TWENTY FIVE YEARS OF FRIENDSHIP WITH A MATHEMATICIAN

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Part I

First of all dear Claude, please receive my warmest congratulations on the occasion of your 60^{th} birthday ! Strangely enough, it coincides with another birthday, that of the beginning of our collaboration which turned also into a cordial friendship ! Twenty five years during which I learned a lot from you, both from the scientific and human points of view !

Mathematicians like very much keywords to set the context of their work. Let me give you, Ladies and Gentlemen, a few keywords that describe some aspects of the rich personality of our Colleague Claude.

Kindhearted. Claude has a generous and tolerant character as I noticed in many occasions, about his family, or his colleagues or his students. A proverb says : 'Far from the eyes, far from the heart' but this does not apply to Claude's friendship. We did'nt meet that much last years, but he always knows everything about me and my relatives as if we left each other only a few hours before ! He always cares about through mails or postcards or indirectly, asking common friends, giving also the latest news about his family, which he is very proud of. I am sure that he does the same with all of his friends, geographically close or not.

Sense of humour. Claude's jokes are famous in the department of mathematics of the University of Liège, where the echos of his laugther are still bouncing on the walls and in the memories. He has a formidable speciality. From time to time, he pronounces a sentence and starts to look at you with shining eyes and a slight smile in his beard. If the sentence is not directly relevant to the current subject of conversation, then be careful : it's probably a spoonerism ! A fine and subtle spoonerism ! Claude has quickly suspected me of being a very poor audience for such a game. Since then, he usually gives rapidly the appropriate permutation revealing the hidden sentence so that we can both appreciate the joke ! I told you, he is tolerant...

Teacher, Encyclopaedic. Claude is a talented lecturer. I attended a good many of his talks, always being fascinated by his great ability to draw up, however technical the subject could be, beautiful landscapes in which every notions and results naturally fit proper places and in clear relationship. Claude's talks are like *stories* that he tells with a true sense of the mise en scene and a great empathy. Their scenarios are built out of the impressive erudition of Claude who has a deep and vast mathematical knowledge as well as a lucid and quite complete vision of mathematical physics.

All these qualities are well demonstrated in the very big book devoted to the Virasoro group and algebra, written with Laurent Gieu, starting from Laurent's Ph. D. and from various lecture notes of Claude.

In fact, Claude's erudition is not limited to these two domains. He has a considerable general knowledge giving to his conversations a particularly nice flavor.

Part II

I could add obvious keywords such as *Mathematician, Researcher,...* each accompanied with some comments but I prefer to devote the second part of my talk to more scientifical considerations, restricting myself to our collaboration.

Claude and I have written eight common papers, one of which with D. Mélotte, all more or less related to a general framework : *quantization.* Most of them are therefore dealing with problems related to *formal deformation theory* of infinite dimensional algebras and, in particular, with questions leading to *computations of cohomology spaces.*

Of course, these papers only constitute a small sample of the scientifical contribution of Claude whose great curiosity led him to investigate many areas of mathematics and mathematical physics.

Arround 1984, Claude already had a great expertise in cohomology and especially cohomologies 'à la Gelfand-Fuks' related to infintite dimensional algebras. He then started to study the *current algebras* as they are related to gauge theories on the one hand and nice examples of Kirillov's local algebras that are not subalgebras of vector fields on the other hand.

At the same time, Marc De Wilde and I just achieved to build star-products over arbitrary symplectic manifolds. So, we had a certain expertise in cohomological computations and in deformation theory. In particular, we were acquainted with the relevant *graded structures*, a concept that was intended to play a more and more important role both for Claude's future research directions and mine. Moreover, my Ph. D., obtained a few years before, was devoted to a kind of current algebras.

Some preparation - A. Let me recall some facts about formal deformations. Let $\mathcal{A}$ be an associative or Lie algebra and denote by μ its product. A *formal deformation* of $\mathcal{A}$ is an associative or Lie product

$\mu_t : \mathcal{A}[[t]] \times \mathcal{A}[[t]] \to \mathcal{A}[[t]]$ of the form

$$\mu_t = \sum_{k=0}^{\infty} t^k \mu_k \tag{1}$$

where the μ_k are bilinear maps on $\mathcal{A}$ and where $\mu_0 = \mu$. It is thus a kind of Taylor expansion of a curve passing trough μ of associative or Lie products on the vector space on which μ is defined.

Of course, depending on the context, one is often led to make some assumptions about the μ's, and about the related mappings, specially when dealing with infinite dimensional algebras related to manifolds. In most cases, $\mathcal{A}$ consists then of sections of some bundle and its product is a bidifferential operator. One usually requires the $\mu's$ to be local maps. This will be understood in what follows.

Let us now explain quickly where graded structures come into play. Let V be the underlying vector space of $\mathcal{A}$. Then there are $\mathbb{Z}$-graded Lie algebras $\mathcal{L}(V)$ such that μ is an associative or Lie algebra structure on V if and only if $\mu \in \mathcal{L}^1(V)$ and

$$[\mu, \mu] = 0$$

This first gives an elegant way to handle equations on multilinear maps on V without having to explicitly write them down at the level of the arguments of the maps. For instance, to express the fact that (1) is a formal deformation, it suffices to expand the condition $[\mu_t, \mu_t] = 0$. This leads to the infinite set of equations ($k = 1, 2, 3, ...$)

$$[\mu, \mu_k] = -\frac{1}{2} \sum_{\substack{p+q=k \\ p,q>0}} [\mu_p, \mu_q] \tag{2}$$

Second, it gives a nice way to understand these equations in terms of appropriate cohomologies, opening the way toward some strategies of resolution.

Indeed, assume that $\mathcal{L}$ is a $\mathbb{Z}$-graded Lie algebra and pick out some $\alpha \in \mathcal{L}^1$ such that $[\alpha, \alpha] = 0$. It then follows from the graded version of the Jacobi identity that

$$(\mathcal{L}, \partial_\alpha := \text{ad } \alpha)$$

is a differential space and that ∂_α is a derivation so that its cohomology $\mathrm{H}^*(\mathcal{L}, \partial_\alpha)$ inherits the graded Lie algebra structure of $\mathcal{L}$.

As shown by equations (2), the cohomology space relevant to the *existence* problem of formal deformations is the *second* cohomology space. The *first* space is involved in the *classification* problem of the deformed structures. In particular, if it vanishes, then all formal deformations of μ are isomorphic (in a quite specific way, moreover). The algebra $\mathcal{A} = (V, \mu)$ is then said to be *rigid*.

Nowadays, for the associative and Lie structures, the graded algebras $\mathcal{L}(V)$ are well known and have been generalized in various ways. In the associative case, it is the *Gersternhaber* algebra $M(V)$ of V and the corresponding cohomoloy is the *Hochshild* cohomology. It is isomorphic to, but not equal to, the graded Lie algebra of derivations of the graded algebra $\otimes^* V^*$. Up to a one-unit shift of the degree, as a vector space, it is the space of multilinear mappings from V into itself. In the Lie case, it is the*Nijenhuis-Richardson* algebra $A(V)$, associated to the *Chevalley-Eilenberg* cohomology. It is isomorphic to the algebra of derivations of the exterior algebra $\wedge^* V^*$. With the same shift of degree, the underlying vector space is the space of multilinear skewsymmetric mappings of V into itself.

Due to the shift of the degree, the cohomology spaces relevant to the study of formal deformations are then the second and the third spaces of the Hochshild and Chevalley-Eilenberg cohomologies.

Some preparation - B. Having in mind to compute the μ's inductively, observe that equation (2) expresses the the coboundary of the next unknown μ_k in terms of the already computed unknowns, instead of providing μ_k itself. This explains why the study of formal deformations of an algebra can be a very difficult task!

This is that direct approach, however, that De Wilde and I used to prove the existence of star products over symplectic manifolds. Later, Fedosov proposed a very nice construction, which is *natural* in the category of symplectic manifolds equipped with a symplectic linear connection. Roughly speaking, he builds a fiber bundle over the manifold, whose fibers naturally bear a sort of Weyl algebra structure. The space of sections of the bundle is thus an associative algebra and the curvature of the connection defines in a canonical way a derivation of that algebra. As a linear space, the kernel of that derivation turns out to be isomorphic to the space of functions of the base manifold and the corresponding associative algebra structure it inherits from the kernel is shown to be a star-product.

Roughly, in our papers devoted to formal deformations, Claude and I used the straight step by step construction of the μ's, with one major exception that I shall describe here. It leads to some difficult but nice open problems. I originally designed it to provide a proof of the existence of star products over an arbitrary *Poisson* manifold, to which Fedosov's construction does not apply, but M. Kontsevich, with his celebrated formality theorem, got a proof before I could complete my program. Nevertheless, I think that the approach still deserves attention as it brings a different point of view on the subject and it opens intriguing perspectives.

Let $\mathcal{L}$ be a $\mathbb{Z}$-graded Lie algebra. The space

$$\wedge_{gr}(\mathcal{L})_q = \oplus_p \wedge_{gr}^p (\mathcal{L})_q$$

of multilinear graded skewsymmetric mappings from $\mathcal{L}$ into itself, which are homogeneous of weight q, is a differential space when equipped with the Chevalley-Eilenberg differential of the adjoint representation of $\mathcal{L}$. Let then c_t be a formal power series of cocycles belonging to $\wedge^p_{gr}(\mathcal{L})_q$, where $p+q=1$. Then for each $\mu \in \mathcal{L}^1$, the following equation

$$\frac{d\mu_t}{dt} + c_t(\mu_t, ..., \mu_t) = 0, \quad \mu_0 = \mu,$$

has a unique solution

$$\mu_t \in \mathcal{L}^1[[t]]$$

and

$$[\mu_t, \mu_t] = 0 \quad \Longleftrightarrow \quad [\mu, \mu] = 0$$

The key point here, is that *the considered cohomology is independent of the deformed structure* μ. In particular, when dealing with a smooth manifold M, it only depends on that manifold and it may be used to deform *each* Poisson structure that M could carry.

It can be shown that all 1-differential formal deformations of the Poisson bracket of a *symplectic* manifold are obtained in this way, with $q=-1$ (see below) while the equivalence problem can be solved using $q=0$. The general existence problem of star products requires to compute part of the third space of the cohomology of cochains of weight $q=-2$, and this is a very hard problem, which remains open.

1-*differential formal deformations.* We apply the above method in our paper with D. Mélotte, in order to construct 1-differentiable formal deformations of the Poisson Lie algebra of a Poisson manifold.

We first provide a preliminary abstract algebraic construction which has its own interest. It applies to a large family of graded subalgebras $\mathcal{L}$ of the Nijenhuis-Richardson algebra $A(V)$, which we call *admissible*[1]. The goal is to produce cocycles of $\wedge_{gr}(\mathcal{L})_{-1}$ using a more convenient differential space.

By restriction to $\mathcal{L}^0$, a subalgebra of $gl(V)$, any element c of that space gives an element $\theta(c)$ of the much simpler space $\wedge(\mathcal{L}^0, V)$ and the idea is to build a right inverse χ of θ. Moreover, we want it to be also *an homomorphism of differential spaces* in order to get cocycles of $\wedge_{gr}(\mathcal{L})_{-1}$ out of cocycles of $\wedge(\mathcal{L}^0, V)$. We showed that such a χ exists but is only defined on some subcomplex $\mathcal{C}_{\mathcal{L}}$ of the latter.

It happens that, V being the space of smooth functions on a smooth manifold M, $A(V)$ has preferred admissible subalgebras. One of them is the space of 1-differential and vanishing on the constant cochains. Up to one-unit shift of the gradings, it is the Schouten algebra of polyvector fields of the manifold. We observed that the de Rham complex of M is a subcomplex of $\mathcal{C}_{\mathcal{L}}$ and that for any smooth form

$$\omega = \frac{1}{p!} \sum_{i_1,\dots,i_p} \omega_{i_1,\dots,i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

1. It is not important here to know their precise definition.

the cochain $\chi(\omega)$ is given by the nice natural formula

$$\chi(\omega)(T_1, ..., T_p) = \sum_{i_1,...,i_p} \omega_{i_1,...,i_p}(\iota_{dx^{i_1}} T_1) \wedge \cdots \wedge (\iota_{dx^{i_p}} T_p)$$

Using this, we were able to construct a huge family of 1-differentiable formal deformations of the Poisson bracket of an arbitrary manifold, that exhausts them in the symplectic case.

Interesting problems related to $\mathcal{C}_{\mathcal{L}}$ are left open. For instance, it would be useful to describe it completely in the above case as well as in the case of the admissible algebra of *local* cochains.

I now will give a taste of the contents of our other common papers.

First papers. At first glance, a *current algebra* is the space of functions from a smooth manifold M into some algebra A (Lie or associative). The product in $C^\infty(M, A)$ is defined pointwise

$$uv : x \mapsto u(x)v(x)$$

More generally, one can also consider the set of sections $\Gamma^\infty(E)$ of a vector-bundle E over M having a cocycle with values in the group of automorphisms of the algebra A. Each fiber of E has then a natural structure of algebra isomorphic to A and the product of sections is again defined pointwise. For instance, E can be the bundle associated to some principal bundle P over M and to the adjoint action of its structure group G acting on its Lie algebra $\mathfrak{g}$. The corresponding current algebra, denoted $\mathfrak{g}_P$, is the kernel of the canonical short exact sequence of Lie algebras

$$0 \to \mathfrak{g}_P \to \text{aut } P \to \text{Vect}(M) \to 0 \tag{3}$$

of P, where aut P is the algebra of infinitesimal automorphisms of P and Vect(M) is the Lie algebra of vector fields of M.

Our very first common paper was devoted to the study of the deformations of $\mathfrak{g}_P$, where $\mathfrak{g}$ is simple over $\mathbb{C}$. The result is very simple, if I can say : this algebra is rigid, but the proof is not that easy as the computation of the relevant cohomology, the core of the paper, is really difficult. I shall quote the following fact (we dealt with *local* cochains) : $\text{H}^2(\mathfrak{g}_P, \mathfrak{g}_P)$ is vanishing if $\mathfrak{g}$ is not of type $A_l, l > 1$, but *it is infinite dimensional otherwise.* In other words, in the latter case, the algebra has infinitely many *infinitesimal* deformations μ_t i.e. such that

$$[\mu_t, \mu_t] = 0 \mod t^2$$

but none of them is a deformation.

Two more papers are devoted to current algebras. One concerns the Lie algebras $\mathfrak{g}_P$ with reductive type $\mathfrak{g}$ and the other describes the deformations of the associative algebra $a_n = C^\infty(M, \mathbb{K}^n_n)$ of n by n matrices with entries in the algebra, a_1, of smooth $\mathbb{K}$-valued functions on M, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

In the reductive case, the behavior of $\mathfrak{g}_P$ is drastically different from the previous one. Indeed, we showed the rather surprising fact that every formal deformation of $\mathfrak{g}_P$ is then isomorphic to a deformation of the form

$$\mu_t = \mu + ta + t^2 b$$

where the cochains a and b are built *out of a local Lie algebra* γ *of* M and a connection ω of P (a gauge field), in a way that I have no place to detail here. Let me just quote the fact that b is a function of the curvature of ω. The result follows again mainly from the explicit computation of $\mathrm{H}^2(\mathfrak{g}_P, \mathfrak{g}_P)$.

To get our description of the deformations of a_n, we proved a cohomological version of the Morita invariance principle for the Hochschild homology proved by Dennis and Igusa, and we used it to compare the spaces of equivalence [2] classes of formal deformations of a_n and a_1. More precisely, any p-linear map c from a_1 into itself extends as a p-linear map $\varphi(c)$ on a_n via

$$\varphi(c) : (u_1 \otimes A_1, ..., u_p \otimes A_p) \to c(u_1, \ldots, c_p) A_1 \cdots A_p$$

$(u_i \in a_1, A_j \in \mathbb{K}_n^n)$. First, φ is shown to induce *an isomorphism*

$$\varphi_\sharp : \mathrm{H}^*(a_1, a_1) \to \mathrm{H}^*(a_n, a_n)$$

as well as a map

$$\Phi : \mathrm{Def}(a_1) \to \mathrm{Def}(a_n)$$

between sets of equivalence classes of formal deformations. The former is then used to show that the latter is a bijection.

Algebras of vector fields. Two papers are devoted to Lie algebras of vector fields.

In the first, we computed the cohomology spaces involved in the study of the deformations of the algebra of infinitesimal automorphisms of a principal bundle. To that end, we used the Hochshild-Serre spectral sequence of (3), leading back to cohomologies of $\mathfrak{g}_P$ for which we had developped tools in our previous papers and to cohomologies 'à la Gelfand-Fuks'. Again, computations are quite difficult. In the case where the structure group G of P is simple, then aut P is rigid. For the case $G = GL(n, \mathbb{C})$, we proved that each infinitesimal deformation of $\mathfrak{g}_P$ gives rise to a complete deformation, under the assumption that the third de Rham cohomology space of the base M vanishes. I still ignore to what extend this condition is necessary.

In the second paper, we solved a problem posed by André Lichnérowicz who asked about the deformations of the Lie algebra of the unimodular vector fields over a manifold oriented by a volume element ω. This fit in his general program devoted to the study of E. Cartan's

2. Equivalence of formal deformations is a kind of isomorphism given by formal power series of linear maps. I shall not detail here.

classical infinite dimensional Lie algebras. The case of unimodular vector fields is particular because the associated sheaf $\mathcal{U}_M$ is not fine (the unimodular vector fields are not stable by multiplication by functions). This makes the computations of the relevant cohomology spaces quite difficult. One starts with a resolution [3]

$$0 \to \mathcal{U}_M \xrightarrow{i} \Omega^{m-1} \xrightarrow{d} \Omega^m \to 0$$

by fine sheaves, where $i(X) = \iota_X \omega$ and d is the de Rham differential. It leads to a short exact sequence of complexes

$$0 \to \mathcal{C}^*(\mathcal{U}_M, \mathcal{U}_M) \xrightarrow{i_*} \mathcal{C}^*(\mathcal{U}_M, \Omega^{m-1}) \xrightarrow{d_*} \mathcal{C}^*(\mathcal{U}_M, \Omega^m) \to 0$$

leading to a long exact sequence in *hypercohomology* which allows us to show, after pages of hard computations, that *if* $\dim M \neq 3$*, then*

$$H^2(\mathcal{U}_M, \mathcal{U}_M) = 0$$

In particular, the algebra of unimodular vector fields is rigid if $\dim \neq 3$. In dimension 3, the later space is one-dimensional ! However, *the algebra is still* rigid but this is far from being obvious...

A singleton. When we wrote that paper, the problem was to provide a well-behaved category of Lie bialgebra modules, having in particular a good cohomology theory.

The question in the background : *how to define modules and cohomology that seems lacking ?* has a nice answer if we think about some tool exposed above : find a graded Lie algebra $\mathcal{L}$ such that $\alpha \in \mathcal{L}^1$ is a Lie bialgebra if and only if $[\alpha, \alpha] = 0$. The adjoint cohomology is then defined by ∂_α, the left multiplication by α. This construction can be extended to get more general modules without too much pain.

The bracket we have built to characterize Lie bialgebras is now called *grand crochet*, 'big bracket', after Yvette Kosmann and her school.

Part III

The last paper I will allude to is devoted to the computation of the cohomology of some modules of the Nijenhuis-Richardson $A(V)$ for finite dimensional V. The result does not really matter at this point. My aim here is to stress important features of my collaboration with Claude.

The contribution of Claude to the above outlined results, is great. Beside being pleasant, to work with him is also quite fascinating and revealed to me wide new mathematical horizons, useful concepts and techniques. One faces some big difficulty, technical or conceptual, then after some preparation, he chooses in his wide knowledge some *deus ex machina*. For instance in this last paper it was some *coinduction lemma*. But remember the Morita invariance principle and hypercohomology, aside numerous other facts which I did'nt mentioned.

3. Ω^p is the sheaves of p-forms on M, and m is the dimension of M.

This last paper also introduced me to super geometry, which I did not really took into consideration immediately, contrary to Claude, my own route following a different direction.

So, that paper is, in a sense a branch point in our scientifical preoccupations after which the shape of our collaboration changed considerably. A triangle emerged : Lyon, Marseille and Liège each center investigating in some directions but all belonging to the same ubiquitous and multiform *quantization* and its avatars.

It is time now to evoke the name of Valentin Ovienko, one of the best mathematicians promoted by Claude, and like Claude, one of my best friends. Valentin shares with Claude the same passion for mathematical physics and proves also a great knowledge in various fields, leading him to numerous fruitful collaborations.

In particular, Valentin has intense and durable collaborations with Claude, Christian Duval and with me, building thus a strong relationship between the three poles.