence the 1-form

$$\widehat{\Theta} = \Theta + dt$$

of $\widehat{K(1)} = K(1) \times \mathbb{R}$. Try and lift $K(1)$ so as to preserve $\widehat{\Theta}$.

The super Virasoro group $\mathrm{Vir}(S^{1|1})$

Now Θ fails to be $K(1)$-invariant; introduce hence the 1-form

$$\widehat{\Theta} = \Theta + dt$$

of $\widehat{K(1)} = K(1) \times \mathbb{R}$. Try and lift $K(1)$ so as to preserve $\widehat{\Theta}$.

Corollary

The group $K(1)$ of contactomorphisms of $(S^{1|1}, [\alpha])$ admits a lift as a group, $\mathrm{Vir}(S^{1|1})$, of automorphisms of $(\widehat{K(1)}, \widehat{\Theta})$, whose group law is

$$(\Phi_1, t_1)(\Phi_2, t_2) = (\Phi_1 \circ \Phi_2, t_1 + t_2 \underbrace{-\frac{1}{2}\int_{S^{1|1}} \mathcal{E}(\Phi_1 \circ \Phi_2)\mathrm{A}(\Phi_2)}_{\mathbf{BT}(\Phi_1,\Phi_2)})$$

with $\mathbf{BT} = \int_{S^{1|1}} (\mathcal{E} \smile \mathrm{A})$ the **Bott-Thurston** cocycle of $K(1)$, [Rad86].

Triple (S, GF, **BT**): super version of the “Trilogy of the moment” [Igl95].

Var. 5
The supercircle $S^{1|N}$

Var. 5
The supercircle $S^{1|N}$

- The Euclidean and affine cocycles of $K(N)$
- The Schwarzian cocycle of $K(2)$

The case of the supercircle $S^{1|N}$

For the supercircle $S^{1|N}$, endowed with the contact 1-form

$$\alpha = dx + \sum_{i,j=1}^{N} \delta_{ij}\, \xi^i d\xi^j$$

the invariants of $\mathrm{E}_+(1|N)$, $\mathrm{A}_+(1|N)$ and $\mathrm{SpO}_+(2|N)$ retain the same form as for $N = 1$. However, the odd invariant $(I_{\mathrm{p}})_1$ is now determined up to the action of $\mathrm{O}(N)$. The corresponding differential invariants are

- Euclidean cocycle $\mathcal{E} : K(N) \to \mathcal{F}_0(S^{1|N})$:

$$\mathcal{E}(\Phi) = \log E_\Phi$$

- Affine cocycle $\mathcal{A} : K(N) \to \Omega^1(S^{1|N})$:

$$\mathcal{A}(\Phi) = d\mathcal{E}(\Phi)$$

The case of the supercircle $S^{1|N}$

For the supercircle $S^{1|N}$, endowed with the contact 1-form

$$\alpha = dx + \sum_{i,j=1}^{N} \delta_{ij}\, \xi^i d\xi^j$$

the invariants of $\mathrm{E}_+(1|N)$, $\mathrm{A}_+(1|N)$ and $\mathrm{SpO}_+(2|N)$ retain the same form as for $N = 1$. However, the odd invariant $(I_\mathrm{p})_1$ is now determined up to the action of $\mathrm{O}(N)$. The corresponding differential invariants are

- Euclidean cocycle $\mathcal{E} : K(N) \to \mathcal{F}_0(S^{1|N})$:

$$\mathcal{E}(\Phi) = \log E_\Phi$$

- Affine cocycle $\mathcal{A} : K(N) \to \Omega^1(S^{1|N})$:

$$\mathcal{A}(\Phi) = d\mathcal{E}(\Phi)$$

Remark: In Cartan's formula [$\Rightarrow$ 1-cocycles of $K(N)$], $\Phi^*[t_1, t_2]$ is *no longer* proportional to $[t_1, t_2]$, up to $O(\varepsilon^3)$, for $N \geq 3$. The Schwarzian, $\mathcal{S}(\Phi)$, is therefore not given by the Cartan formula if $N \geq 3$. But ...

Theorem

The even cross-ratio $(I_p)_0$, and the Cartan formula yield the projective 1-cocycle $\mathcal{S} : K(2) \to \mathcal{Q}(\mathrm{S}^{1|2})$

$$\mathcal{S}(\Phi) = \frac{1}{6}\alpha^2 \left(D_1 D_2 \mathrm{S}_{12} + \frac{1}{2}\mathrm{S}_{12}^2 \right) + \frac{1}{2}\alpha(\beta^1 D_2 + \beta^2 D_1)\mathrm{S}_{12} + \beta^1 \beta^2 \, \mathrm{S}_{12}$$

with $\mathrm{S}_{12} = 2\,\mathrm{S}(\Phi)\alpha^{-1}$ where

$$\boxed{\mathrm{S}(\Phi) = \left(\frac{D_2 D_1 E_\Phi}{E_\Phi} - \frac{3}{2}\frac{D_2 E_\Phi D_1 E_\Phi}{E_\Phi^2} \right) \alpha}$$

The projection of quadratic differentials to 1-densities of $S^{1|2}$ returns the above Schwarzian derivative $\mathrm{S} : K(2) \to \mathcal{F}_1(\mathrm{S}^{1|2})$.

The kernels of these cocycles coincide; they are isomorphic to $\mathrm{PC}(2|2) = \mathrm{SpO}(2|2)/\{\pm \mathrm{Id}\}$.

Coda

- Classification of the geometries of $(S^{1|N}, [\alpha])$, where $N \geq 2$.
- Construction of the Bott cocycle of $K(2)$ in the same vein.
- Detailed study of the Möbius supercircle $S_+^{1|1}$.[6]
- ...

[6] Its superfunctions are defined as the smooth superfunctions of $\mathbb{R}^{1|1}$ invariant under $(x, \xi) \mapsto (x + 2\pi, -\xi)$.