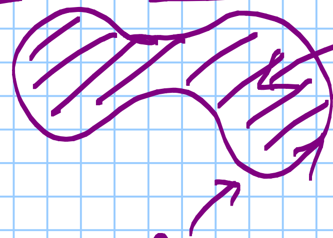
Cours 6. Théorème de Green-Riemann

6.1. Lien : intégrales doubles et intégrales curvilignes



$\iint_D$ — intégrale double

$\oint$ — l'intégrale curviligne sur un circuit

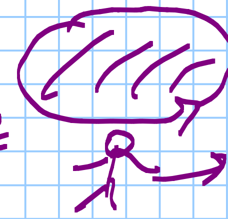
Thm Green-Riemann

Soient $D \subset \mathbb{R}^2$ avec le bord $C = \partial D$ (bord)

et $P, Q : D \to \mathbb{R}$

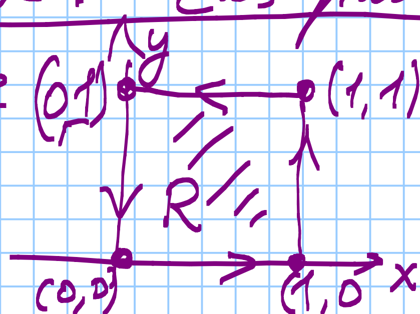
$$\boxed{\oint_{C^+ = \partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy}$$

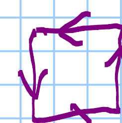
orienté de sort que si on parcours C^+ alors D est à gauche



6.2. Preuve : cas particulier

Preuve



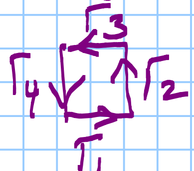
$\partial R =$ 

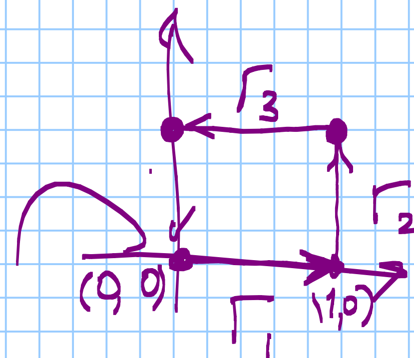
On veut m.q.

$$\oint_{\partial R} P(x,y)\,dx = -\iint_R \frac{\partial P}{\partial y}\,dx\,dy \quad \left(\begin{array}{c}\text{Choisi}\\ Q(x,y)=0\end{array}\right)$$

Coté gauche:

$$\int_{\partial R} P\,\underline{dx} = \int_{\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4} P\,dx$$





$\gamma_1 : [0,1] \to \Gamma_1 , \quad t \mapsto (t,0)$ $\qquad dx = 1\cdot dt$

$\underline{x=t,\ y=0}$ $\qquad dy = 0\,dt$

$\gamma_2 : [0,1] \to \Gamma_2 , \quad t \mapsto (1,t)$ $\qquad \left.\begin{array}{l} dx = 0\cdot dt \\ dy = 1\cdot dt \end{array}\right]$

$\gamma_3 : \overrightarrow{[0,1]} \to \Gamma_3 , \quad t \mapsto (1-t,1)$ $\qquad \underline{dx = -dt}$

$dy = 0\cdot dt$

$\gamma_4 : [0,1] \to \Gamma_4 , \quad t \mapsto (0,1-t)$ $\qquad \left.\begin{array}{l} dx = 0\cdot dt \\ dy = -dt \end{array}\right]$

$$\int_{\partial R} P(x,y)\,dx = \int_{\Gamma_1} P\,dx + \cancelto{0}{\int_{\Gamma_2} P\,dx} + \int_{\Gamma_3} P\,dx + \cancelto{0}{\int_{\Gamma_4} P\,dx}$$

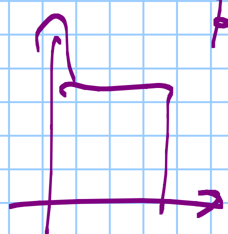
$$= \int_0^1 P(t,0)\,dt - \int_0^1 P(1-t,1)\,dt$$

$$\boxed{\int_{\partial R} P(x,y)\,dx = \int_0^1 P(t,0)\,dt - \int_0^1 P(t,1)\,dt} \quad (*)$$

$$\int_0^1 P(1-t,1)\,dt \underset{s=1-t}{=} -\int_1^0 P(s,1)\,ds = +\int_0^1 P(s,1)\,ds$$

<u>Côté droit</u>

$$-\iint_R \frac{\partial P}{\partial y}\,dx\,dy = -\int_0^1 \left(\int_0^1 \frac{\partial P}{\partial y}\,dy \right) dx$$



$$= -\int_0^1 \Big[P(x,y) \Big]_{y=0}^{1} dx$$

$$= -\int_0^1 \big[P(x,1) - P(x,0) \big]\,dx$$

$$= \int_0^1 P(x,0)\,dx - \int_0^1 P(x,1)\,dx$$

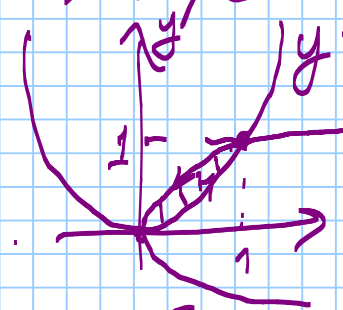
Du coup $\displaystyle \oint P(x,y)\,dx = -\iint \frac{\partial P}{\partial y}\,dx\,dy$

De même

$$\oint_{\partial R} Q(x,y)\,dy = \iint_R \frac{\partial Q}{\partial x}\,dx\,dy$$

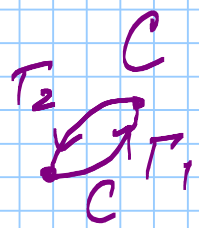
6.3 Calcule

exemple.



$y = x^2$

$$I := \int_C (2xy - x^2)\,dx + (x+y^2)\,dy$$

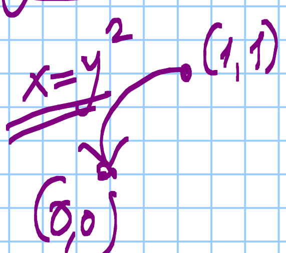


$y = x^2$

$x = y^2$

$$\gamma_1 : \underset{t \in}{[0,1]} \longmapsto \Gamma_1 \qquad \begin{cases} x = t \\ y = t^2 \end{cases}$$

$$\begin{cases} dx = dt \\ dy = 2t\,dt \end{cases}$$



$$\gamma_2 : \underset{s \in}{[0,1]} \longmapsto \Gamma_2 \qquad \begin{cases} x = (1-s)^2 \\ y = 1-s \end{cases}$$

$s = 0 \quad (x,y) = (1,1)$

$s = 1 \quad (x,y) = (0,0)$

$$\underbrace{x = y^2}$$

$$\Updownarrow$$

$$\begin{cases} x = 1-u \\ y = \sqrt{1-u} \end{cases}$$

$$\begin{cases} dx = (-2+2s)\,ds \\ dy = -ds \end{cases} \rightsquigarrow$$

$$I := \int_C (2xy - x^2)\,dx + (x+y^2)\,dy = \int_{\Gamma_1 \cup \Gamma_2} (2xy - x^2)\,dx + (x+y^2)\,dy$$

$$= \int_0^1 \left(2\cdot t \cdot t^2 - t^2\right)dt + \left(t + (t^2)^2\right)\cdot 2t\,dt$$

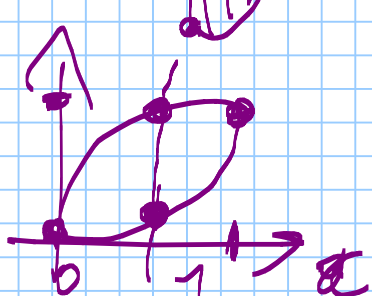
$$+ \int_0^1 2(1-s)^2(1-s) - (1-s)^4\Big)\left(-2+2s\right)ds$$

$$+ \left((1-s)^2 + (1-s)\right)ds \;\Leftarrow$$

$$\int_C \underbrace{(2xy - x^2)}_{P}\,dx + \underbrace{(x+y^2)}_{Q}\,dy$$

$$\overset{\text{thm GR}}{=} \iint \left(\frac{\partial(x+y^2)}{\partial x} - \frac{\partial(2xy - x^2)}{\partial y}\right)dx\,dy$$

$$= \iint (1 - 2x)\,dx\,dy = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} (1-2x)\,dy\right)dx$$



$$= \int_0^1 \Big[(1-2x)\,y\Big]_{y=x^2}^{\sqrt{x}}\,dx$$

$$= \int_0^1 (1-2x)\cdot\left(\sqrt{x} - x^2\right)dx$$

$$= \int_0^1 \left[x^{1/2} - 2x^{3/2} - x^2 + 2x^3\right] dx$$

$$= \left[\frac{x^{3/2}}{3/2} - 2\frac{x^{5/2}}{5/2} - \frac{x^3}{3} + 2\frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{3} - \frac{4}{5} - \frac{1}{3} + \frac{1}{2} = \frac{10-24+15}{30}$$

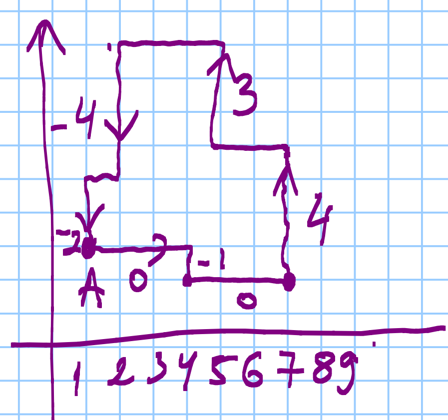
$$= \boxed{\frac{1}{30}}$$

6.4) Application du thm de Green-Riemann : aire

$$\cdot \quad \text{Aire } D := \iint_D dx\,dy$$

$$\text{"}\ \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1}\ \text{"}$$

$$\text{Aire } D := \frac{1}{2}\int_{\partial D} -\underset{\uparrow P}{y}\,dx + \underset{\uparrow Q}{x}\,dy = -\int_{\partial D} y\,dx = \underbrace{\int_{\partial D} x\,dy}$$



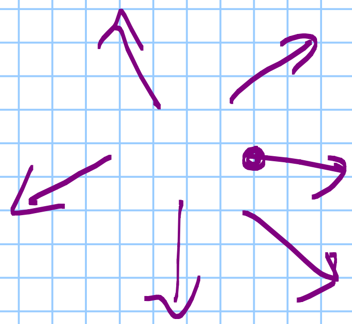
$x = 4$
$dy = -1$

$$0 + 4\cdot(-1) + 7\times 4 + 3\times 5$$
$$+ 2(-4) + 1\cdot(-2) =$$

Aire $\overset{?}{=} 3 + 6 + 6 + 5 + 9$

6.5 Thm. de Poincaré

Pour les champs de gradient



$\big(P(x,y),\, Q(x,y)\big)$ – vecteur au pt (x,y)

$\underline{\underline{(P,Q)(x,y)}}$ – champ de vecteur

Champs de gradient

$$f(x,y) \longmapsto \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)(x,y)$$

$$\boxed{P = \frac{\partial f}{\partial x}, \quad Q = \frac{\partial f}{\partial y}}$$

$$\int_{\partial D} P\,dx + Q\,dy \overset{GR.}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

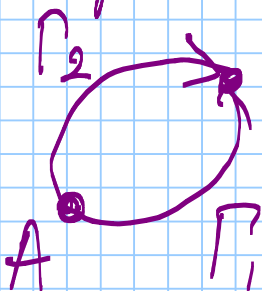
$$= \iint_D \left(\frac{\partial \left(\frac{\partial f}{\partial y}\right)}{\partial x} - \frac{\partial \left(\frac{\partial f}{\partial x}\right)}{\partial y}\right) dx\,dy$$

Thm de Schwarz $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

et alors $\boxed{\oint_{\partial D} P\,dx + Q\,dy = 0}$

$\boxed{\text{Si } (P,Q) = \overrightarrow{\text{grad}}\, f}$ pour une certaine fonction f.

Rq Pour un champ de gradient



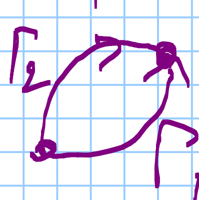
deux chemins différent de A vers B donne la même valeur de l'intégrale

$$\boxed{\int_{\Gamma_1} P\,dx + Q\,dy = \int_{\Gamma_2} P\,dx + Q\,dy}$$

dépend que des valeurs en A et B

$$P = \frac{\partial f}{\partial x} \quad \text{et} \quad Q = \frac{\partial f}{\partial y}$$

$$\int_{\Gamma_1 - \Gamma_2} P\,dx + Q\,dy = \int_{\substack{\text{circuit} \\ \text{fermé}}} P\,dx + Q\,dy \overset{\substack{(P,Q) = \overrightarrow{\text{grad}} f \\ \downarrow}}{=} 0$$



$$\text{"} \int_{-\Gamma_2} = -\int_{\Gamma_2} \text{"} \qquad \text{"} \int_a^b = -\int_b^a \text{"}$$

Comment savoir si (P, Q) est un champ de gradient ?

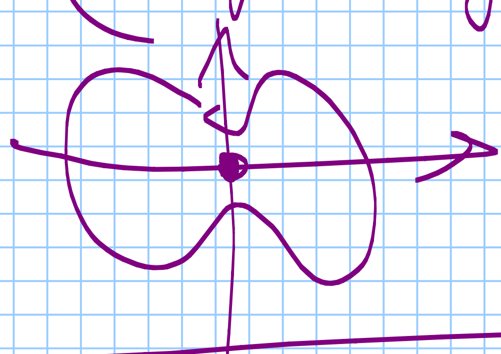
Condition necessaire :

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$ sur un domaine "étoilé" sans trou

$\boxed{\text{Comment trouver } f \text{ ?}}$

Thm de Poincaré : (P, Q) est un champ de gradient sur un **domaine étoilé** ssi $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$$\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$



Comment on cherche f si on sait que $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$P = x - y + 2xy \quad Q = y^2 - x + x^2$

$$\frac{\partial P}{\partial y} = -1 + 2x \qquad \frac{\partial Q}{\partial x} = -1 + 2x$$

On cherche f t.q.

$$\begin{cases} \dfrac{\partial f}{\partial x} = P = \underbrace{x - y + 2xy}_{} \quad (1) \\ \dfrac{\partial f}{\partial y} = Q = y^2 - x + x^2 \quad (2) \end{cases}$$

$$(1) \Rightarrow f(x,y) = \int (x - y + 2xy)\,dx$$

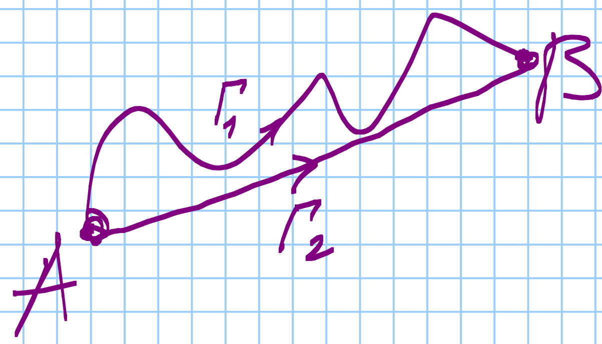
$$f(x,y) = \frac{x^2}{2} - yx + \frac{2x^2y}{2} + \varphi(y)$$

$$\frac{\partial\left(\frac{x^2}{2} - yx + \frac{\cancel{2}x^2y}{\cancel{2}} + \varphi(y)\right)}{\partial y} = y^2 - x + x^2$$

$$0 \underbrace{- x + x^2}_{} + \frac{\partial \varphi}{\partial y} = y^2 \underbrace{- x + x^2}_{}$$

$$\frac{\partial \varphi}{\partial y} = y^2 \Leftrightarrow \varphi(y) = \frac{y^3}{3} + C$$

$$\boxed{f(x,y) = \frac{x^2}{2} - yx + x^2y + \frac{y^3}{3} + C}$$



$$\int_\Gamma P\,dx + Q\,dy$$

$$(P, Q) = \text{grad} f$$

L'intégrale curviligne d'un champs de gradient entre deux pts A et B ne dépend pas de parcours – que de pts A et B.

En effet, $\Gamma_1 - \Gamma_2$ fait un circuit fermé. Donc

$$\int_{\Gamma_1 - \Gamma_2} \overrightarrow{\text{grad}} f \, ds = 0$$

$$\Rightarrow \int_{\Gamma_1} \overrightarrow{\text{grad}} f \, ds = \int_{\Gamma_2} \overrightarrow{\text{grad}} f \, ds .$$