"Polynomial and Elliptic Algebras of "small dimensions."

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To 60-th anniversary of my friend Claude Roger

Plan

1. Introduction
   - Poisson algebras associated to elliptic curves.
2. Examples of regular leave algebras, $n = 3$
   - Elliptic algebras
   - "Mirror transformation"
3. Heisenberg invariancy, Poisson structures on Moduli spaces, Odesskii-Feigin-Polishchuk
4. Cremona transformations and Poisson morphisms of $\mathbb{P}^4$

A Poisson structure on a manifold M(smooth or algebraic) is given by a bivector antisymmetric tensor field $\pi \in \Lambda^2(TM)$ defining on the corresponded algebra of functions on M a structure of (infinite dimensional) Lie algebra by means of the Poisson brackets

$$\{f, g\} = \langle \pi, df \wedge dg \rangle.$$

The Jacobi identity for this brackets is equivalent to an analogue of (classical) Yang-Baxter equation namely to the "Poisson Master Equation": $[\pi, \pi] = 0$, where the brackets $[,] : \Lambda^p(TM) \times \Lambda^q(TM) \mapsto \Lambda^{p+q-1}(TM)$ are the only Lie super-algebra structure on $\Lambda^{\cdot}(TM)$.

Nambu-Poisson

Let us consider $n-2$ polynomials Q_i in $\mathbb{C}^n$ with coordinates $x_i, i = 1, ..., n$. For any polynomial $\lambda \in \mathbb{C}[x_1, ..., x_n]$ we can define a bilinear differential operation

$$\{,\} : \mathbb{C}[x_1, ..., x_n] \otimes \mathbb{C}[x_1, ..., x_n] \mapsto \mathbb{C}[x_1, ..., x_n]$$

by the formula

$$\{f, g\} = \lambda \frac{df \wedge dg \wedge dQ_1 \wedge ... \wedge dQ_{n-2}}{dx_I \wedge dx_2 \wedge ... \wedge dx_n}, \quad f, g \in \mathbb{C}[x_1, ..., x_n]. \quad (1)$$

Sklyanin algebra

The case $n = 4$ in (1) corresponds to the classical (generalized) Sklyanin quadratic Poisson algebra. The very Sklyanin algebra is associated with the following two quadrics in $\mathbb{C}^4$:

$$Q_1 = x_1^2 + x_2^2 + x_3^2, \tag{2}$$

$$Q_2 = x_4^2 + J_1 x_1^2 + J_2 x_2^2 + J_3 x_3^2. \tag{3}$$

The Poisson brackets (1) with $\lambda = 1$ between the affine coordinates looks as follows

$$\{x_i, x_j\} = (-1)^{i+j} det\left(\frac{\partial Q_k}{\partial x_l}\right), l \neq i, j,\ i > j. \tag{4}$$

A wide class of the polynomial Poisson algebras arises as a quasi-classical limit $q_{n,k}(\mathcal{E})$ of the associative quadratic algebras $Q_{n,k}(\mathcal{E},\eta)$. Here $\mathcal{E}$ is an elliptic curve and n, k are integer numbers without common divisors ,such that $1 \leq k < n$ while η is a complex number and $Q_{n,k}(\mathcal{E},0) = \mathbb{C}[x_1, ..., x_n]$.

Feigin-Odesskii-Sklyanin algebras

Let $\mathcal{E} = \mathbb{C}/\Gamma$ be an elliptic curve defined by a lattice $\Gamma = \mathbb{Z} \oplus \tau\mathbb{Z}, \tau \in \mathbb{C}, \Im\tau > 0$. The algebra $Q_{n,k}(\mathcal{E}, \eta)$ has generators $x_i, i \in \mathbb{Z}/n\mathbb{Z}$ subjected to the relations

$$\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{j-i-r}(-\eta)\theta_{kr}(\eta)} x_{j-r} x_{i+r} = 0$$

and have the following properties:

Basic properties

- $Q_{n,k}(\mathcal{E},\eta) = \mathbb{C} \oplus Q_1 \oplus Q_2 \oplus \ldots$ such that $Q_\alpha * Q_\beta = Q_{\alpha+\beta}$, here $*$ denotes the algebra multiplication. The algebras $Q_{n,k}(\mathcal{E},\eta)$ are $\mathbb{Z}$ - graded;
- The Hilbert function of $Q_{n,k}(\mathcal{E},\eta)$ is $\sum_{\alpha\geq 0} \dim Q_\alpha t^\alpha = \frac{1}{(1-t)^n}$.
- $Q_{n,k}(\mathcal{E},\eta) \simeq Q_{n,k'}(\mathcal{E},\eta)$, if $kk' \equiv 1 \pmod n$;
- The maps $x_i \mapsto x_{i+1}$ et $x_i \mapsto \varepsilon^i x_i$, where $\varepsilon^n = 1$, define automorphisms of the algebra $Q_{n,k}(\mathcal{E},\eta)$;
- We see that the algebra $Q_{n,k}(\mathcal{E},\eta)$ for fixed $\mathcal{E}$ is a flat deformation of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$.

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$Q_{2m,1}(\mathcal{E}, \eta)$ as an ACIS

A. Odesskii and V.R prove in 2004 the following

Theorem

The elliptic algebra $Q_{2m,1}(\mathcal{E}, \eta)$ has m commuting elements of degree m.

Let $q_{n,k}(\mathcal{E})$ be the correspondent Poisson algebra. The algebra $q_{n,k}(\mathcal{E})$ has $l = \gcd(n, k+1)$ Casimirs. Let us denote them by $P_\alpha, \alpha \in \mathbb{Z}/l\mathbb{Z}$. Their degrees $\deg P_\alpha$ are equal to n/l.

Quintic elliptic Poisson algebra

Let us consider the algebra $q_{5,1}(\mathcal{E})$:

Example

We have the polynomial ring with 5 generators $x_i, i \in \mathbb{Z}/5\mathbb{Z}$ enabled with the following Poisson bracket:

$$\begin{aligned} \{x_i, x_{i+1}\}_{5,1} &= \left(-\frac{3}{5}k^2 + \frac{1}{5k^3}\right) x_i x_{i+1} - 2\,\frac{x_{i+4}x_{i+2}}{k} + \frac{{x_{i+3}}^2}{k^2} \\ \{x_i, x_{i+2}\}_{5,1} &= \left(-\frac{1}{5}k^2 - \frac{3}{5k^3}\right) x_{i+2} x_i + 2\, x_{i+3} x_{i+4} - k\, {x_{i+1}}^2 \end{aligned} \quad (5)$$

Here $i \in \mathbb{Z}/5\mathbb{Z}$ and $k \in \mathbb{C}$ is a parameter of the curve $\mathcal{E}_\tau = \mathbb{C}/\Gamma$, i.e. some function of τ.

Casimir of degree 5

The center $Z(q_{5,1}(\mathcal{E}))$ is generated by the polynomial

$$P_{5,1} = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 +$$

$$(1/k^4 + 3k)\left(x_0^3 x_1 x_4 + x_1^3 x_0 x_2 + x_2^3 x_1 x_3 + x_3^3 x_2 x_4 + x_2^3 x_0 x_3\right) +$$

$$+\left(-k^4 + 3/k\right)(x_0^3 x_2 x_3 + x_1^3 x_3 x_4 + x_2^3 x_0 x_4 + x_3^3 x_1 x_0 + x_4^3 x_1 x_2) +$$

$$+(2k^2 - 1/k^3)\left(x_0 x_1^2 x_4^2 + x_1 x_2^2 x_0^2 + x_2 x_0^2 x_4^2 + x_3 x_1^2 x_0^2 + x_4 x_1^2 x_2^2\right) +$$

$$+(k^3 + 2/k^2)\left(x_0 x_2^2 x_3^2 + x_1 x_3^2 x_4^2 + x_2 x_0^2 x_4^2 + x_3 x_1^2 x_0^2 + x_4 x_1^2 x_2^2\right) +$$

$$+(k^5 - 16 - 1/k^5) x_0 x_1 x_2 x_3 x_4.$$

It is easy to check that for any $i \in \mathbb{Z}/5\mathbb{Z}$

$$\{x_{i+1}, x_{i+2}\}\{x_{i+3}, x_{i+4}\} + \{x_{i+3}, x_{i+1}\}\{x_{i+2}, x_{i+4}\}+$$

$$+\{x_{i+2}, x_{i+3}\}\{x_{i+1}, x_{i+4}\} = 1/5\frac{\partial P}{\partial x_i}.$$

Second elliptic Poisson structure for $n = 5$

It follows from the description of Odesskii-Feigin that there are two essentially different elliptic algebras with 5 generators: $Q_{5,1}(\mathcal{E},\eta)$ and $Q_{5,2}(\mathcal{E},\eta')$. The corresponding Poisson counter-part of the latter is $q_{5,2}(\mathcal{E})$:

Example

$$\begin{aligned} \{y_i, y_{i+1}\}_{5,2} &= \left(\frac{2}{5}\lambda^2 + \frac{1}{5\lambda^3}\right) y_i y_{i+1} + \lambda y_{i+4} y_{i+2} - \frac{{y_{i+3}}^2}{\lambda} \\ \{y_i, y_{i+2}\}_{5,2} &= \left(-\frac{1}{5}\lambda^2 + \frac{2}{5\lambda^3}\right) y_{i+2} y_i - \frac{y_{i+3} y_{i+4}}{\lambda^2} + {y_{i+1}}^2 \end{aligned} \tag{6}$$

where $i \in \mathbb{Z}_5$. The center $Z(q_{5,2}(\mathcal{E}) = \mathbb{C}[P_{5,2}]$.'

Artin-Tate elliptic Poisson algebra

Let

$$P(x_1, x_2, x_3) = 1/3(x_1^3 + x_2^3 + x_3^3) + kx_1x_2x_3, \tag{7}$$

then

$$\begin{aligned} \{x_1, x_2\} &= kx_1x_2 + x_3^2 \\ \{x_2, x_3\} &= kx_2x_3 + x_1^2 \\ \{x_3, x_1\} &= kx_3x_1 + x_2^2. \end{aligned}$$

The quantum counterpart of this Poisson structure is the algebra $Q_3(\mathcal{E}, \eta)$, where $\mathcal{E} \subset \mathbb{C}P^2$ is an elliptic curve given by $P(x_1, x_2, x_3) = 0$.

Non-algebraic Poisson transformation

The interesting feature of this algebra is that their polynomial character is preserved even after the following changes of variables: Let

$$y_1 = x_1, y_2 = x_2 x_3^{-1/2}, y_3 = x_3^{3/2}. \tag{8}$$

The polynomial P in the coordinates (y_1, y_2, y_3) has the form

$$P^{\vee}(y_1, y_2, y_3) = 1/3 \left(y_1^3 + y_2^3 y_3 + y_3^2\right) + k y_1 y_2 y_3 \tag{9}$$

Second elliptic singularity normal form

The Poisson bracket is also polynomial (which is not evident at all!) and has the same form:
$\{y_i, y_j\} = \frac{\partial P^\vee}{\partial y_k}$, where $(i, j, k) = (1, 2, 3)$.
Put $\deg y_1 = 2, \deg y_2 = 1, \deg y_3 = 3$ then the polynomial $P^\vee$ is also homogeneous in (y_1, y_2, y_3) and defines an elliptic curve $P^\vee = 0$ in the weighted projective space $\mathbb{W}P_{2,1,3}$.

Second "mirror" - third elliptic normal form

Now let $z_1 = x_1^{-3/4} x_2^{3/2}, z_2 = x_1^{1/4} x_2^{-1/2} x_3, z_3 = x_1^{3/2}$.
The polynomial P in the coordinates (z_1, z_2, z_3) has the form $P(z_1, z_2, z_3) = 1/3\left(z_3^2 + z_1^2 z_3 + z_1 z_2^3\right) + kz_1 z_2 z_3$ and the Poisson bracket is also polynomial (which is not evident at all!) and has the same form: $\{z_i, z_j\} = \frac{\partial P}{\partial z_k}$, where $(i, j, k) = (1, 2, 3)$.
Put $\deg z_1 = 1, \deg z_2 = 1, \deg z_3 = 2$ then the polynomial P is also homogeneous in (z_1, z_2, z_3) and defines an elliptic curve $P = 0$ in the weighted projective space $\mathbb{W}P_{1,1,2}$.

Comments

The origins of the strange non-polynomial change of variables (8) lie in the construction of "mirror" dual Calabi - Yau manifolds and the torus (7) has (9) as a "mirror dual". Of course, the mirror map is trivial for 1-dimensional Calabi - Yau manifolds. Curiously, mapping (8) being a Poisson map if we complete the polynomial ring in a proper way and allow the non - polynomial functions gives rise to a new "relation" on quantum level: the quantum elliptic algebra $Q_3(\mathcal{E}^\vee)$ corresponded to (9) has complex structure $(\tau + 1)/3$ when (7) has τ. Hence, these two algebras are different. The "quantum" analogue of the mapping (8) is still obscure and needs further studies.

Heisenberg group

Consider an $n-$dimensional vector space V and fixe a base $v_0, \ldots, v_{n-1}$ of V then the Heisenberg group of level n in the Schrödinger representaion is the subgroup $H_n \subset GL(V)$ generated by the operators

$$\sigma : (v_i) \to v_{i-1}; \qquad \tau : v_i \to \varepsilon_i v_i, \, (\varepsilon_i)^n = 1, 0 \leq i \leq n-1.$$

This group has order n^3 and is a central extension

$$1 \to \mathbb{U}_n \to H_n \to \mathbb{Z}_n \times \mathbb{Z}_n \to 1,$$

where $\mathbb{U}_n$ is the group of $n-$th roots of unity.

This action provides the automorphisms of the Sklyanin algebra which are compatible with the grading and defines also an action on the "quasiclassical" limit of the Sklyanin algebras $q_{n,k}(\mathcal{E})$- the elliptic quadratic Poisson structures on $\mathbb{P}^{n-1}$ which are identified with Poisson structures on some moduli spaces of the degree n and rank $k+1$ vector bundles with parabolic structure (= the flag $0 \subset F \subset \mathbb{C}^{k+1}$ on the elliptic curve $\mathcal{E}$)

Odesskii-Feigin description-1

Odesskii-Feigin(1995-2000):
Let $\mathcal{M}_{n,k}(\mathcal{E}) = \mathcal{M}(\xi_{0,1}, \xi_{n,k})$ be the moduli space of $k+1$-dimensional bundles on the elliptic curve $\mathcal{E}$ with 1-dimensional sub-bundle. $\xi_{0,1} = \mathcal{O}_{\mathcal{E}}$, $\xi_{n,k}$ - indecomposable bundle of degree n and rank k. This moduli space is a space of exact sequences:

$$0 \longrightarrow \xi_{0,1} \longrightarrow F \longrightarrow \xi_{n,k} \longrightarrow 0$$

up to an isomorphism.

Odesskii-Feigin description-2

Theorem

$$\mathcal{M}_{n,k}(\mathcal{E}) \cong \mathbb{P}Ext^1(\xi_{n,k}; \xi_{0,1}) \cong \mathbb{C}P^{n-1}.$$

The Poisson structure $q_{n,k}(\mathcal{E})$ in the "classical limit" ($\eta \to 0$) of $Q_{n,k}(\mathcal{E}, \eta)$ is a homogeneous quadratic on $\mathbb{C}^n$ and define a Poisson structure on $\mathbb{C}P^{n-1}$ which coincides with the intrinsic Poisson structure on the moduli space of parabolic bundles $\mathcal{M}_{n,k}(\mathcal{E})$.

Polishchuk description-1

A. Polishchuk(1999-2000):
There exists a natural Poisson structure on the moduli space of triples (E_1, E_2, Φ) of stable vector bundles over $\mathcal{E}$ with fixed ranks and degrees, where $\Phi : E_2 \to E_1$ a homomorphism. For $E_2 = \mathcal{O}_{\mathcal{E}}$ and $E_1 = E$ this structure is exactly the Odesskii-Feigin structure on $\mathbb{P}Ext^1(E, \mathcal{O}_{\mathcal{E}})$.

Polishchuk description-2

Theorem

Let $\mathcal{M}_{n,k}(\mathcal{E}) \cong \mathbb{P}Ext^1(E, \mathcal{O}_{\mathcal{E}})$ where E is a stable bundle with fixed determinant $\mathcal{O}(nx_0)$ of rank k, $(n, k) = 1$. Suppose in addition that $(n+1, k) = 1$. Then there is a birational transformation (compatible with Poisson structures = "birational Poisson morphism")

$$\mathcal{M}_{n,k}(\mathcal{E}) \to \mathcal{M}_{n,\phi(k):=-(k+1)^{-1}}(\mathcal{E}) \cong \mathbb{P}H^0(F),$$

where F is a stable vector bundle of degree n and rank $k+1$. Moreover, the composition

$$\mathcal{M}_{n,k}(\mathcal{E}) \to \mathcal{M}_{n,\phi(k)}(\mathcal{E}) \to \mathcal{M}_{n,\phi^2(k)}(\mathcal{E}) \to \mathcal{M}_{n,k}(\mathcal{E})$$

is the identity.

Odesskii-Feigin "quantum" homomorphisms for 5-generator algebras

Let $Q_{5,1}(\mathcal{E},\eta)$ and $Q_{5,2}(\mathcal{E},\eta)$ be "quantum" elliptic Sklyanin algebras corresponded to $q_{5,1}(\mathcal{E})$ and to $q_{5,2}(\mathcal{E})$.

Example

(Odesskii-Feigin,1988)

- The algebra $Q_{5,2}(\mathcal{E},\eta)$ is a subalgebra in $Q_{5,1}(\mathcal{E},\eta)$ generated by 5 elements with 10 quadratic relations.
- In its turn, the algebra $Q_{5,1}(\mathcal{E},\eta)$ is a subalgebra in $Q_{5,2}(\mathcal{E},\eta)$.
- The compositions of embeddings $Q_{5,1}(\mathcal{E},\eta) \to Q_{5,2}(\mathcal{E},\eta) \to Q_{5,1}(\mathcal{E},\eta)$ transforms the generators $x_i \to P_{5,1}x_i$ and $Q_{5,2}(\mathcal{E},\eta) \to Q_{5,1}(\mathcal{E},\eta) \to Q_{5,2}(\mathcal{E},\eta)$ transforms the generators $y_i \to P_{5,2}y_i$.

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General (naive) definitions

Consider $n+1$ homogeneous polynomial functions φ_i in $\mathbb{C}[x_0, \cdots, x_n]$ of the same degree which are non identically zero. One can associated the rational map:

$$\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n, [x_0 : \cdots : x_n] \mapsto [\varphi_0([x_0, \cdots, x_n) : \cdots : \varphi_n([x_0, \cdots, x_n)].$$

The family of polynomial φ_i or φ is called a birational transformation of $\mathbb{P}^n$ if there exists a rational map $\psi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ such that $\psi \circ \varphi$ is the identity. A birational transformation is also called a Cremona transformation.

Let $(\lambda : \mu) \in \mathbb{P}^1$ such that $k = \lambda/\mu$ and $\mathcal{E}_{\lambda,\mu}$ is given by the set of the quadrics

$$C_i^0(x) = \lambda\mu x_i^2 - \lambda^2 x_{i+2}x_{i+3} + \mu^2 x_{i+1}x_{i+4}, \qquad i \in \mathbb{Z}_5, \qquad (10)$$

(These quadrics are 4x4 Pfaffians of the Klein syzygy 5x5 skew-symmetric matrix of linear forms.)

The elliptic quintic scroll $Q_{\lambda,\mu}(z)$ is given by the set of cubics

$$Q_i^0(z) = \lambda^2\mu^2 z_i^3 + \lambda^3\mu(z_{i+1}^2 z_{i+3} + z_{i+2} z_{i+4}^2) - \lambda\mu^3(z_{i+1} z_{i+2}^2 + z_{i+3}^2 z_{i+4}) - \\ -\lambda^4 z_i z_{i+1} z_{i+4} - \mu^4 z_i z_{i+2} z_{i+3}, \qquad i \in \mathbb{Z}_5. \tag{11}$$

The transformation $v_\pm : \mathbb{P}^4(x) \mapsto \mathbb{P}^4(z)$ is given in coordinates by
$v_+ : z_i \to x_{i+2}x_{i+4}^2 - x_{i+1}^2 x_{i+3}$ and by
$v_- : z_i \to x_{i+1}x_{i+2}^2 - x_{i+3}^2 x_{i+4}$.
The incidence variety $I_{\lambda,\mu}$ is the elliptic scroll over the curve $\mathcal{E}_{\lambda,\mu}$ which is transformed by the Cremona transformation to the scroll $S \subset \mathbb{P}^4(w)$.

Theorem

- *The quadro-cubic Cremona transformations (10) and (11) are Poisson morphisms of $\mathbb{P}^4$ which transform $q_{5,1}(\mathcal{E})$ to $q_{5,2}(\mathcal{E})$ and vice versa.*
- *These Cremona transformations are "quasi-classical limits" of Odesskii-Feigin "quantum" homomorphisms $Q_{5,1}(\mathcal{E}, \eta) \to Q_{5,2}(\mathcal{E}, \eta)$ and vice versa.*
- *The Casimir quintics $P_{5,1,2}$ are given by Jacobians of these quadro-cubic Cremona transformations (11,10) and their zero levels $P_{5,1,2} = 0$ are Calabi-Yau 3-folds in $\mathbb{P}^4$.*

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- *The quadro-cubic Cremona transformations (10) and (11) are Poisson morphisms of* $\mathbb{P}^4$ *which transform* $q_{5,1}(\mathcal{E})$ *to* $q_{5,2}(\mathcal{E})$ *and vice versa.*
- *These Cremona transformations are "quasi-classical limits" of Odesskii-Feigin "quantum" homomorphisms* $Q_{5,1}(\mathcal{E}, \eta) \rightarrow Q_{5,2}(\mathcal{E}, \eta)$ *and vice versa.*
- *The Casimir quintics* $P_{5,1,2}$ *are given by Jacobians of these quadro-cubic Cremona transformations (11,10) and their zero levels* $P_{5,1,2} = 0$ *are Calabi-Yau 3-folds in* $\mathbb{P}^4$.

Cremona transformations in $\mathbb{P}^4$

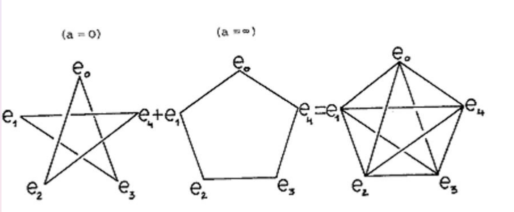
$$
\begin{tikzcd}
 & & & \mathbb{P}^4(x) \times \mathbb{P}^4(z) \arrow[rr, dashed, "\Phi"] & & \mathbb{P}^4(w) \subset \mathbb{P}^{14} \\
 & & I_{\lambda,\mu} \arrow[ur, hook] \arrow[rrr, "\Phi"] \arrow[d, "\Phi"'] \arrow[dr] & & & S_{\lambda,\mu}(w) \arrow[u, hook] \\
\mathrm{graph}(v_+) = \Gamma_{\lambda,\mu} \arrow[urr, hook] \arrow[rrr] & & \mathcal{E}_{\lambda,\mu}(x) \arrow[dr] & \mathcal{E}_{\lambda,\mu}(w) \arrow[urr, hook] & & Q_{\lambda,\mu}(z) \approx S^2(\mathcal{E}_{\lambda,\mu}) \arrow[u] \arrow[lll] \\
\mathcal{E}_{\lambda,\mu}(x) \arrow[u] \arrow[urr, equal] \arrow[rrr, "v_+"] \arrow[d, hook] & & & \mathcal{E}'_{\lambda,\mu}(z) \arrow[u] \arrow[urr, hook] \arrow[d, hook] & & \\
S_{15} \subset \mathbb{P}^4(x) \arrow[rrr, dashed, "v_+"] & & & S_{45} \subset \mathbb{P}^4(z) & &
\end{tikzcd}
$$

The conditions under which a general Cremona transformation 10 on $\mathbb{P}^4$ gives the Poisson morphism from $q_{5,1}(\mathcal{E})$ to some $H-$invariant quadratic Poisson algebra read like the following algebraic system:

$$\begin{cases} -a^3\lambda + 4\,\lambda^4 a + 2\,\lambda^5 a^2 + 2\,\lambda^3 - 2\,a^2 + a^6\lambda^4 = 0 \\ -1 + 2\,a^2\lambda^2 - a^3\lambda^3 + 2\,a\lambda = 0 \end{cases} \tag{12}$$

The system has two classes of solutions: $a\lambda = -1$ and $a = \frac{3\pm\sqrt{5}}{2\lambda}$ for each λ satisfies to the equation $\lambda^{10} + 11\lambda - 1 = 0$.

These exceptional solutions correspond to the vertexes of the Klein icosahedron inside $\mathbb{S}^2 = \mathbb{P}^1$ and the associated singular curves forms pentagons (the following figures belong to K. Hulek):



Each pentagon corresponds to a degeneration of the Odesskii-Feigin-Sklyanin algebra $q_{5,2}(\mathcal{E})$ which is (presumably) new examples of $H-$invariant quadratic Poisson structures on $\mathbb{C}^5$.

FIN

THANK YOU FOR YOUR ATTENTION!