Explicit solutions for integrable systems and applications

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Introduction : two theorems

Two types of “integrable” systems in this talk

- A vector field $\dot{x} = f(x)$ on a smooth manifold M
- A PDE $u_t = F(u, u_x, u_{xx}, \dots)$

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"Explicit" solutions

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- Formal solutions ; Laurent series
- Univalent, periodic, quasi-periodic solutions
- Solitons, $\cdots$

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Applications

- Abelian varieties, moduli spaces
- Random permutations, brownian motions
- Minimal surfaces, $\cdots$

The Liouville theorem

- (M, ω) a symplectic manifold of dimension $2n$
- $(F_1, \dots, F_n)$ independent functions in involution

Then for a generic point m_0 in M, the integral curve (solution) of each $\mathcal{X}_{F_i}$ starting from m can be determined by quadratures.

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Functions in involution : all the Poisson brackets $\{F_i, F_j\} = 0$, where

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}$$

in terms of canonical coordinates, $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$.

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Independent functions : the open subset of M where the differentials $dF_1, dF_2, \ldots, dF_n$ are independent is *dense* in M

The Hamiltonian vector fields $\mathcal{X}_H$: the ω-duals to the dH :

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2. Inverse function theorem.
3. Integration (of an exact differential form).

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3. Integration (of an exact differential form).

The integration :

1. Where the differentials are independent the vector fields $\mathcal{X}_{F_i}$ define an involutive, hence integrable, distribution.
2. On the integral manifolds, the forms ω_i, dual to the $\mathcal{X}_{F_i}$ are closed, hence locally exact, $\omega_i = dt_i$.
3. By integration, then local inversion, the coordinates on the integral manifolds can be expressed in terms of the t_i.

The Adler-Kostant-Symes theorem

- $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ Lie algebra splitting
- $\langle \cdot \mid \cdot \rangle$ symmetric, non-degenerate, ad-invariant
- $\mathcal{A}$ the algebra of ad-invariant functions

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Then

(1) A Poisson bracket on $C^\infty(\mathfrak{g})$ is defined by

$$\{F, G\}(x) := \langle x \,|\, [(\nabla_x F)_+, (\nabla_x G)_+] - [(\nabla_x F)_-, (\nabla_x G)_-] \rangle$$

(2) For $H \in \mathcal{A}$, the Hamiltonian vector field $\mathcal{X}_H$:

$$\dot{x} = [x, (\nabla_x H)_-]$$

(3) For $x_0 \in \mathfrak{g}$ and for small $|t|$, let $g_+(t)$ and $g_-(t)$ be the smooth curves in $\mathbf{G}_+$ resp. $\mathbf{G}_-$ such that

$$\exp(-t\nabla_{x_0} H) = g_+(t)^{-1} g_-(t)$$

with $g_\pm(0) = e$. Then the integral curve of $\mathcal{X}_H$ starting from x_0 is given for small $|t|$ by

$$x(t) = Ad_{g_-(t)} x_0.$$

The Euler top : elliptic solutions

Rigid body spinning around a fixed point, which is its center of gravity, with moments of inertia I_1, I_2, I_3. In terms of the angular velocity Ω :

$$\begin{aligned} I_1\dot{\Omega}_1 &= (I_2 - I_3)\Omega_2\Omega_3 \\ I_2\dot{\Omega}_2 &= (I_3 - I_1)\Omega_1\Omega_3 \\ I_3\dot{\Omega}_3 &= (I_1 - I_2)\Omega_1\Omega_2 \end{aligned}$$

Without parameters :

$$\begin{aligned} \dot{u}_1 &= u_2 u_3 \\ \dot{u}_2 &= u_1 u_3 \\ \dot{u}_3 &= u_1 u_2 \end{aligned}$$

Constants of motion : $H_1 := u_1^2 - u_2^2$ and $H_2 := u_1^2 - u_3^2$.

One uses the constants of motion to integrate : let $u_1^2 - u_2^2 = a$ and $u_1^2 - u_3^2 = b$ then

$$(\dot{u}_1)^2 = u_2^2 u_3^2 = (u_1^2 - a)(u_1^2 - b).$$

$$sn(u_1) = \int \frac{du_1}{\sqrt{(u_1^2 - a)(u_1^2 - b)}} = t.$$

This integral is an elliptic integral and its inverse is an elliptic function, $u_1 = sn^{-1}(t)$.

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Main fact : the elliptic function $sn^{-1}(t)$ is doubly periodic ...over **C**. It is a meromorphic function on the elliptic curve / elliptic Riemann surface

$$y^2 = (x^2 - a)(x^2 - b).$$

The Toda lattice : Moser's integration

Lax equation

$$\dot{L} = [L, B]$$

where

$$L = \begin{pmatrix} b_1 & a_1 & & & 0 \\ a_1 & b_2 & & & \\ & & \ddots & & \\ & & & b_{n-1} & a_{n-1} \\ 0 & & & a_{n-1} & b_n \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & a_1 & & & 0 \\ -a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_{n-1} \\ 0 & & & -a_{n-1} & 0 \end{pmatrix}$$

Let $f(\lambda) := (\lambda I_n - L)^{-1}_{nn}$ then

$$f(\lambda) = \sum_{k=1}^{n} \frac{r_k^2}{\lambda - \lambda_k}$$

with $r_k > 0$ and $\sum r_k^2 = 1$. This defines

$$L = (a_1, \ldots, a_n, b_1, \ldots, b_{n-1}) \leftrightarrow (\lambda_1, \ldots, \lambda_n, r_1, \ldots, r_n)$$

bijection and

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Inverse map : with continued fractions

$$f(\lambda) = \cfrac{1}{\lambda - b_n - \cfrac{a_{n-1}^2}{\lambda - b_{n-1} - \cfrac{a_{n-2}^2}{\lambda - b_{n-2} - \cfrac{\ddots}{\ddots \; \cfrac{a_1^2}{\lambda - b_1}}}}}$$

For example : if $n = 2$ then

$$\begin{aligned} b_1 &= -r_1^2\lambda_2 - r_2^2\lambda_1 \\ b_2 &= r_1^2\lambda_1 + r_2^2\lambda_2 \\ a_1 &= r_1 r_2(\lambda_1 - \lambda_2) \end{aligned}$$

KdV solitons

Korteweg-de Vries equation : $u_t = uu_x + u_{xxx}$.

Lax form : $L_t = [L_+^{3/2}, L]$ where $L = \frac{\partial^2}{\partial x^2} + u$.

The KdV hierarchy : for n odd : $L_{t_n} = [L_+^{n/2}, L]$

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n-solitions : For $J \subset I := \{1, \dots, n\}$ denote $c_J := \prod_{j \in J} c_j$ and $k_J := \prod_{i<j \in J} \left(\frac{k_i - k_j}{k_i + k_j}\right)^2$ where $c_1, \dots, c_n$ and $k_1, \dots, k_n$ constants.

$$\tau(t_1, t_3, \dots) := \sum_{J \subset I} c_J k_J \exp\left(2 \sum_{i \in J} \sum_{j=0}^{\infty} k_i^{2j+1} t_{2j+1}\right)$$

The function $u := 2\frac{\partial^2}{\partial x^2} \log \tau(t_1, t_3, \dots,)$ is a solution of KdV.

KdV solitons and vertex operators

The n-soliton can be written in terms of vertex operators :

$$X(p,q) = \exp\left(\sum_{j=1}^{\infty}(p^j - q^j)t_j\right) \exp\sum_{j=-1}^{\infty}\left(\frac{p^j - q^j}{j}\frac{\partial}{\partial t_{-j}}\right).$$

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$$\tau = e^{c_1 X(k_1,-k_1)} \dots e^{c_n X(k_n,-k_n)} 1$$

Hence the vertex operator $X(p,-p)$ permits to add a solition !

KP : theta solutions

KP = generalization of KdV

$$u_{yy} = (u_t - uu_x - u_{xxx})_x$$

There exists for every Riemann surface Γ a solution of KP :

$$u(x, y, t) := \frac{\partial^2}{\partial x^2} \log \vartheta(ax + by + ct \mid Z)$$

Here, the theta function of $Z \in Mat(g \times g)$ (symmetric, $\Im Z > 0$)

$$\vartheta(z \mid Z) = \sum_{l \in \mathbf{Z}^r} e^{\pi i \langle l, Zl \rangle} e^{2\pi i \langle l, z \rangle}.$$

The matrix Z associated to Γ : the period matrix

$$Z_{ij} := \int_{\gamma_i} \omega_j.$$

Application : Schotky problem (1903)

Charaterize the variety of matrices Z which are the period matrix of a Riemann surface.

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Novikov conjecture (1981) : Z comes from a Riemann surface iff

$$\frac{\partial^2}{\partial x^2} \log \vartheta(ax + by + ct \mid Z)$$

is a solution to the KP equation for some a, b, c.

In 1986 Shiota proves the conjecture.

Reduction : the Mumford system

Phase space M_g :

$$L(\lambda) = \begin{pmatrix} V(\lambda) & W(\lambda) \\ U(\lambda) & -V(\lambda) \end{pmatrix}$$

with U and W monic polynomials and

$$\deg V < \deg U = \deg W - 1 = g.$$

The integrable vector fields (for $i = 1, 2, \ldots, g$) :

$$\frac{d}{dt_{2i-1}} L(\lambda) = \left[L(\lambda), \left(\frac{L(\lambda)}{\lambda^{g-i+1}} \right)_+ - \begin{pmatrix} 0 & 0 \\ U_{g-i} & 0 \end{pmatrix} \right]$$

The moment map

$$
\begin{array}{rccl}
H & : & M & \longrightarrow & \mathbf{C}^{2g+1} \\
 & & L(\lambda) & \mapsto & \det(L(\lambda)-\mu) = \mu^2 - U(\lambda)W(\lambda) - V(\lambda)^2
\end{array}
$$

For $f(\lambda) = \prod_{i=1}^{5}(\lambda - \lambda_i)$, generic, the fiber above $\mu^2 - f(\lambda)$ is

$$\mathrm{Jac}(\mu^2 = f(\lambda)) \setminus \Theta.$$

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Generic solution

$$\begin{aligned} U(\lambda_k) &= c_k \left(\frac{\vartheta \begin{bmatrix} \delta_k \\ \epsilon_k \end{bmatrix} (A\vec{t} + b)}{\vartheta(A\vec{t} + b)} \right)^2 \\ V(\lambda) &= \frac{d}{dt_1} U(\lambda) \\ W(\lambda) &= \frac{f(\lambda) - V^2(\lambda)}{U(\lambda)} \end{aligned}$$

Link KdV-Mumford

Let $(U(\lambda,\vec{t}), V(\lambda,\vec{t}), W(\lambda,\vec{t}))$ solution to the Mumford system, $U(\lambda) = \lambda^g + U_{g-1}\lambda^{g-1} + \cdots + U_0$, etc.

Then

$$u(\vec{t}) := 2\frac{\partial^2}{\partial t_1^2}\log U_{g-1}(\vec{t})$$

is a solution to the KdV hierarchy (with $x = t_1$). For $f(\lambda)$ such that $\mu^2 = f(\lambda)$ is smooth, one recovers the solutions in terms of theta functions.

Algebraic integrability

In 1980 Adler and van Moerbeke introduced the notion of algebraic integrability : an a.c.i. system is a complex integrable system such that

(1) the generic fibers of the (complex) momentum map are affine parts of complex algebraic tori $\mathbf{C}^r/\Lambda_c$

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Example : the Mumford system is a.c.i.

Laurent solutions for a.c.i. systems

Theorem [Adler, van Moerbeke, V., 2004]

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Laurent solution of $\dot{x}_i = f(x_1, \ldots, x_n)$:

$$x_i(t) = \sum_{j=k_i}^{\infty} \alpha_{ij} t^j \qquad i = 1, \ldots, n$$

convergent for $t \in B(0; \epsilon) \setminus \{0\}$.

Application : obstruction to algebraic integrability

Let $e_0, e_1, \ldots, e_\ell \in \mathbf{R}^{\ell+1}$ such that

- $e_0, e_1, \ldots, e_\ell$ are dependent
- $\forall i : e_0, \ldots, \widehat{e_i}, \ldots, e_\ell$ are independent.

Let $A = (a_{ij})$ be its Cartan matrix

$$a_{ij} := \frac{2\langle e_i | e_j \rangle}{\langle e_j | e_j \rangle}$$

On $\mathbf{C}^{2(\ell+1)}$ the vector field $\mathcal{V}$:

$$\dot{x} = x \cdot y \qquad \dot{y} = Ax.$$

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Theorem [Adler, van Moerbeke]
If $\mathcal{V}$ is a.c.i., then A is the Cartan matrix of an affine (twisted) Lie algebra.

Application : projective embeddings of Kummer surfaces (with L. Piovan)

For the Riemann surface Γ :

$$\mu^2 = \prod_{i=1}^{5}(\lambda - \lambda_i) = \sum_{i=0}^{5} \sigma_i \lambda^{5-i},$$

$\mathrm{Kum}(\Gamma) := \mathrm{Jac}(\Gamma)/(-1)$. Surface with 16 singular points.

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Embeddings as quartic surface in $\mathbf{P}^3$:

$$\begin{aligned}
&a_1^2(\theta_0^2\theta_2^2 + \theta_1^2\theta_3^2) + a_2^2(\theta_0^2\theta_1^2 + \theta_2^2\theta_3^2) + a_0^6(\theta_0^2\theta_3^2 + \theta_1^2\theta_2^2) \\
&\quad +2a_1a_2(\theta_0\theta_1 - \theta_2\theta_3)(\theta_0\theta_2 - \theta_1\theta_3) \\
&\quad +2a_0^3a_2(\theta_0\theta_1 + \theta_2\theta_3)(\theta_0\theta_3 - \theta_1\theta_2) \\
&\quad -2a_0^3a_1(\theta_0\theta_2 + \theta_1\theta_3)(\theta_0\theta_3 + \theta_1\theta_2) \\
&\quad +2\delta\theta_0\theta_1\theta_2\theta_3 = 0,
\end{aligned}$$

where $a_0^2 = \lambda_{ij},\ a_1^2 = \lambda_{ik}\lambda_{im}\lambda_{in}$ and $a_2^2 = \lambda_{jk}\lambda_{jm}\lambda_{jn}$.

In terms of the σ_i

$$
\begin{aligned}
0 = {} & (4\sigma_3\sigma_5 - \sigma_4^2)\theta_0^4 \\
& +2[-2\sigma_2\sigma_5\theta_1 + (\sigma_2\sigma_4 - 2\sigma_1\sigma_5)\theta_2 + 2\sigma_5\theta_3]\theta_0^3 \\
& +[4\sigma_1\sigma_5\theta_1^2 - (2\sigma_4 + \sigma_2^2)\theta_2^2 + (4\sigma_5 - 2\sigma_1\sigma_4)\theta_1\theta_2 \\
& \quad -2\sigma_4\theta_1\theta_3 + 4\sigma_3\theta_2\theta_3]\theta_0^2 \\
& +2[-2\sigma_5\theta_1^3 + 2\sigma_4\theta_1^2\theta_2 + (\sigma_1\sigma_2 - 2\sigma_3)\theta_1\theta_2^2 \\
& \quad +\sigma_2\theta_2^3 + 2\theta_2\theta_3^2 - \sigma_2\theta_1\theta_2\theta_3 - 2\sigma_1\theta_2^2\theta_3]\theta_0 \\
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& \quad +\sigma_2\theta_2^3 + 2\theta_2\theta_3^2 - \sigma_2\theta_1\theta_2\theta_3 - 2\sigma_1\theta_2^2\theta_3]\theta_0 \\
& -(\theta_2^2 - \sigma_1\theta_1\theta_2 + \theta_1\theta_3)^2.
\end{aligned}
$$

Embedding as quartic in $\mathbf{P}^3$ with 6 singuliar points :

$$\sum_{i=1}^{4} \sum_{\substack{1\le j<k\le 4 \\ j,k\neq i}} \lambda_{ij}^2\lambda_{jk}^2\lambda_{ki}^2\lambda_{im}\lambda_{in}\theta_i^2\theta_j\theta_k = 0,$$

where $\{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$.

Rational solutions (Mumford and KdV)

Before : if the curve $\mu^2 = f(\lambda)$ is smooth, solution to Mumford and KdV in terms of theta fonctions.

Here : the other extreme : the very singular curve $\mu^2 = \lambda^{2g+1}$. The corresponding fiber of the momentum map $H^{-1}(0)$: the affine variety of $(U(\lambda), V(\lambda), W(\lambda))$ such that $U(\lambda)W(\lambda) - V(\lambda)^2 = 0$.

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Theorem [Inoue, V., Yamazaki, 2009]

- $H^{-1}(0)$ is stratified by $g+1$ manifolds of dimension $0, 1, \ldots, g$, generated by the integrable flows ;
- The $g+1$ corresponding solutions are (explicit !) rational functions ;
- Each stratum compactifies into the generalized Jacobian of the curve $\mu^2 = \lambda^{2i+1}$ where $i = 0, \ldots, g$;
- One recovers the rational solutions to the KdV hierarchy (for every g).

The rational solutions of KdV for small g

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$$\tau_4(\vec{t}) = \frac{t_1^{10}}{4725} - \frac{t_1^7 t_3}{105} - t_1 t_3^3 + \frac{t_1^5 t_5}{15} + t_1^2 t_3 t_5 - t_5^2 - \frac{t_1^3 t_7}{3} + t_3 t_7$$

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General formula : via Schur polynomials or via the theory of generalized Jacobians.

Witten's conjecture / Kontsevich's theorem

$\mathcal{M}_{g,n}$ moduli space of smooth curves of genus g with n marked points. $\overline{\mathcal{M}}_{g,n}$ is an algebraic variety of dimension $3g-3+n$.
Every marked point defined a line bundle $\mathcal{L}_i$ on $\mathcal{M}_{g,n}$. For $\Sigma = (\Sigma, (p_i)) \in \mathcal{M}_{g,n}$ the fiber of $\mathcal{L}_i$ at Σ is $T^*_{p_i}\Sigma$.
For $d_1 + \cdots + d_n = 3g-3+n$ one defines the intersection number

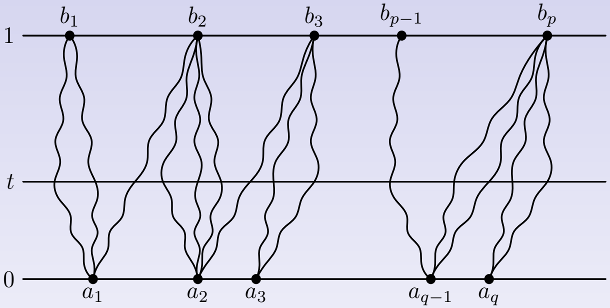
$$\left\langle \tau_{d_1}\tau_{d_2}\dots\tau_{d_n} \right\rangle := \int_{\overline{\mathcal{M}}_{g,n}} c_1(\mathcal{L}_1)^{d_1} \wedge \cdots \wedge c_1(\mathcal{L}_n)^{d_n}$$

Generating function

$$F(\vec{t}) = \sum_{(n_0,n_1,\dots n_\ell)} \frac{t_1^{n_0}}{n_0!}\frac{t_3^{n_1}}{n_1!}\dots\frac{t_{2\ell+1}^{n_\ell}}{n_\ell!}\left\langle \tau_0^{n_0}\tau_1^{n_1}\dots\tau_\ell^{n_\ell}\right\rangle$$

Theorem [Witten/Kontsevich (1990)]
The function $u := \partial^2 F/\partial t_1^2$ is a solution to KdV.

Non-intersecting Brownian motions



$$\mathcal{P}(t,E) \sim \int_{E^N} \det(p(t,a_i,x_j))\det(p(1-t,x_i,b_j))\prod_{i=1}^{N} dx_i$$

$$p(t,x,y) = \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}}$$

A solution to KP

After coalescence and some transformations :

$$\mathcal{P}(t,E) \sim \det\left(\left(\int_E y^{i+j} e^{-y^2/2} e^{(\tilde{a}_\alpha+\tilde{b}_\beta)y}\,dy\right)_{\substack{0\leqslant i<m_\alpha\\ 0\leqslant j<n_\beta}}\right)_{\substack{1\leqslant\alpha\leqslant q\\ 1\leqslant\beta\leqslant p}}$$

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Deformation :

$$\begin{aligned}
\tilde{a}_\alpha y &\rightarrow \tilde{a}_\alpha y - \sum_{k=1}^{\infty} s_{\alpha,k} y^k \\
\tilde{b}_\beta y &\rightarrow \tilde{b}_\beta y - \sum_{k=1}^{\infty} t_{\beta,k} y^k
\end{aligned}$$

A solution to KP

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Deformation :

$$\begin{aligned} \tilde{a}_\alpha y &\rightarrow \tilde{a}_\alpha y - \sum_{k=1}^{\infty} s_{\alpha,k} y^k \\ \tilde{b}_\beta y &\rightarrow \tilde{b}_\beta y - \sum_{k=1}^{\infty} t_{\beta,k} y^k \end{aligned}$$

Theorem [Adler, van Moerbeke, V., 2008] One obtains a solution to the $(p+q)$-KP hierarchy, where the $s_{\alpha,k}$ and the $t_{\beta,k}$ are time variables.