Math 872 Exam 1 Topics

Homotopy Theory.

Motivation: understand all continuous functions $f : X \to Y$, since it is functions to/from 'model' spaces that allow us to explore a space.

E.g., paths = $\gamma : I = [0,1] \to X$. How many 'essentially distinct' paths are there from $(-1,0)$ to $(1,0)$ in $\mathbb{R}^2 \setminus \{(0,0)\}$? What is inessetial? Deformations.

Two maps $f, g : X \to Y$ are *homotopic* if one can be deformed to the other (through continuous maps). Formally, there is a cts map $H : X \times I \to Y$ so that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. We write: $f \simeq g$ (via H).

Note: $\gamma_x : t \mapsto H(x,t)$ is a cts path in Y, for every x.

Notation: $f : (X,A) \to (Y,B)$ means $A \subseteq X$, $B \subseteq Y$ and $f(A) \subseteq B$.

Two maps $f, g : (X,A) \to (Y,B)$ are homotopic rel A if $H : X \times I \to Y$ also satisfies $H(a,t) = f(a) = g(a)$ for all $a \in A$, $t \in I$. [So, in part, $f|_A = g|_A$.]

Basic example: any two maps $f, g : X \to \mathbb{R}^n$ are homotopic, via a *straight-line homotopy*: $H(x,t) = (1-t)f(x) + tg(x)$.

Homotopy is an equivalence relation: $f \simeq f$ (via $H(x,t) = f(x)$), $f \simeq g$ implies $g \simeq f$ (via $K(x,t) = H(x, 1-t)$); $f \simeq g$ and $g \simeq h$ implies $f \simeq h$ (via doubling the speed; $M(x,t) = H(x,2t)$ for $t \leq 1/2$ and $= K(x, 2t-1)$ for $t \geq 1/2$).

This allows us to introduce a new notion of equivalence of topological spaces. X and Y are *homotopy equivalent* [we write $X \simeq Y$] if there are $f : X \to Y$ and $g : Y \to X$ so that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$.

Homotopy equivalence is an equivalence relation! Note: a homeomorphism is a homotopy equivalence! [$g \circ f = \mathrm{Id}_X \simeq \mathrm{Id}_X$].

The homotopy viewpoint.

The basic idea is that homotopy equivalence (= 'h.e.') allows us to move past/around 'unimportant' differences in spaces. For example, $\mathbb{R}^2 \setminus \{(0,0)\} \cong S^1 \times \mathbb{R} \simeq S^1 \times I \simeq S^1$ means that maps into $\mathbb{R}^2 \setminus \{(0,0)\}$ 'behave like' maps into S^1 (which we can more readily understand?).

Algebraic topology seeks to understand topological spaces through algebraic invariants. An algebraic invariant assigns to each space X an algebraic object $A(X)$ and to each map $f : X \to Y$ a homomorphism $A(f) : A(X) \to A(Y)$. If X and Y are the 'same', then $A(X)$ and $A(X)$ will be isomorphic. Usually, 'same' means homeomorphic, but we will often find that homotopy equivalent spaces will same the same invariants, due the the methods that we use to build them.

This can be both bad and good, 'homotopy invariance' of a invariant means that it will not be able to distinguish h.e. spaces that are not homeomorphic. But it also means that when computing an algebraic invariant, we can replace a space X with $Y \simeq Y$, which may streamline a computation.

A *retraction* of X onto $A \subseteq X$ is a map $r : X \to A$ so that $r(a) = a$ for all $a \in A$. [A is a *retract* of X]. A is a *deformation retract* of X if $\iota \circ r : X \to A \to X$ is $\simeq \mathrm{Id}_X$ [r is a *deformation retraction*]. and r is a *strong deformation retraction* if $\iota \circ r : (X,A) \to (X,A)$ is $\simeq \mathrm{Id}_X$ rel A (i.e., $H(a,t) = a$ for all $a \in A$). We write $X \searrow A$.

For example, $r : \mathbb{R}^n \searrow \{\vec{0}\}$, since $\iota \circ r \simeq \mathrm{Id}_{\mathbb{R}^n}$ via a straight-line homotopy
$$H(x,t) = (1-t)\iota \circ r(\vec{x}) + t\mathrm{Id}_{\mathbb{R}^n}(\vec{x}) = t\vec{x}\ .$$

A space X is *contractible* if $X \simeq \{*\}$.

Mapping cylinders: If $f : X \to Y$, then $M_f = X \times I \coprod Y/\sim$, where $(x,1) \sim f(x)$. [Idea: we glue $X \times \{1\}$ to Y using f.] Then since $X \times I \searrow X \times \{1\}$, we have $M_f \searrow Y$.

Fact: $f : X \to Y$ is a homotopy equivalence $\Leftrightarrow M_f \searrow X \times \{0\}$. This means that $X \simeq Y \Leftrightarrow$ there is a space Z with $X, Y \subseteq Z$ and $Z \searrow X$, $Z \searrow Y$.

The Fundamental Group.

Idea: find the essentially distinct paths between points in X. How? Turn this into a group! How? The concatenation $\gamma * \eta$ of two paths is a path. But: only if the first ends where the second begins (so that, by the Pasting Lemma, the resulting map is cts). So we either have a partial multiplication (= groupoid!), or we focus on loops $\gamma : (I, \partial I) \to (X, x_0)$ based at a fixed point x_0 9we'll do the second).

Elements of the *fundamental group* $\pi_1(X, x_0)$ 'are' loops; the inverse will be the reverse $\overline{\gamma}(t) = \gamma(1-t)$, since $\gamma * \overline{\gamma} \simeq c_{x_0}$, and the identity element will be the constant map c_{x_0}. But! to make $\gamma * \overline{\gamma}$ equal c_{x_0}, we need to work with *homotopy classes* of loops. So elements really are equivalence classes $[\gamma]$ of loops, under $\simeq$ rel ∂I.

Then by building homotopies (mostly working on the domain I, i.e., building $K = \gamma \circ H : I \times I \to I \to X$) we can see that $[\gamma][\eta] = [\gamma * \eta]$ is well defined, $[\gamma]^{-1} = [\overline{\gamma}]$ is the inverse, and $([\gamma][\eta])[\omega] = [\gamma]([\eta][\omega])$, so under $*$, $\pi_1(X, x_0)$ is a group. [Most of the proofs that needed maps (like $(\gamma * \eta) * \omega$ and $\gamma * (\eta * \omega)$ (which are the same concatenations, except at 4,4, and 2 times speed, versus 2,4, and 4 times speed) are homotopic can be given 'picture' proofs, in addition to explicit analytic formulas.

Given a map $f : (X, x_0) \to (Y, y_0)$, we get an induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ via $f_*[\gamma] = [f \circ \gamma]$. This is well-defined, and a homomorphism.

Basic computations: $\pi_1(\{*\}, *) = \{1\}$, as are $\pi_1(\mathbb{R}^n, \vec{0})$ and $\pi_1([0,1]^n, x_0)$ for any x_0. More generally, any contractible space has trivial fundamental group.

Since $(f \circ g)_* = f_* \circ g_*$, and $(\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X, x_0)}$, then $X \cong Y$ via f implies $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.

More generally, if $f : X \to Y$ is a h.e., then f_* is an isomorphism, but, because of basepoint issues, the inverse of f_* is generally not g_* for g a homotopy inverse. Th is is because under a homotopy $H : X \times I \to X$ of $g \circ f$ to Id, the basepoint x_0 traces out a path η from $g(f(x_0)) = x_1$ to x_0, and $[g \circ f \circ \gamma] = [\overline{\eta} * \gamma * \eta]$. This map $[\gamma] \mapsto [\overline{\eta} * \gamma * \eta]$ from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ is a *change of basepoint isomorphism*, which we might call η_* ? The fact that homotopies can drag basepoints around will be a theme we will return to many times moving forward.

If X is path connected, then, up to isomorphism, $\pi_1(X, x_0)$ is independent of x_0 (we can always find a path to effect an isomorphism), and so we will often write $pi_1(X)$, when X is path-connected, when we only care about the abstract group.

$\pi_1(S^1, (1,0)) \cong \mathbb{Z}$. The main ingredients:

Writing $S^1 \subseteq \mathbb{C}$ and $\gamma_n(t) = e^{2\pi int}$ is the loop traversing S^1 n times counterclockwise at uniform speed, then (1) every loop γ at $(1,0)$ is $\simeq \gamma_n$ for some n.

We define $w : \pi_1(S^1,(1,0)) \to \mathbb{Z}$ by $w[\gamma] = n$ if $[\gamma] = [\gamma_n]$. This is well-defined: (2) if $\gamma_n \simeq \gamma_m$ rel endpoints, then $n = m$.

w is a bijective homomorphism!

The proof of (1) amounted to making a general γ progressively nicer, via homotopy. This involved

Lebesgue Number Theorem: If (X,d) is a compact metric space and $\{U_\alpha\}$ is an open covering of X, then there is an $\epsilon > 0$ so that for every $x \in X$ there is an $\alpha = \alpha(x)$ so that we have $N_d(x,\epsilon) \subseteq U_\alpha$.

Then by covering S^1 by the 'top 2/3rds' and 'bottom 2/3rds' subsets and taking inverse images under $\gamma : (I,\partial I) \to (S^1,(1,0))$, the LNT will partition I into finitely many intervals each mapping into top or bottom. Creating subpaths by restricting to each subinterval, and inserting 'hairs' to points $(1,0),(-1,0)$ in the intersection of top and bottom, we can then homotope the subpaths to standard paths $t \mapsto e^{\pm 2\pi it}$. Cancelling pairs the reverse direction give us our 'normal forms' γ_n.

The proof of (2) amounted to using an 'extra' coordinate $(\cos t, \sin t, t)$ to keep track of how many times we wind around the circle. To do this correctly, we really use the map $p : t \mapsto (\cos t, \sin t, t) \mapsto (\cos t, \sin t)$ and then <u>lift</u> paths $\gamma : I \to S^1$ to paths $\widetilde{\gamma} : I \to \mathbb{R}$ with $\gamma = p \circ \widetilde{\gamma}$. This agin uses the LNT to partition I into subintervals mapping into top and bottom, and the fact that the inverse image of top and bottom are a disjoint union of open sets mapped homeomorphically under p to the top and bottom. [This is the *evenly covered property*.]

More than this, homotopies $H : I \times I \to S^1$ can also be lifted; this enables us to show that loops homotopic rel endpoints, when lifted both starting at the same point, will end at the same point. Since γ_n when lifted starting at 0 will end at n, the result follows.

Applications. This single computation has many applications! First, there is no retraction $r : \mathbb{D}^2 \to \partial\mathbb{D}^2$. This is because if there were one, then $r_* : \pi_1(\mathbb{D}^2,(1,0)) \to \pi_1(S^1,(1,0))$ would be a surjection, which is impossible.

This in turn gives the *Brouwer Fixed Point Theorem*: Every countinuous map $f : \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point. For if not, we can then manufacture a retraction $r : \mathbb{D}^2 \to \partial\mathbb{D}^2$.

Finally, we can prove the *Fundamental Theorem of Algebra*: Every non-constant polynomial p has a complex root. For if not, then for large enough N the map

$$t \mapsto f(Ne^{2\pi it}) \mapsto f(Ne^{2\pi it})/||f(Ne^{2\pi it})||$$

from I to S^1 is homotopic to both $c_{(1,0)} = \gamma_0$ and γ_n for $n =$ the degree of f, a contradiction.

Group presentations.

Free groups: $\Sigma =$ a set; a *reduced word* on Σ is a (formal) product $a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}$ with $a_i \in \Sigma$ and $\epsilon_i = \pm 1$, and either $a_i \neq a_{i+1}$ or $\epsilon_i \neq -\epsilon_{i+1}$ for every i. (I.e., no $aa^{-1}, a^{-1}a$ in the product.)

The free group $F(\Sigma) =$ the set of reduced words, with multiplication = concatenation followed by reduction; remove all possible $aa^{-1}, a^{-1}a$ from the site of concatenation. Identity element = the empty word, $(a_1^{\epsilon_1} \cdots a_n^{\epsilon_n})^{-1} = a_n^{-\epsilon_n} \cdots a_1^{-\epsilon_1}$. $F(\Sigma)$ is generated by Σ, with no relations among the generators other than the "obvious" ones.

Important property of free groups: any function $f : \Sigma \to G$, G a group, extends uniquely to a homomorphism $\phi : F(\Sigma) \to G$.

If $R \subseteq F(\Sigma)$, then $< R >^N$ = normal subgroup generated by $R = \{\prod_{i=1}^{n} g_i r_i g_i^{-1} : n \in \mathbb{N}_0, g_i \in F(\Sigma), r_i \in R\}$ = smallest normal subgroup containing R.

$F(\Sigma)/ < R >^N$ = the group with *presentation* $< \Sigma | R >$. Any homom $\varphi : F(\Sigma) \to G$ with $F(R) = \{1_G\}$ induces $\overline{\varphi} :< \Sigma | R > \to G$.

If $G_1 =< \Sigma_1 | R_1 >$ and $G_2 =< \Sigma_2 | R_2 >$, then their *free product* $G_1 * G_2 =< \Sigma_1 \coprod \Sigma_2 | R_1 \cup R_2 >$. Any pair of homoms $\phi_i : G_i \to G$ extends uniquely to a homom $\phi : G_1 * G_2 \to G$

Gluing groups: given groups $G_1 =< \Sigma_1 | R_1 >$ and $G_2 =< \Sigma_2 | R_2 >$, a group H and homomorphisms $\phi_1 : H \to G_i$, the largest group "generated" by G_1 and G_2, in which $\phi_1(h) = \phi_2(h)$ for all $h \in H$ is $G_1 *_H G_2 =< \Sigma_1 \coprod \Sigma_2 | R_1 \cup R_2 \cup \{\phi_1(h)(\phi_2(h))^{-1} : h \in H\} >$.

Important special cases : $G *_H \{1\} = G/ < \phi(H) >^N =< \Sigma | R \cup \phi(H) >$, and $G_1 *_\{ 1\} G_2 \cong G_1 * G_2$.

Seifert-van Kampen Theorem. If we express a topological space as the union $X = X_1 \cup X_2$, then we have inclusion-induced homomorphisms

$$j_{1*} : \pi_1(X_1) \to \pi_1(X) \ , \ j_{2*} : \pi_1(X_2) \to \pi_1(X)$$

This in turn gives a homomorphism $\phi : \pi(X_1) * \pi_1(X_2) \to \pi_1(X)$. Under the hypotheses X_1, X_2 are open, and $X_1, X_2, X_1 \cap X_2$ are path-connected (choose a basepoint in $X_1 \cap X_2$ and) this homom is onto, and has kernel $H =< i_{1*}(\gamma)(i_{2*}(\gamma))^{-1} : \gamma \in \pi_1(A) >^N$, so we have the *Seifert - van Kampen Theorem:* $\pi_1(X) \cong \pi_1(X_1) *_{\pi_1(A)} \pi_1(X_2)$.

[Why? Lebesgue number theorem! Any loop into X, using the inverse images of X_1, X_2 as an open cover, can be partitioned into subloops alternately mapping into X_1, X_2, which makes ϕ surjective. Partitioning a null homotopy, using H to change which of $\pi(X_1), \pi_1(X_2)$ a loop lies in, yields the result.]

Generalization (of sorts): if $X = C \cup D$ closed sets, with C, D having nbhds U, V which deformation retract to C, D (and $U \cap V$ def retracts to $C \cap D = A$, then $\pi_1(X) \cong \pi_1(C) *_{\pi_1(A)} \pi_1(D)$.

Applications.

Fundamental groups of graphs: Choosing a maximal tree T in a graph Γ, $\Gamma \simeq \Gamma/T$ =a bouquet of circles, which by SvK has fundamental gorup free on the number of loops.

Gluing on a 2-disk: If X is a topological space and $f : \partial\mathbb{D}^2 \to X$ is continuous, then we can construct the quotient space $Z = (X \coprod \mathbb{D}^2)/\{x \sim f(x) : x \in \partial\mathbb{D}^2\}$, the result of gluing $\mathbb{D}^2$ to X along f. Then SvK (with some delicacy choosing the basepoint), treating f as a loop in X, gives $\pi_1(Z) \cong \pi_1(X) *_{\mathbb{Z}} \{1\} = \pi_1(X_2)/ < \mathbb{Z} >^N \cong \pi_1(X_2)/ < [f] >^N$. So the effect of gluing on a 2-disk onto a space, on the fundamental group, is to add a new relator, namely the word represented by the attaching map (adjusting for basepoint). All of this applies equally well to attaching several 2-disks; each adds a new relator. This in turn opens up huge possibilities for the computation of $\pi_1(X)$. For example, for cell complexes (see below!), we can inductively compute π_1 by starting with the 1-skeleton, with free fundamental group, and attaching the 2-cells one by one, which each add a relator to the presentation of $\pi_1(X)$.

Knot complements $X = S^3 \setminus K$ deformation retract onto a 2-complex which can be built from a planar diagram of the knot. From this, a presentation for $\pi_1(X)$ can be built, with a generator for each strand of the knot diagram, and a (length 4) relator for each crossing, expressing the relation that the overstrand conjugates one understrand at the crossing to the other. (The particular form of the relator ($xax^{-1} = b$ or $x^{-1}ax = b$ is determined by an orientation for the knot.)

CW complexes: The "right" spaces to do algebraic topology on. The basic idea: CW complexes are built inductively, by gluing disks onto lower-dimensional strata. $X = \bigcup X^{(n)}$, where $X^{(0)}$ = a disjoint union of points, and, inductively, $X^{(n)}$ is built from $X^{(n-1)}$ by gluing n-disks D_i^n along their boundaries. $X = \bigcup X^{(n)}$ is given the *weak topology*; that is, $C \subseteq X$ is closed $\Leftrightarrow C \cap X^{(n)}$ is closed for all n. Each disk D_i^n has a *characteristic map* $\phi_i : D_i^n \to X$ given by $D_i^n \to X^{(n-1)} \cup (\coprod D_i^n) \to X^{(n)} \subseteq X$. $f : X \to Y$ is cts $\Leftrightarrow f \circ \phi_i : D_i^n \to X \to Y$ is cts for all D_i^n.

A *CW pair* (X, A) is a CW complex X and a *subcomplex* A. If (X, A) is a CW pair, then X/A admits a CW structure whose cells are $[A]$ and the cells of X not in A. We can glue two CW complexes X, Y along isomorphic subcomplexes $A \subseteq X, Y$, yielding $X \cup_A Y$.

Perhaps the most important property of CW complexes (for algebraic topology, anyway) is the *homotopy extension property*; given a CW pair (X, A), a map $f : X \to Y$, and a homotopy $H : A \times I \to Y$ such that $H|_{A\times 0} = f|_A$, there is a homotopy (extension) $K : X \times I \to Y$ with $K|_{A\times I} = H$. This is because $B = X \times \{0\} \cup A \times I$ is a retract of $X \times I$; K is the composition of this retraction and the "obvious" map from B to Y. Consequence: if (X, A) is a CW pair and A is contractible, then $X/A \simeq X$.

Covering spaces: A map $p : E \to B$ is called a covering map if for every point $x \in B$, there is a neighborhood $\mathcal{U}$ of x (an *evenly covered neighborhood*) so that $p^{-1}(\mathcal{U})$ is a disjoint union $\mathcal{U}_\alpha$ of open sets in E, each mapped homeomorphically onto $\mathcal{U}$ by (the restriction of) p . B is called the *base space* of the covering; E is called the *total space*.

The disjoint union of 42 copies of a space, each mapping homeomorphically to a single copy, is an example of a *trivial covering*. The famous exponential map $p : \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi it} = (\cos(2\pi t), \sin(2\pi t))$ is a covering map. We can build many finite-sheeted (every point inverse is finite) coverings of a bouquet of two circles, by assembling n points over the vertex, and then, on either side (the red/blue sides?), connecting the points by n (oriented) arcs, one with one red/blue arcs going in/out of each vertex. Covering spaces of more "interesting" graphs can be assembled similarly. Our basic theme: covering spaces of a (suitably nice) space X have a very close relationship to $\pi_1(X, x_0)$.

Homotopy Lifting Property: If $p : \widetilde{X} \to X$ is a covering map, $H : Y \times I \to X$ is a homotopy, $H(y, 0) = f(y)$, and $\widetilde{f} : Y \to \widetilde{X}$ is a *lift* of f (i.e., $p \circ \widetilde{f} = f$), then there is a unique lift $\widetilde{H}$ of H with $\widetilde{H}(y, 0) = \widetilde{f}(y)$.

The idea: using the Lebesgue Number Theorem, we build the homotopy a little bit at a time, using inverse images of evenly-covered neighborhoods. In particular, applying this property in the case $Y = \{*\}$, we get the

Path Lifting Property: Given a covering map $p : \widetilde{X} \to X$, a path $\gamma : I \to X$ with $\gamma(0) = x_0$, and a point $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique path $\widetilde{\gamma}$ lifting γ with $\widetilde{\gamma}(0) = \widetilde{x}_0$.

An immediate consequence: If $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering map, then the induced homomorphism $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective. Even more, $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))) \subseteq \pi_1(X, x_0)$ is precisely the elements given by loops at x_0, whose lifts to paths starting at $\widetilde{x}_0$, are loops.

The cardinality of a point inverse $p^{-1}(y)$ is, by the evenly covered property, constant on (small) open sets, so the set of points of x whose point inverses have any given cardinality is open. Consequently, if X is connected, this number is constant over all of X, and is called the number of *sheets* of the covering $p : \widetilde{X} \to X$. If X and $\widetilde{X}$ are path-connected, then the number of sheets of a covering map equals the index of the subgroup $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$ in $G = \pi_1(X, x_0)$. The idea: loops representing elements in the same coset have lifts at $\widetilde{x}_0$ which end at the same point.

The path lifting property (because $\pi([0,1],0) = \{1\}$) is actually a special case of a more general **lifting criterion**: If $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering map, and $f : (Y, y_0) \to (X, x_0)$ is a map, where Y is path-connected and locally path-connected, then there is a lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f (i.e., $f = p \circ \widetilde{f}$) $\Leftrightarrow f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Furthermore, two lifts of f which agree at a single point are equal.

Universal covering spaces: A covering space $(\widetilde{X}, \widetilde{x}_0)$ determines a subgroup of $\pi_1(X, x_0)$. Does it go the other way? A particularly important covering space to identify is one which is simply connected. Such a covering is essentially unique: if X is *locally path-connected* and has two connected, simply connected covering spaces $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$, then there is a homeomorphism $h : X_1 \to X_2$ with $p_2 \circ h = p_1$.

Not every (locally path-connected) space X has a universal covering; a (further) necessary condition is that X be *semi-locally simply connected*: every point $x \in X$ has a nbhd $x \in \mathcal{U} \subseteq X$ with $\iota_* : \pi_1(U, x) \to \pi(X, x)$ is trivial. Conversely, every path connected, locally path connected, and semi-locally simply connected (S-LSC) space X has a universal covering. $\widetilde{X}$ is the space whose points are (equivalence classes $[\gamma]$ of) based paths $\gamma : (I, 0) \to (X, x_0)$, where two paths are equivalent if they are homotopic rel endpoints. The projection map is $p([\gamma]) = \gamma(1)$.

This in turn is the key to building covering spaces corresponding to any subgroup H of $\pi_1(X)$. [This can, alternatively, be done by mimicking the construction above, except paths γ, η are equivalent when $[\gamma * \overline{\eta}] \in H$.] The key to this is the *deck transformation group (Deckbewegungsgruppe)* of a covering space $p : \widetilde{X} \to X$; this is the set of all homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $p \circ h = p$.

By definition, these h permute each of the point inverses of p. Since h is a lift of the projection map p, by the lifting criterion h is determined by which point in $p^{-1}(x_0)$ it takes the basepoint $\widetilde{x}_0$ of $\widetilde{X}$ to. A deck transformation sending $\widetilde{x}_0$ to $\widetilde{x}_1$ exists $\Leftrightarrow p_*(\pi_1(\widetilde{X}, \widetilde{x}_0) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)$ [we need one inclusion to give h, and the opposite inclusion to ensure it is a bijection].

In general, these two groups $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0), p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)$ are always *conjugate*, by the projection of a path from $\widetilde{x}_0$ to $\widetilde{x}_1$. Paths in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$ are in 1-to-1 corresp with the cosets of $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$ in $\pi_1(X, x_0)$; so deck transformations are in 1-to-1 corresp with cosets whose representatives conjugate H to itself. The set of such elements in G is called the *normalizer of* H *in* G, and denoted $N_G(H)$ or simply $N(H)$. The deck transformation group is therefore isomorphic to the group $N(H)/H$ under $h \mapsto$ the coset with representative the projection of the path from $\widetilde{x}_0$ to $h(\widetilde{x}_0)$.

Applying this to the universal covering space $p : \widetilde{X} \to X$, in this case $H = \{1\}$, so $N(H) = \pi_1(X, x_0)$. So the deck transformation group is isomorphic to $\pi_1(X, x_0)$. For example, this gives the quickest possible proof that $\pi_1(S^1) \cong \mathbb{Z}$, since $\mathbb{R}$ is a contractible covering space, whose deck transformations are the translations by integer distances.

Thus $\pi_1(X)$ acts on its universal cover as a group of homeomorphisms. And since this action is *simply transitive* on point inverses [there is exactly one (that's the simple part) deck transformation carrying any one point in a point inverse to any other one (that's the transitive part)], the quotient map from $\widetilde{X}$ to the orbits of this action is the projection map p to X.

But! Given $G = \pi_1(X, x_0)$ and its action on a univ cover $\widetilde{X}$, we can, instead of modding out by G, mod out by any subgroup H of G, to build $X_H = \widetilde{X}/H$. This is a space with $\pi_1(X_H) \cong H$, having $\widetilde{X}$ as univ covering. And since the quotient (covering) map $p_G : \widetilde{X} \to X = \widetilde{X}/G$ factors through $\widetilde{X}/H$, we have an induced map $p_H : \widetilde{X}/H \to X$, which is a covering map. So every subgroup of G is the fundamental group of a covering of X. Even more:

The Galois correspondence: Two coverings $p_1 : X_1 \to X$, $p_2 : X_2 \to X$ are *isomorphic* if there is a homeo $h : X_1 \to X_2$ with $p_1 = p_2 \circ h$. Isomorphic coverings give, under projection, conjugate subgroups of $\pi_1(X, x_0)$. For a path-connected, locally path-connected, semi-locally simply-connected space X, the image of the induced homomorphism on π_1 gives a one-to-one correspondence between [isomorphism classes of (connected) coverings of X] and [conjugacy classes of subgroups of $\pi_1(X)$].

So, for example, if you have a group G that you are interested in, you know of a (nice enough) space X with $\pi_1(X) \cong G$, and you know enough about the coverings of X, then you can gain information about the subgroup structure of G.

For example, a free group $F(\Sigma)$ is π_1 of a bouquet of circles X. Since every covering of X is a graph, we have: every subgroup of a free group is free. A subgroup H of index n in $F(\Sigma)$ corresponds to a n-sheeted covering $\widetilde{X}$ of X. If $|\Sigma| = m$, then $\widetilde{X}$ will have n vertices and nm edges. Collapsing a maximal tree, having $n-1$ edges, to a point, leaves a bouquet of $nm - n + 1$ circles, so $H \cong F(nm - n + 1)$.

Given a free group $G = F(a_1, \dots a_n)$ and a collection of words $w_1, \dots w_m \in G$, we can determine the rank and ndex of the subgroup it H they generate by building the corresponding cover. The idea is to start with a bouquet of m circles, each subdivided and labelled to spell out the words w_i. Then we repeatedly identify edges sharing on common vertex if they are labelled precisely the same (same letter *and* same orientation). This process is known as *folding*. When done, we have (by adding trees if needed), constructed the covering corresponding to $< w_1, \dots w_m > \subseteq G$.

With work, this same process can be applied to subgroups of finitely presented groups (to be certain it stops, one usually needs a priori knowledge that the subgroup has finite index). In so doing, it yields a presentation for the subgroup!

Postscript: why care about covering spaces? The preceding discussion probably makes it clear that covering places play a central role in (combinatorial) group theory. It also plays a role in embedding problems; a common scenario is to have a map $f : Y \to X$ which is injective on π_1, and we wish to know if we can lift f to a finite-sheeted covering so that the lifted map $\widetilde{f}$ is homotopic to an embedding. Information that is easier to obtain in the case of an embedding can then be passed down to gain information abut the original map f