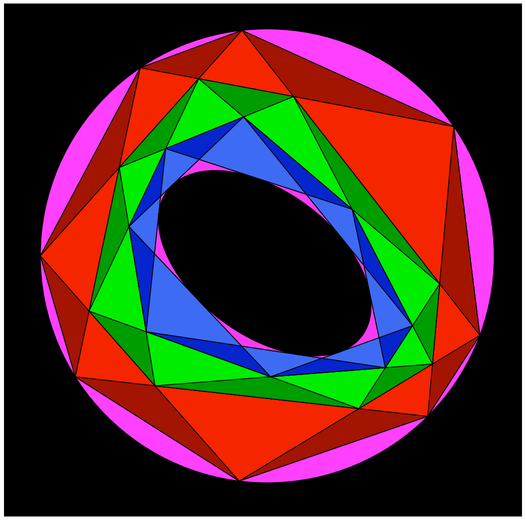
Pentagramma Mirificum
old wine into new wineskins

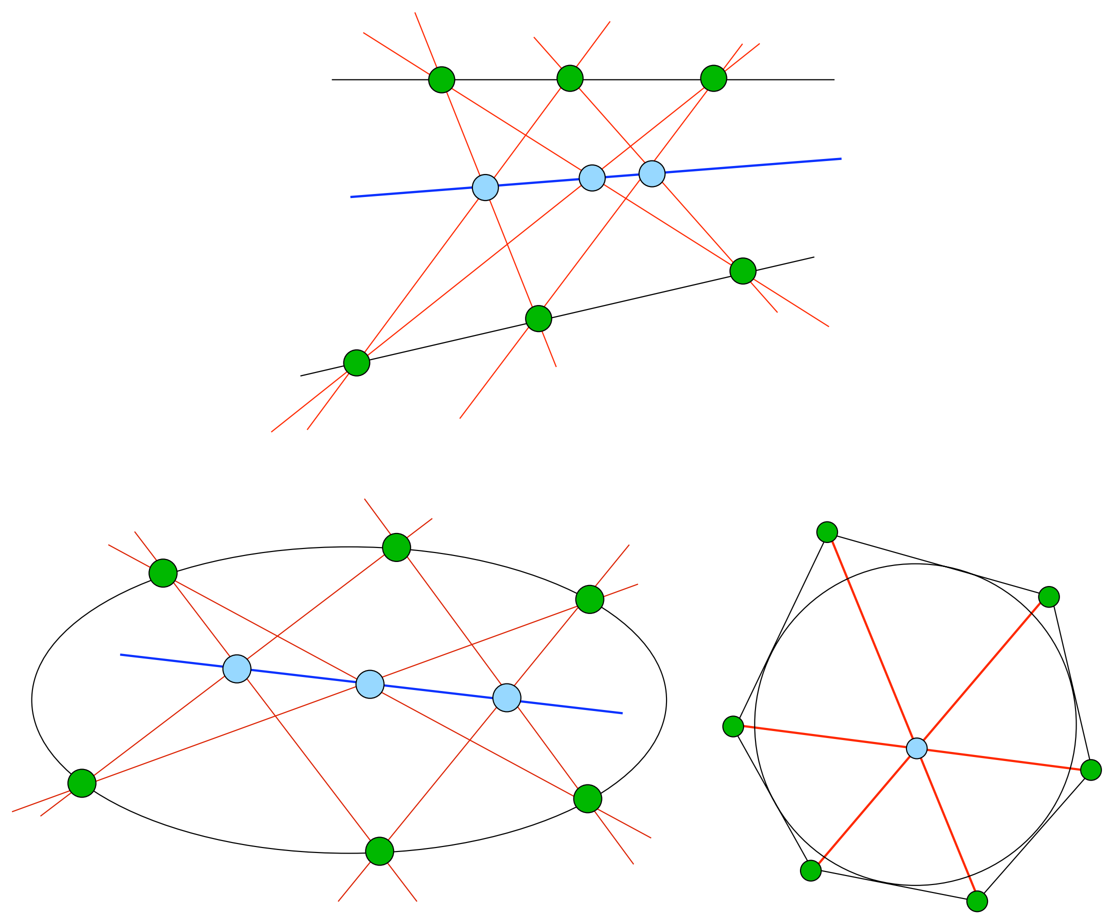
R. Schwartz and S. Tabachnikov

(arXiv:0910.1952; Math. Intelligencer, to appear)

Lyon, November 2009, Claude's B-Day



Three classical theorems, Pappus ($\sim$ 300 A.D.), Pascal (1639) and Briançon (1806):



Concerning the subtitle, a New Testament parable, in Luke's version, reads:

> And no one puts new wine into old wineskins; otherwise the new wine will burst the skins and will be spilled, and the skins will be destroyed. But new wine must be put into fresh wineskins. And no one after drinking old wine desires new wine, but says, "The old is good".

We discovered eight (new?) configuration theorems of projective geometry ('old wine') by computer experimentation ('new wineskins'); most of the proofs are also by a computer algebra system. And we believe, this old (wine) is good.

A *polygon* $P \subset \mathbf{RP}^2$ is a cyclically ordered collection $\{p_1, ..., p_n\}$ of points, its vertices. The sides $\{l_1, ..., l_n\}$ are the lines $l_i = \overline{p_i p_{i+1}}$ in $\mathbf{RP}^2$. The *dual polygon* $P^* \subset (\mathbf{RP}^2)^*$ has vertices $\{l_1, ..., l_n\}$; the sides of the dual polygon are $\{p_1, ..., p_n\}$ (considered as lines in $(\mathbf{RP}^2)^*$. One has: $(P^*)^* = P$.

Let $\mathcal{P}_n$ and $\mathcal{P}_n^*$ be the sets of n-gons in $\mathbf{RP}^2$ and $(\mathbf{RP}^2)^*$. Define the k-diagonal map $T_k : \mathcal{P}_n \to \mathcal{P}_n^*$: for $P = \{p_1, ..., p_n\}$, let

$$T_k(P) = \{\overline{p_1 p_{k+1}}, \overline{p_2 p_{k+2}}, \ldots, \overline{p_n p_{k+n}}\}.$$

Then T_k is an *involution*. The map T_1 is the projective duality.

Extend the notation: $T_{ab} = T_a \circ T_b, T_{abc} = T_a \circ T_b \circ T_c$, and so on.

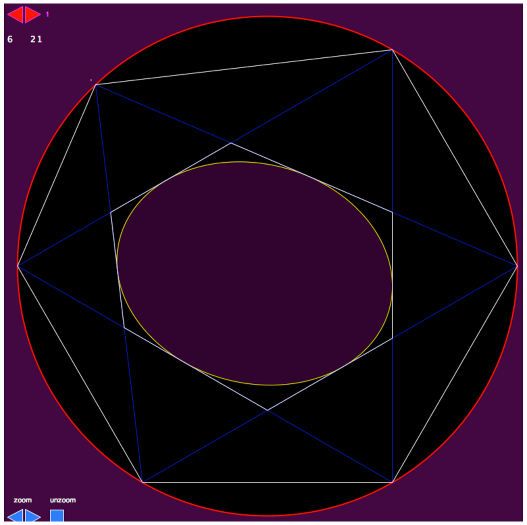
Theorem 1:

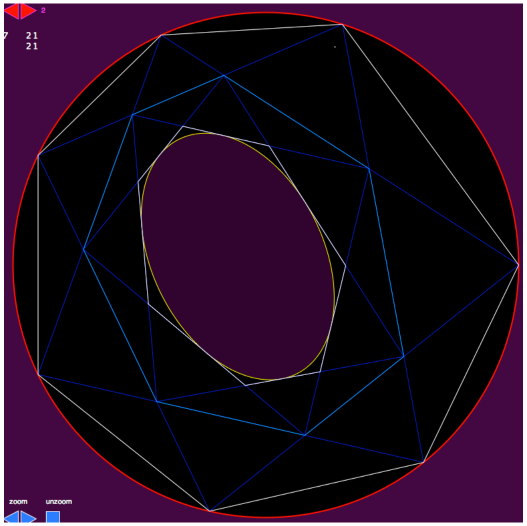
(i) If P is an inscribed 6-gon, then $P \sim T_2(P)$.

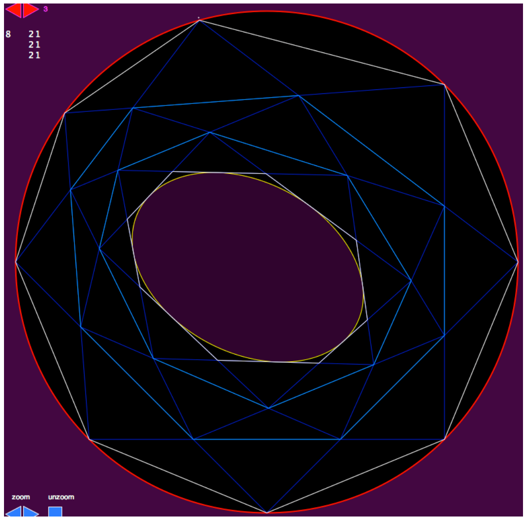
(ii) If P is an inscribed 7-gon, then $P \sim T_{212}(P)$.

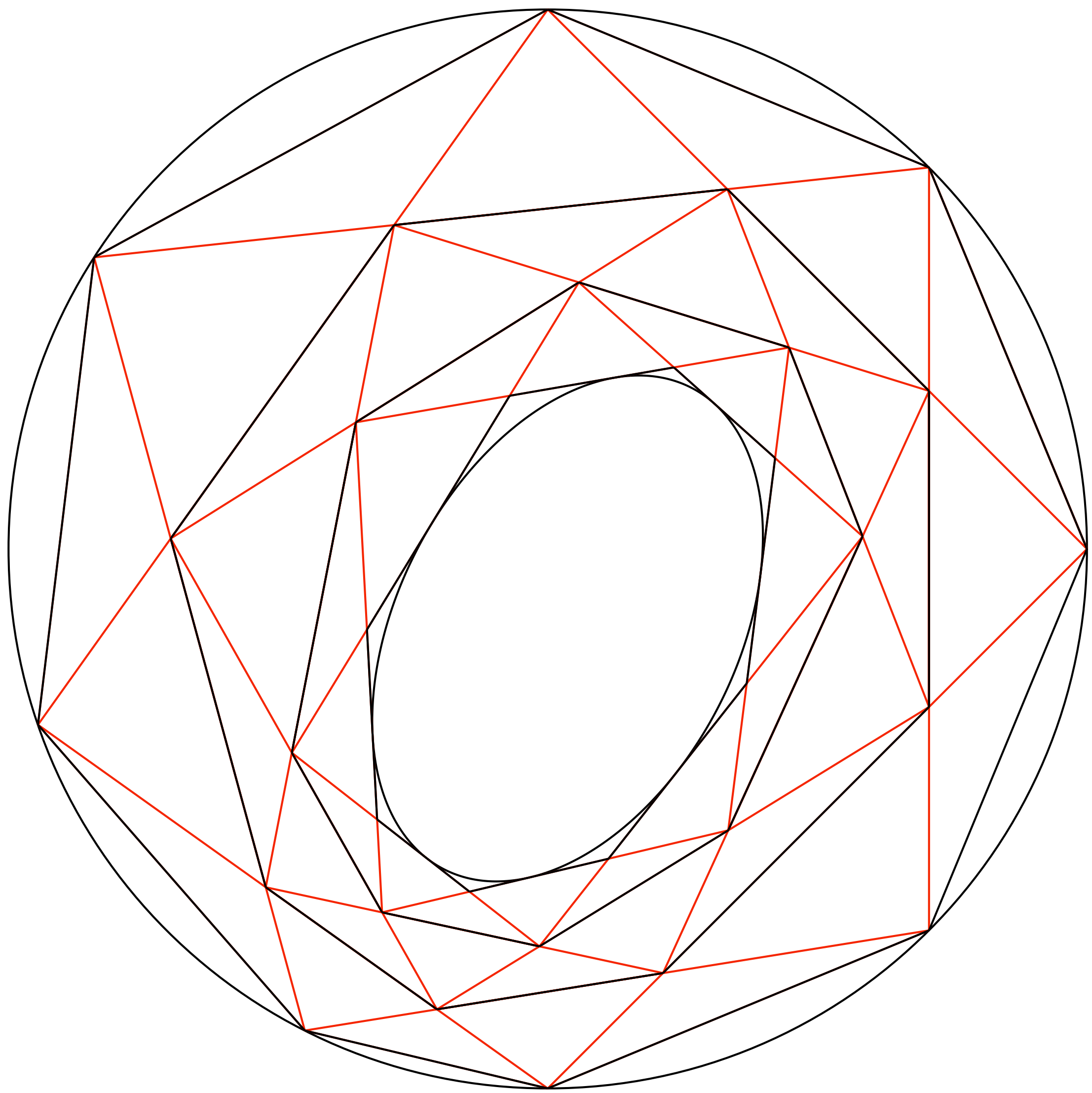
(iii) If P is an inscribed 8-gon, then $P \sim T_{21212}(P)$.

(and it doesn't continue!)



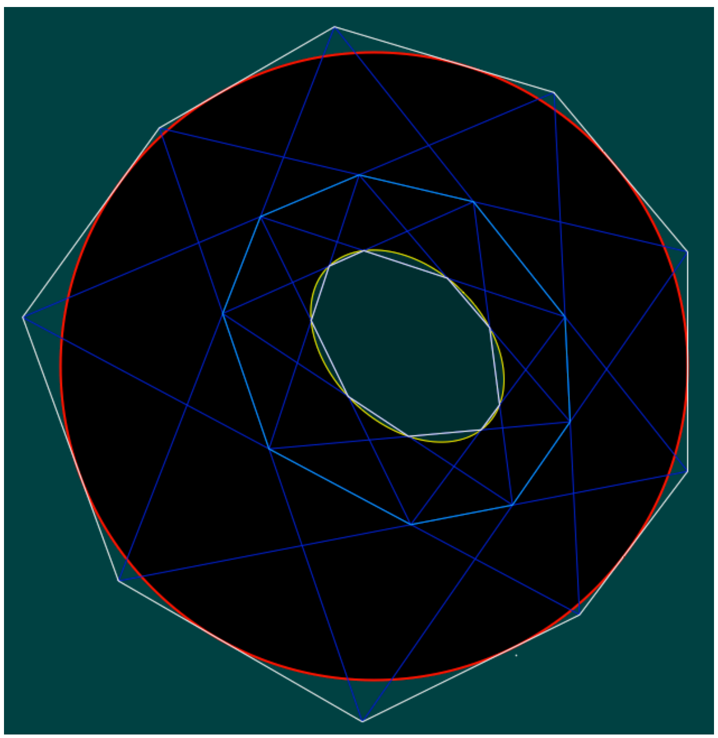


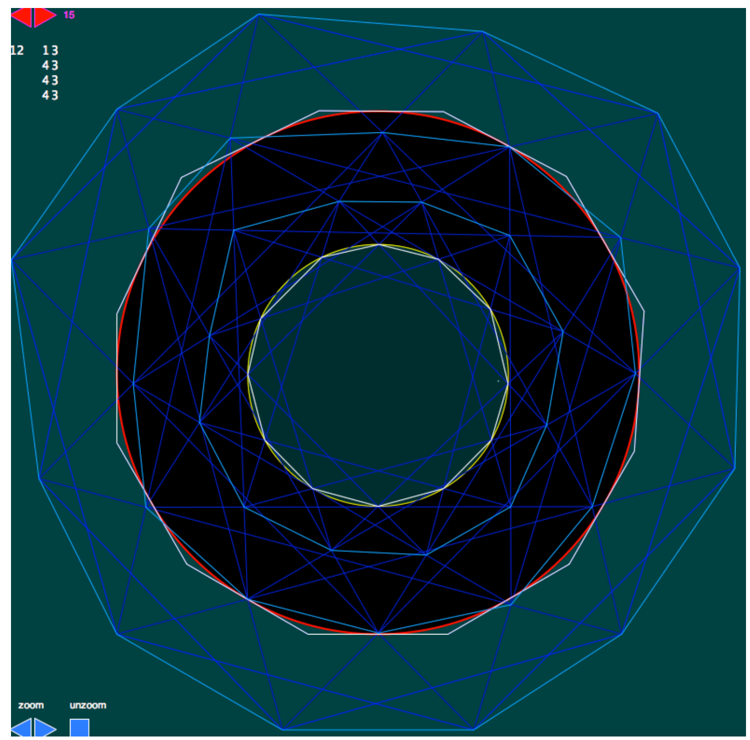




Theorem 2: If P is a circumscribed 9-gon, then $P \sim T_{313}(P)$.

Theorem 3: If P is an inscribed 12-gon, then $P \sim T_{3434343}(P)$.





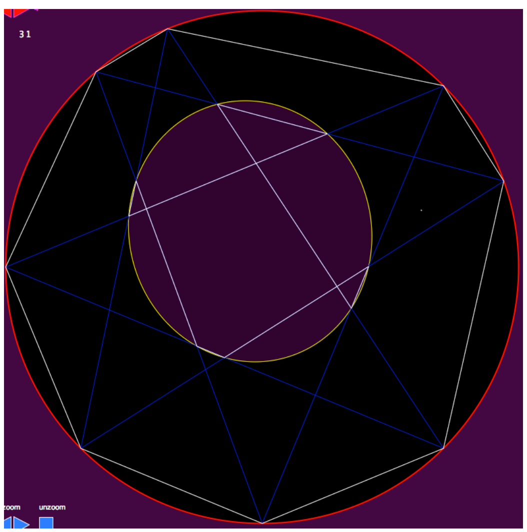
Theorem 4:

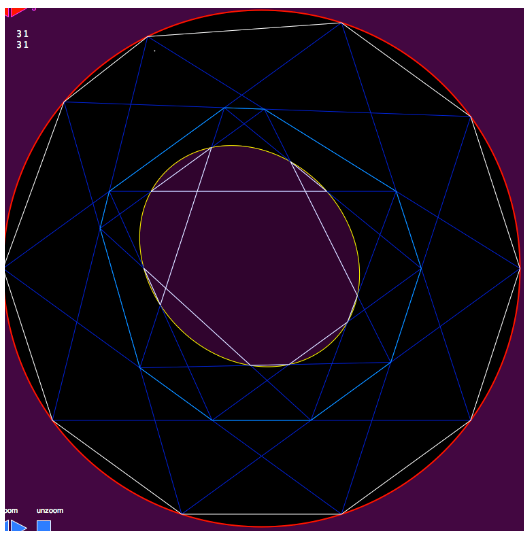
(i) If P is an inscribed 8-gon, then $T_3(P)$ is circumscribed.

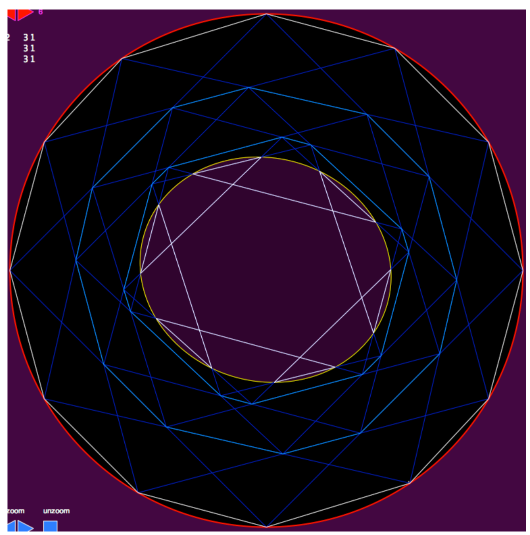
(ii) If P is an inscribed 10-gon, then $T_{313}(P)$ is circumscribed.

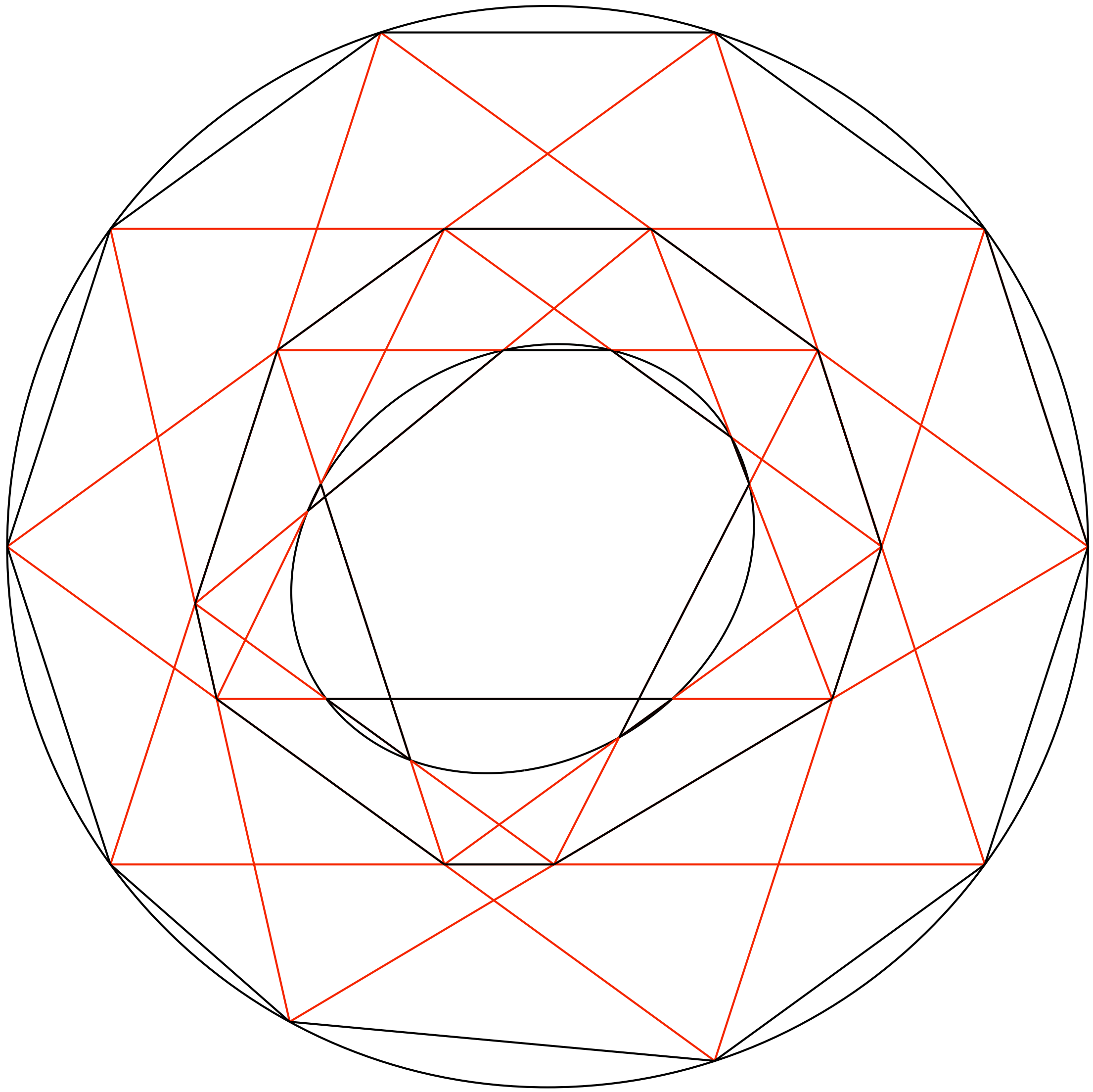
*(iii)** If P is an inscribed 12-gon, then $T_{31313}(P)$ is circumscribed.

(and it doesn't continue either!)









Odds and Ends

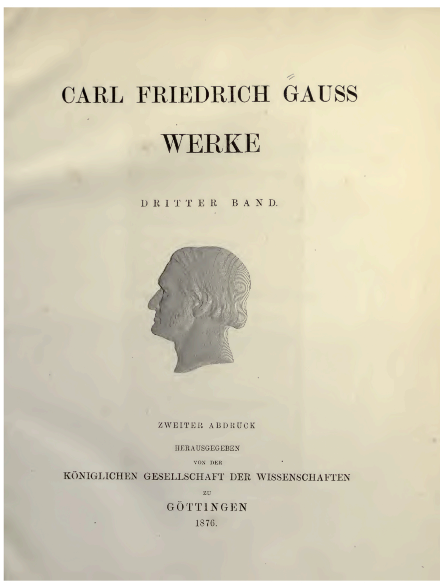
(i) The Pentagram Map: T_{12}.

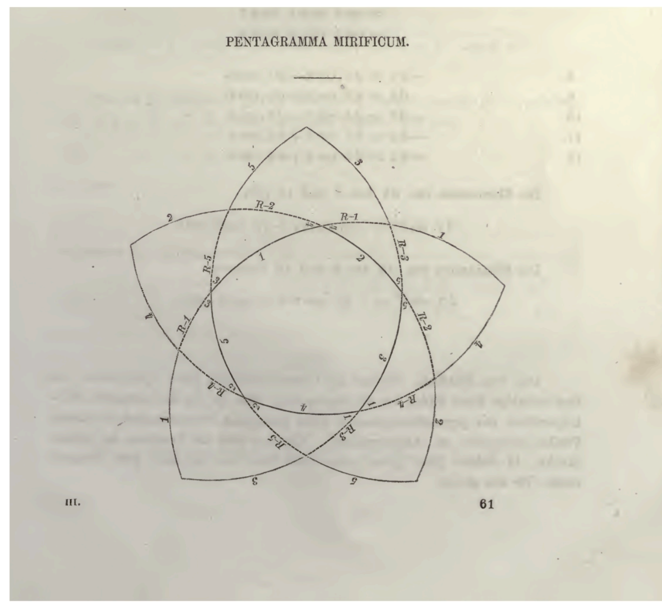
Studied by R. Schwartz in
The pentagram map, Experim. Math. **1** (1992), 71–81;
The pentagram map is recurrent, Experim. Math. **10** (2001), 519–528;
Discrete monodromy, pentagrams, and the method of condensation, J. Fixed Point Th. Appl. **3** (2008), 379–409;

and by V. Ovsienko, R. Schwartz and S. Tabachnikov in
Quasiperiodic motion for the pentagram map, E.R.A. **16**, 2009, 1–8;
The pentagram map: a discrete integrable system, arXiv:0810.5605.

(ii) Pentagons:
1) Every pentagon is inscribed and circumscribed.
2) Every pentagon is projectively equivalent to its dual.
3) The pentagram map is the identity: $T_{12}(P) = P$.
Thus 5-gons can be added to Theorem 1.

C.-F. Gauss studied self-polar pentagons in:





(iii) Palindroms: in all the theorems, the words w are palindromic and the maps T_w are involutions.

(iv) Self-dual polygons: are classified in
D. Fuchs, S. Tabachnikov. *Self-dual polygons and self-dual curves*, Funct. Anal. & Other Math., 2 (2009), 203–220.

For odd n, the space of projective equivalence classes is $n-3$-dimensional, just like the moduli space of inscribed (or circumscribed) n-gons. Theorem 1 (ii) can be rephrased:

P is an inscribed heptagon iff $T_2(P)$ is projectively self-dual.

Theorem 2 can be also rephrased:

P is an inscribed nonagon iff $T_3(P)$ is projectively self-dual.

(v) Relabeling: Relabel the vertices of a dodecagon as follows: $i \mapsto 5i$. Then Theorem 4 (iii), which reads:
If P is an inscribed dodecagon then $T_{131313}(P)$ is also inscribed, transforms to:
If P is an inscribed dodecagon then $T_{535353}(P)$ is also inscribed.

And so on.

(vi) A similar theorem: R. Schwartz has a theorem in *Discrete monodromy, pentagrams, and the method of condensation* that can be rephrased as follows:

If P is a $4n$-gon inscribed into a degenerate conic then

$$(T_1 T_2 T_1 T_2 \ldots T_1)(P) \qquad (4n-3 \text{ terms})$$

is also inscribed into a degenerate conic.

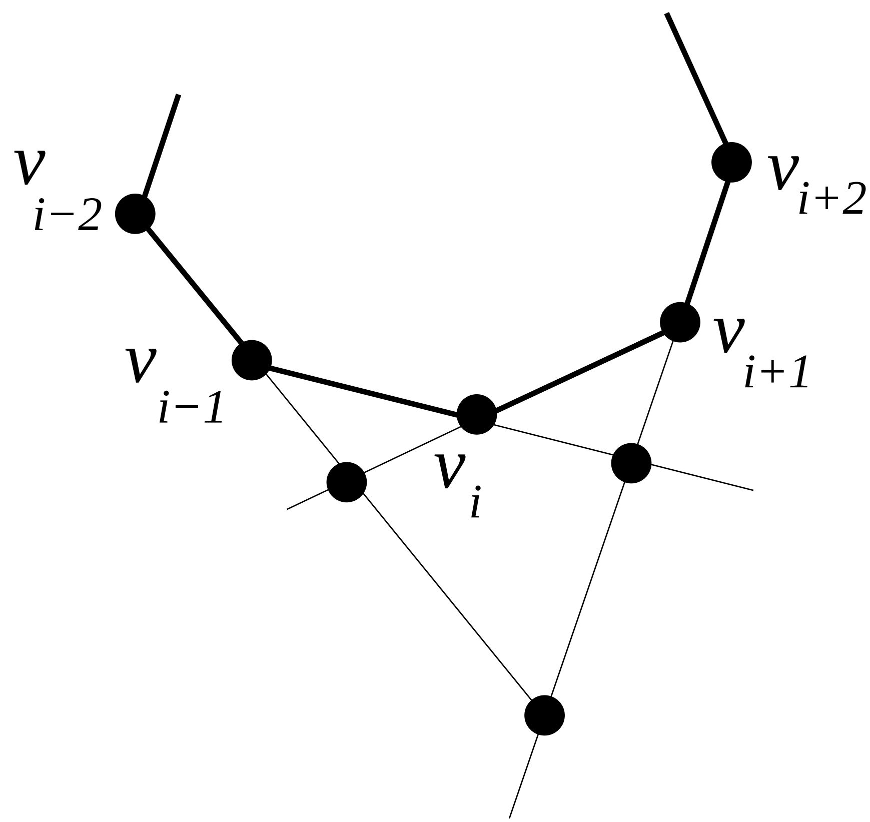
The Pentagram Map

Two spaces: $\mathcal{P}_n$, space of closed n-gons; dim $= 2n-8$;
$\mathcal{T}_n$, space of *twisted* n-gons; dim $= 2n$:
$\phi : \mathbf{Z} \to \mathbf{RP}^2$ s.t. $\phi(k+n) = M \circ \phi(k)$; $\forall k$.
M is the *monodromy*.

Twisted polygons ϕ_1 and ϕ_2 are equivalent if there is $\Psi \in PGL(3, \mathbf{R})$ such that $\phi_2 = \Psi \circ \phi_1$. Then $M_2 = \Psi M_1 \Psi^{-1}$.

Main Theorem: The Pentagram Map on $\mathcal{T}_n$ is completely integrable:
1). There are $2[n/2]+2$ *algebraically independent integrals.*
2). There is an invariant Poisson structure of corank 2 if n *is odd, and corank 4 if* n *is even, such that the integrals Poisson commute.*

Corner coordinates: left and right cross-ratios $x_1, y_1, \ldots, x_n, y_n$.



Cross-ratio: $[t_1, t_2, t_3, t_4] = \frac{(t_1-t_2)\,(t_3-t_4)}{(t_1-t_3)\,(t_2-t_4)}$.

The map $T_{12}(x, y) = (x^*, y^*)$ is given by:

$$x_i^* = x_i \, \frac{1 - x_{i-1}\, y_{i-1}}{1 - x_{i+1}\, y_{i+1}}, \qquad y_i^* = y_{i+1} \, \frac{1 - x_{i+2}\, y_{i+2}}{1 - x_i\, y_i}.$$

Two consequences:

1). Hidden *scaling symmetry*

$$(x_1, y_1, ..., x_n, y_n) \mapsto (tx_1, t^{-1}y_1, ..., tx_n, t^{-1}y_n)$$

commutes with the map.

2). "Easy" invariants:

$$O_n = \prod_{i=1}^{n} x_i, \quad E_n = \prod_{i=1}^{n} y_i,$$

and, for even n,

$$O_{n/2} = \prod_{i \text{ even}} x_i + \prod_{i \text{ odd}} x_i, \quad E_{n/2} = \prod_{i \text{ even}} y_i + \prod_{i \text{ odd}} y_i.$$

These are the Casimir functions of the Poisson bracket.

Monodromy invariants:

$$\frac{O_n^2 E_n (\mathrm{Tr}\ M)^3}{\det M} = \sum_{k=1}^{[n/2]} O_k$$

are polynomials in (x_i, y_i), decomposed into homogeneous components; likewise, for E_k with M^{-1} replacing M (with negative weights).

They are algebraically independent.

Poisson bracket:

$$\{x_i, x_{i+1}\} = -x_i\, x_{i+1}, \quad \{y_i, y_{i+1}\} = y_i\, y_{i+1},$$

and the rest $= 0$. The Jacobi identity is automatic.

Inscribed polygons

Computer experiments suggest: *if a (twisted) polygon is inscribed into a conic then* $E_k = O_k$ *for all* k (and the same for circumscribed polygons). Why??

Theorem: the space of inscribed n-gons is a coisotropic subspace of the Poisson manifold $\mathcal{T}_n$.

This leads to a Poisson mapping of the moduli space of twisted n-gons in $\mathbf{RP}^1$ which takes the left corner coordinates x_i to the right ones y_i (this is also given by rational functions).

Conjecture: this map is completely integrable for each n.

Continuous limit

Object of study: the space $\mathcal{P}$ of non-degenerate twisted parameterized curves in $\mathbf{RP}^2$ modulo projective equivalence:

$$\gamma(x+1) = M(\gamma(x)).$$

Lift so that

$$\left| \Gamma(x)\,\Gamma'(x)\,\Gamma''(x) \right| = 1.$$

Then

$$\Gamma'''(x) + u(x)\,\Gamma'(x) + v(x)\,\Gamma(x) = 0.$$

Thus $\mathcal{P}$ identifies with the space of linear differential operators on $\mathbf{R}$:

$$A = \left(\frac{d}{dx}\right)^3 + u(x)\,\frac{d}{dx} + v(x),$$

where u and v are smooth 1-periodic functions.

Rewrite as

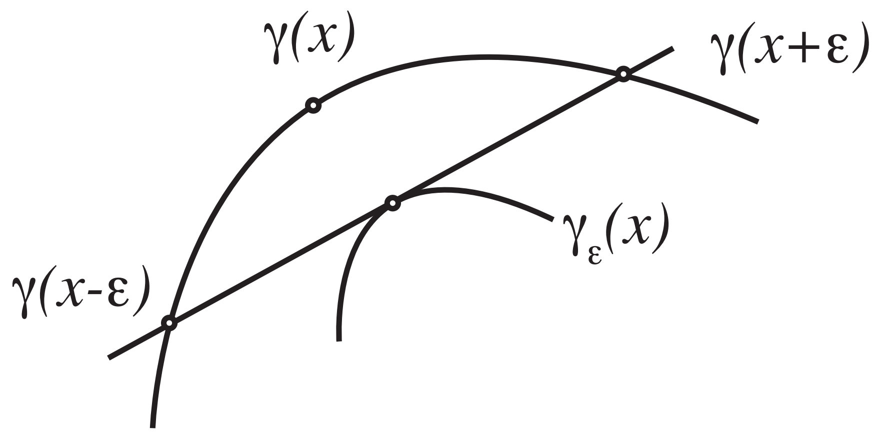
$$A = \left(\frac{d}{dx}\right)^3 + \frac{1}{2}\left(u(x)\,\frac{d}{dx} + \frac{d}{dx}\,u(x)\right) + w(x)$$

where $w(x) = v(x) - \frac{u'(x)}{2}$ (sum of a skew-symmetric and zero-order symmetric operators).

The functions u and w are projective-differential invariants of the curve γ ("projective curvature" and "projective length element").

Want to study $\mathcal{P}_n \to \mathcal{P}$ as $n \to \infty$.

Construction:



It turns out that

$$u_\varepsilon = u + \varepsilon^2 \widetilde{u} + (\varepsilon^3), \quad w_\varepsilon = w + \varepsilon^2 \widetilde{w} + (\varepsilon^3),$$

giving the flow: $\dot{u} = \widetilde{u}$, $\dot{w} = \widetilde{w}$.
A computation reveals :

$$\dot{u} = w', \quad \dot{w} = -\frac{u\,u'}{3} - \frac{u'''}{12},$$

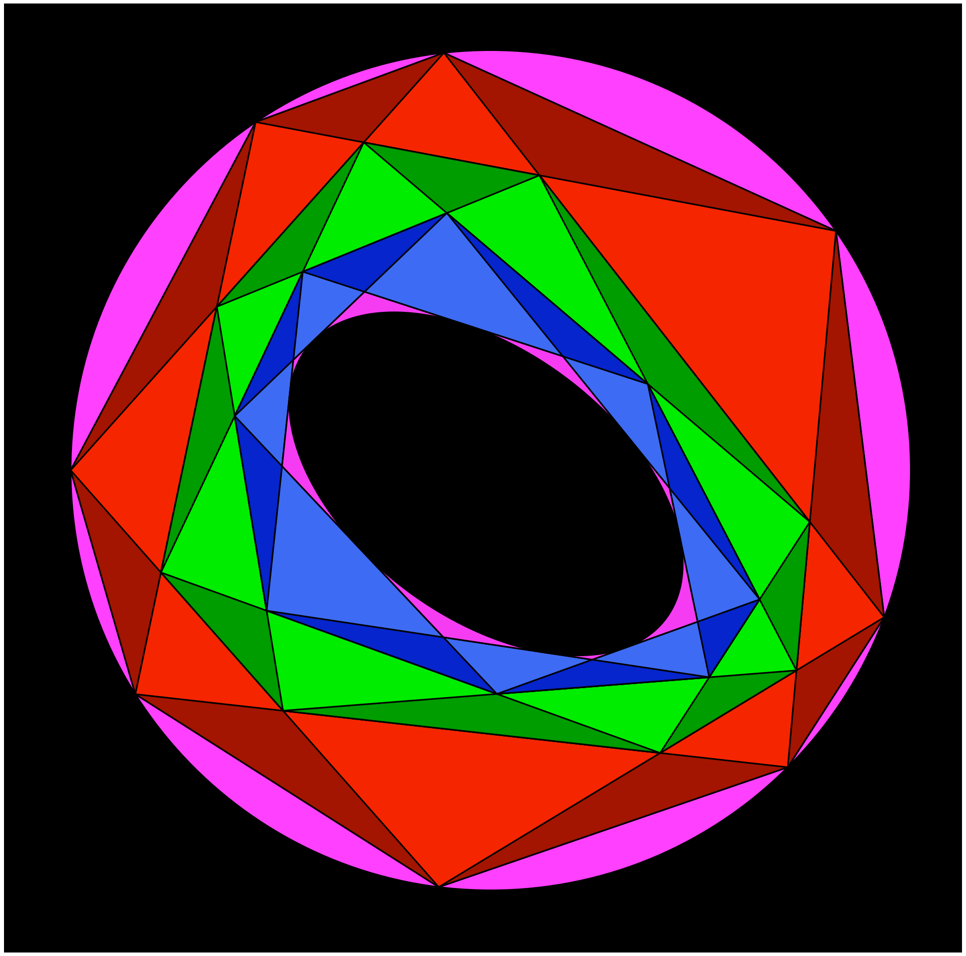
or

$$\ddot{u} + \frac{\left(u^2\right)''}{6} + \frac{u^{(IV)}}{12} = 0,$$

the Boussinesq equation!

The continuous limit of the scaling is:

$$u(x) \mapsto u(x), \quad w(x) \mapsto w(x) + t.$$



Happy Birthday, Claude!