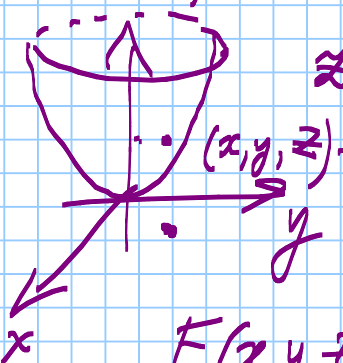
Cours 7. Surfaces

7.1. Surfaces de $\mathbb{R}^3$

1. Explicitement $\subset$ 2. Implicitement

$z = f(x,y)$ $\qquad\qquad$ $F(x,y,z)=0$



$z = x^2+y^2$ $\qquad\qquad$ $x^2+y^2+z^2-R^2=0$

$F(x,y,z) = f(x,y)-z=0$

3. Paramétrique

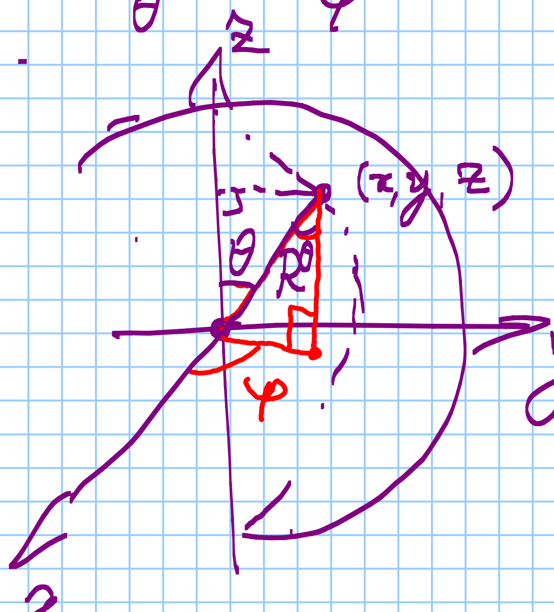
$$D \subset \mathbb{R}^2 \xrightarrow{g} S \subset \mathbb{R}^3$$

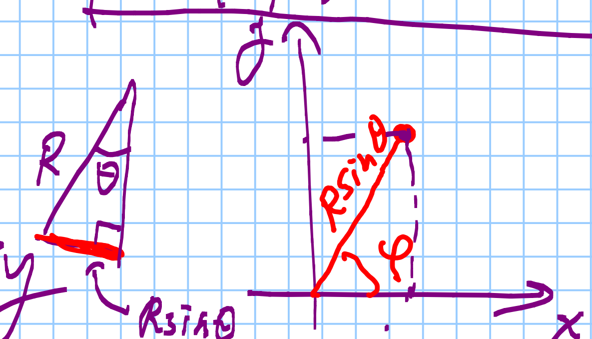
$$(u,v) \longmapsto (x(u,v), y(u,v), z(u,v))$$

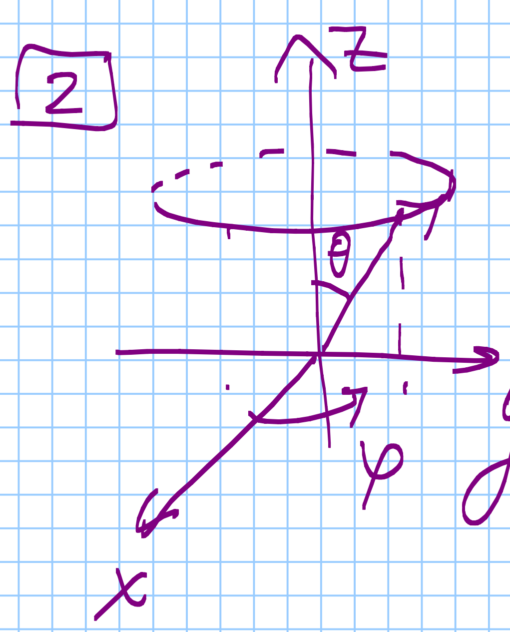
Exemple

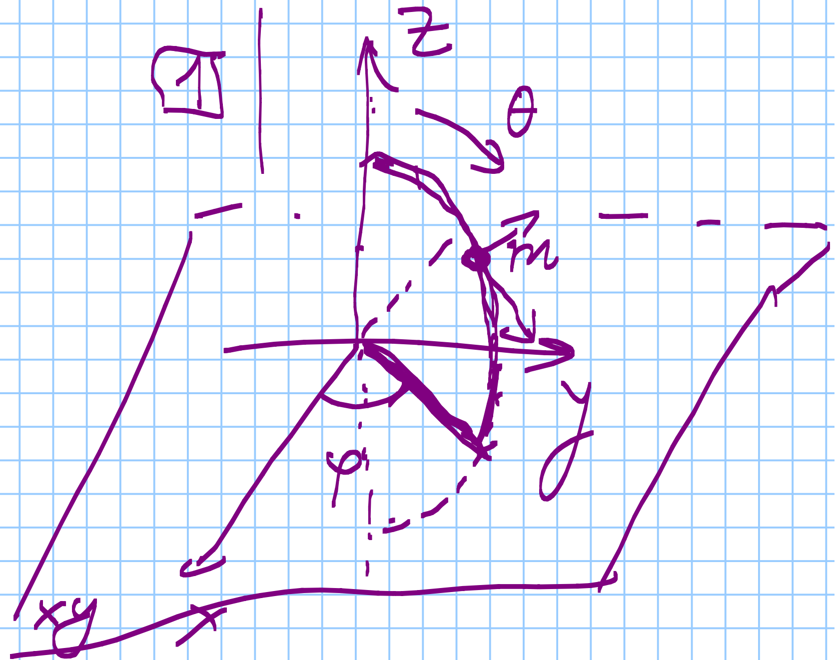
$g: [0,\pi] \times [0,2\pi[\longmapsto$
$\theta \qquad \varphi$

$$\boxed{\begin{cases} x(\theta,\varphi) = R\sin\theta \cdot \cos\varphi \\ y(\theta,\varphi) = R\sin\theta \sin\varphi \\ z(\theta,\varphi) = R\cos\theta \end{cases}}$$









$\boxed{1}$ φ fixé, θ varie $[0, \pi]$

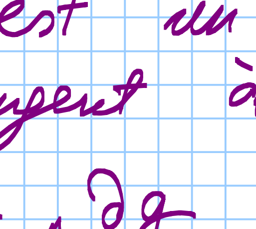
Vect. tangent à demi-cercle

est $\dfrac{\partial g}{\partial \theta} = (R\cos\theta\cos\varphi,\ R\cos\theta\sin\varphi,\ -R\sin\theta)$

$\boxed{2}$ θ fixé, φ varie $[0, 2\pi]$

$$\frac{\partial g}{\partial \varphi} = (-R\sin\theta\sin\varphi,\ R\sin\theta\cos\varphi,\ 0)$$

Vecteur normal à une surface



au pt. m est un vecteur orthogon. au plan tangent à pt. m.

$N = \dfrac{\partial g}{\partial \theta} \wedge \dfrac{\partial g}{\partial \varphi}$ ~ vecteur normal de longueur $|v_1||v_2|\sin t$

à la sphère.

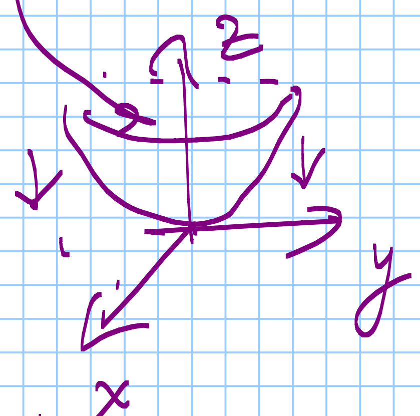
L'élément d'aire

$$\boxed{dA := \left\| \frac{\partial \vec{g}}{\partial u} \wedge \frac{\partial \vec{g}}{\partial v} \right\| du\, dv}$$

En particulier si surface

$z = h(x,y)$

$$dA = \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}\, dx\, dy$$



7.2) Intégrale d'une fonction sur surface

$f : V \longrightarrow \mathbb{R}$, $V \subset \mathbb{R}^3$

S - surface $\subset V \subset \mathbb{R}^3$

$g : (u,v) \longrightarrow (x,y,z)$, $(u,v) \in D \subset \mathbb{R}^2$

(Sphère $D = [0,\pi] \times [0,2\pi]$)

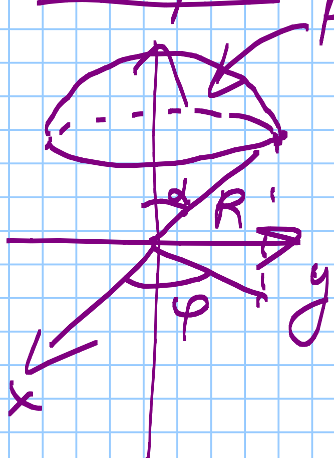
$f(x,y,z)$ sur la surface devient une fonction de u et v

$$f(x(u,v), y(u,v), z(u,v)) = f(g(u,v))$$

$$I := \iint_D f(g(u,v)) \underbrace{\|\vec{N}(u,v)\| \, du\, dv}_{dA}$$

Rque : si f est $\equiv 1$ $I = \int_D dA = \text{Aire } S$ surface

Exemple



Calotte sphérique S

$$\boxed{0 \leq \theta \leq \alpha}$$
$$0 \leq \varphi < 2\pi$$

$$\vec{N}(\theta,\varphi) = \frac{\partial \vec{g}}{\partial \theta} \wedge \frac{\partial \vec{g}}{\partial \varphi}$$

$$= \left(R^2 \sin^2\theta \cos\varphi,\ R^2 \sin^2\theta \sin\varphi,\ R \sin\theta \cos\theta\right)$$

$$\|\vec{N}(\theta,\varphi)\| = R^2 \sin\theta \qquad (\sin^2 t + \cos^2 t = 1)$$

$$A(S) = \int_0^\alpha \int_0^{2\pi} R^2 \sin\theta \, d\theta \, d\varphi = R^2 \int_0^\alpha \sin\theta \, d\theta \int_0^{2\pi} d\varphi$$

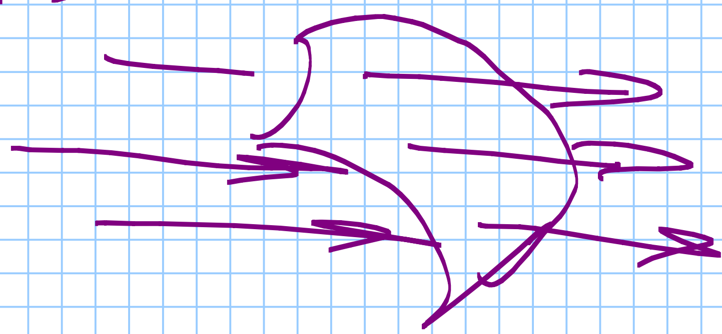
$$= 2\pi R^2 (1 - \cos\alpha)$$

Si $\alpha = \pi$ $\quad A(\text{Sphère}) = 4\pi R^2$

7.3 Intégrale de surface de champs de vecteurs.

Champs de vecteurs sur S :

$\vec{W}$ ⇝ le flux de $\vec{W}$ à travers de S



$dA = \|\vec{N}(u,v)\|\,du\,dv$

$\vec{dA} = \vec{N}(u,v)\,du\,dv$

$= \vec{n}^+ \|\vec{N}(u,v)\|\,du\,dv$

$$\boxed{J := \iint_S \vec{W} \cdot \vec{dA}}$$

$$\vec{W}(x,y,z) = \left(P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\right)$$

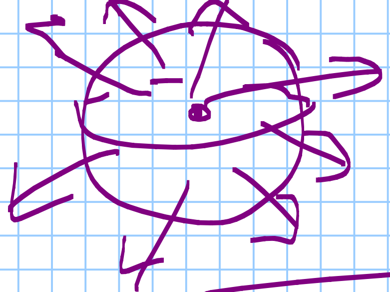
$$J = \iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$

Sur S $\quad g(u,v) = \left(x(u,v), y(u,v), z(u,v)\right)$

dx, dy, dz ⇝ fns de $du\,dv$

7.4. Formule de la divergence (thm d'Ostrogradski - Gauss)

Relie un flux de champ à travers d'une surface fermée



Soit E un domaine de $\mathbb{R}^3$

et $S := \partial E$ le bord de E

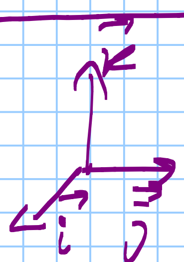
$$\boxed{\iint_{\partial E} \vec{W}\cdot d\vec{A} = \iiint_{E} \underline{\underline{\operatorname{div}}}\, \vec{W}\, dx\, dy\, dz}$$

$\vec{W}(x,y,z) = (P, Q, R)(x,y,z)$

$$\boxed{\operatorname{div} \vec{W} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

div : Champ $\longrightarrow$ Fonction

$$\boxed{\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}}$$



$$\operatorname{div} \vec{W} = \vec{\nabla} \cdot \vec{W}$$

(__Rque__ : $\operatorname{grad} f = \vec{\nabla} f$

$\overrightarrow{\operatorname{rot}}\, \vec{W} = \vec{\nabla} \wedge \vec{W}$)

__Exemple__ E - boule de $\mathbb{R}^3$ de centre O, de rayon r.

$\vec{W} = P\vec{i} + Q\vec{j} + R\vec{k}$ $\quad P = x$

$Q = y$

$\vec{W} = (x, y, 2z)$ $\quad R = 2z$

$$J := \iint_{S^+} \vec{W}\cdot \vec{N}(\theta, \varphi)\, d\theta\, d\varphi$$

$$\vec{W}(\theta,\varphi) = (r\cdot\sin\theta\cos\varphi, r\cdot\sin\theta\sin\varphi, 2r\cos\theta)$$

$$\vec{N}(\theta,\varphi) = (r^2\sin^2\theta\cos\varphi, r^2\sin^2\theta\sin\varphi, 2r\cos\theta)$$

$$\vec{W}\cdot\vec{N} = r^3(\sin\theta + \cos^2\theta\sin\theta)$$

$$J = r^3\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta + \cos^2\theta\sin\theta\, d\theta$$

$$= \underline{\frac{16\pi r^3}{3}}$$

$$\operatorname{div}\vec{W} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 1+1+2 = 4$$

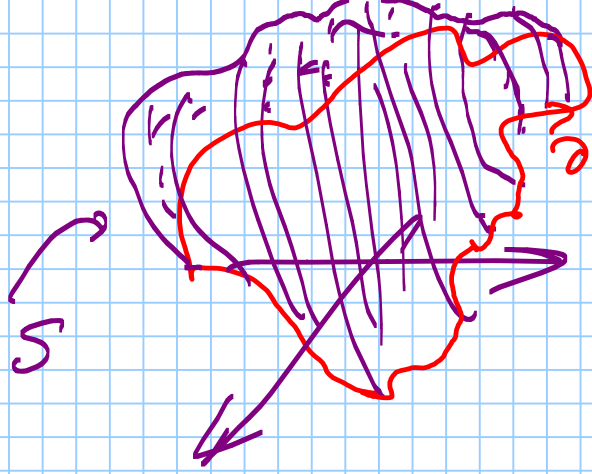
$$\iiint_E \operatorname{div}\vec{W}\,dx\,dy\,dz = 4\iiint_E dx\,dy\,dz$$

$$= 4\cdot \text{Volume}(E) = 4\cdot\frac{4}{3}\pi r^3 = \underline{\frac{16\pi r^3}{3}}$$

7.5 Formule de rotationnel

(= thm. de Stokes)

Relie l'intégrale curviligne d'un champ de vecteurs $\vec{W}$ sur un circuit fermé avec le flux de rotationnel de $\vec{W}$ à travers d'une surface dont ce circuit est le bord



$\partial S = C^+$

$$\boxed{\oint_{\partial S} \vec{W}\cdot\vec{d\ell} = \iint_S \vec{\text{rot}}\,\vec{W}\cdot\vec{dA}}$$

$$\vec{\text{rot}}\,\vec{W} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x},\ \underline{\underline{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}}\right)$$

$$\vec{W} = (P, Q, R)$$

$$\vec{\nabla}\wedge\vec{W} = \begin{vmatrix} \vec{i} & \frac{\partial}{\partial x} & P \\ \vec{j} & \frac{\partial}{\partial y} & Q \\ \vec{k} & \frac{\partial}{\partial z} & R \end{vmatrix}$$

de la formule de Green-Riemann

$$\left(0,\ 0,\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\boxed{\begin{aligned}\oint_{\partial S} P\,dx + Q\,dy + \cancel{R\,dz} = \iint_S & \cancel{\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)} dy\,dz + \cancel{\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)} dx\,dz \\ & + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy\end{aligned}}$$

si pas de composante z

On a $R \equiv 0$. $\frac{\partial}{\partial z}$(de n'importe quelle fn) $= 0$

En $\mathbb{R}^2$ | thm de Stokes (de rotationnel) devient le thm de Green-Riemann.