LIE ALGEBROIDS AS SUPERMANIFOLDS

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Abstract

The notions of

- a Lie algebroid
- a Lie-Rinehart pair
- a supermanifold with a homological vector field
- an odd Poisson manifold
- a Gerstenhaber algebra
- a Q-manifold

turn out to describe more or less the same objects. This talk is an attempt to establish the equivalences.

1. Introduction.

The purpose of this talk is to show that several notions floating around are in fact one and the same. The notions which we discuss are: Lie algebroids [3] and supermanifolds with homological vector fields [9]. There is also a purely algebraic reformulation of Lie algebroids which does not require the presence of a manifold: Lie-Rienhart pair [6].

The notion of a Lie algebroid is an analogue of the Lie algebra of a Lie group for differentiable groupoids. It combines the properties of a Lie algebra and a tangent bundle to a manifold. Lie algebroids recently became quite popular in symplectic geometry, representation theory etc. There are certain interesting examples so it is bit mroe than just a complicated way to talk about the Lie structure on the tangent bundle.

We will not make it explicite here that Lie algebroids are Q- manifolds, but hope it should be clear to the conosseurs of [1].

Definition 1.1. Lie algebroid is a smooth vector bundle $A \to M$ with

- a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of A,
- a bundle map $\alpha : A \to TM$ called an anchor of the Lie algebroid A, such that
  - the anchor α defines a Lie algebra morphism $\Gamma(A) \to \mathcal{X}(M)$, vector fields (i.e. sections of TM, the Lie algebra structure given by the commutator);

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– for $f \in C^\infty(M), X, Y \in \Gamma(A)$ a sort of a Leibniz rule is required:

$$[X, fY] = f[X, Y] + \alpha(X)(f)Y.$$

This way the sections of A get a C^∞-module structure. Moreover the anchor defines the morphism of Lie algebras as C^∞-modules.

2. Examples

- *Lie algebra.* M is a point. Then A is just a Lie algebra, the anchor here is 0.
- *Bundle of Lie algebras.* The anchor is again 0.
- *Cotangent Lie algebroid.* Let M be a Poisson manifold with the Poisson bivector field: $\pi \in \Lambda^2 TM$. Then π defines an anchor map $\tilde{\pi} : T^*M \to TM$ by $\tilde{\pi}_p(\xi_p) = \xi_p \lrcorner \pi_p$. the bracket is defined as follows:
$$[\xi, \eta] = d\phi(\xi, \eta) + \tilde{\pi}(\xi) \lrcorner d\eta - \tilde{\pi}(\eta) \lrcorner d\xi$$
- *Atiyah algebroid.* [3]
- *Lie algebroid of a differentiable groupoid.* Given a Lie groupoid (G, s, t), consider the normal bundle along the base of the groupoid, with section right -invariant s-vertical vector fields. Lie bracket then comes from the Lie bracket on the $Lie(G)$ and the anchor is given by Tt whatever it is.
- *Tangent Lie algebroid.* TM is a Lie algebroid with $\alpha \equiv id$. It can be seen as the Lie algebroid of the Lie groupoid $M \times M$. Notice that a Lie algebroid determines and is determined by a neighborhood of the identity section in the groupoid.
- *Transformation algebroid.* [3]

Generalization: Lie-Rinehart pair $(\mathfrak{g}, C)$, where $\mathfrak{g}$ is a Lie algebra and a C-module, while C is an associative commutative algebra and a $\mathfrak{g}$-module, plus certain compatibility conditions. Lie algebroid $A \to M$ is a Lie-Rinehart pair with $\mathfrak{g} = \Gamma(M, A)$ and $C = C^\infty(M)$.

3. Lie algebroid cohomology

The anchor together with a Lie bracket defines the Lie algebroid differential

$$d_A : \Gamma(\Lambda^{k-1} A^*) \to \Gamma(\Lambda^k A^*)$$

Indeed, let $\xi \in \Gamma(\Lambda^{k-1} A^*)$ then

$$\begin{aligned} d_A \xi(X_1 \wedge \cdots \wedge X_k) &= \sum_i (-1)^i \alpha(X_i) \xi(X_1 \wedge \cdots \wedge \check{X}_i \wedge \cdots \wedge X_k) \\ &\quad + \sum_{i<j} (-1)^{i+j-1} \xi(X_1 \wedge \cdots \check{X}_i \cdots \check{X}_j \cdots \wedge X_k) \end{aligned}$$

It is not too difficult to verify that $d_A^2 = 0$. This differential defines the Lie algebroid cohomology with trivial coefficients. Lie algebroid cohomology is a reasonable definition. In the case of a Lie algebra considered as a Lie algebroid, the Lie algebroid cohomology coincide with the Chevalley-Eilenberg cohomology. The tangent Lie algebroid cohomology coincide with the deRham cohomology of the underlying manifold. For a Poisson manifold the Lie algebroid cohomology of the cotangent algebroid give the Poisson cohomology.

Remark 3.1. The Lie algebroid structure on the bundle $A \to M$ is equivalent to the derivation d_A of the exterior algebra $\Gamma(\Lambda A^*)$ of degree 1 and square 0.

4. Super-stuff

Consider a super-space, that is a space with a $\mathbb{Z}_2$ grading $V = V_0 \oplus V_1$. Elements of V_0 have degree 0, and they are called even, while elements of V_1 have degree 1, and they are called odd. Basic principle for a super-objects is the rule of signs [5]. If in some formula of usual algebra there are monomials with interchanged terms, then in the corresponding formula in super-algebra every interchange of neighboring terms , say x and y, is accompanied by the multiplication of the monomial by the factor $(-1)^{\tilde{x}\tilde{y}}$, where $\tilde{x} = \deg x, \tilde{y} = \deg y$

What are algebraic functions on a super-space V? Symmetric polynomials in even and odd variables:

$$\begin{aligned}\mathbb{S}(V_0 \oplus V_1) &= T(V_0 \oplus V_1)/\{xy - (-1)^{\tilde{x}\tilde{y}}yx = 0\}\\ &= TV_0/\{xy - yx = 0\} \otimes TV_1/\{xy + yx = 0\}\\ &= SV_0 \otimes \Lambda V_1\end{aligned}$$

If we take analytic functions instead of algebraic we get $C^\infty(V_0+V_1) = C^\infty(V_0)\otimes\Lambda V_1$ by the Taylor decomposition since the elements of V_1 are nilpotent.

A smooth m-manifold can be defined as an object obtained from domains in $\mathbb{R}^m$ pasted together by means of smooth transformations. This definition can be formulated in a purely algebraic way. Namely, one can identify a domain $U \subset \mathbb{R}^m$ with the algebra $C^\infty(U)$ of all smooth functions on U and a smooth map of U to V with a homomorphism of $C^\infty(V)$ to $C^\infty(U)$. Such an algebraic construction is generalized as follows. By definition we identify an $(m|n)$-super-domain U_n with a $\mathbb{Z}_2$-graded algebra $C^\infty(U) \otimes \Lambda^n$, where U is a domain in $\mathbb{R}^m$ and Λ^n is a Grassman algebra in n generators $\xi^1, \dots, \xi^n$. (This is a passage from [8]).

There is one important functor on the category of super-spaces – the parity change functor Π. It is defined $(\Pi V)_0 = V_1, (\Pi V)_1 = V_0$

Now back to algebroids. Here we follow [7]. We will view $\Gamma(\Lambda^{k-1}A^*)$ as the algebra of functions on the super-manifold ΠA, where Π is a parity change functor applied to each fibre.

Let $\{x^i\}_{i=1,\dots,\dim M}$ be a coordinate chart on $U \subset M$, and $\{e^a\}_{a=1,\dots,\mathrm{rk} A}$ be a local basis of sections of A^* over U (dual to a basis $\{e_a\}$ of sections of A). Denote by ξ^a the corresponding generators of the Grassman algebra $\Gamma(U, \Lambda A^*)$. Then $\{(x^i, \xi^a)\}$ give a coordinate chart on ΠA with the transformation law inherited from the vector bundle A^*. The derivation d_A can be viewed as an (odd) vector field on ΠA, satisfying

$$[d_A, d_A] = 2d_A^2 = 0,$$

where the bracket denotes a super-commutator. Such vector field is called homological.

Definition 4.1. The odd vector field X on a super-manifold is called homological if $[X, X] = 2X^2 = 0$.

Many objects in various branches of mathematics can be described and studied in terms of homological vector fields.

Example 4.2. differential forms on M can be viewed as functions on ΠTM. Polyvector fields on M can be viewed as functions on ΠT^*M. On ΠTM there is an odd global vector field D such that for each differential form ω on M viewed as a function on ΠTM, $D\omega = d\omega$. Here d is just the deRham differential. Clearly, $D^2 = 0$. The deRham differential is a homological vector field on a super-manifold $\Pi TM = (M, \Lambda T^*M)$.

Since a super-manifold is defined algebraically there exist different notations for a super-manifold: ΠA and $(M, \Lambda A^*)$.

Theorem 4.3 (Vaintrob [9]). *Consider two super-manifolds associated with a bundle* $A \to M$: $\Pi A = (M, \Lambda A^*)$ *and* $\Pi A^* = (M, \Lambda A)$. *then the following three classes of objects:*

1. *Lie algebroid structure on a vector bundle* $A \to M$
2. *homological vector fields of degree* 1 *on a super-manifold* ΠA
3. *odd linear Poisson structures on* ΠA^*

Any vector field on ΠA of degree 1, consists of two terms: derivations along the even directions and derivation along odd directions. The condition $d_A^2 = 0$ is equivalent to:

- The derivation along the even directions defines the anchor,
- The derivation along the odd directions defines the Lie algebra structure on the sections of the Lie algebroid

It could be verified directly. In local coordinates, we have

$$d_A = \xi^a A_a^i(x)\partial_x - \frac{1}{2}C_{ab}^c(x)\xi^a\xi^b\partial_{\xi^c}$$

where repeated indexes assume summation. Then the local expressions for the anchor and the Lie bracket on A:

$$\begin{aligned} \alpha(e_a) &= A_a^i(x)\ \partial_x \\ [e_a, e_b] &= C_{ab}^c(x)\ e_c \end{aligned}$$

Conversely, for a bracket $[\cdot, \cdot]$ and for an anchor map $\alpha : A \to TM$ we can find the functions $C_{ab}^c(x)$ and $A_a^i(x)$ which define the vector field d_A.

The canonical duality between ΠA and ΠA^* transforms ξ^a into ∂_{e_a} and ∂_{ξ^a} into e_a. the vector field d_A becomes the odd bivector field on ΠA^* :

$$\pi = A_a^i(x)\partial_{e_a} \wedge \partial_{x^i} - \frac{1}{2}C_{ab}^c(x)e_c\partial_{e_a} \wedge \partial_{e_b}.$$

A direct computation shows that π defines an odd Poisson structure on $\Pi A^a st$. Another name for an odd Poisson structure is a Gerstenhaber algebra.

WHY THE WAY OF LOOKING AT LIE ALGEBROIDS AS SUPER-MANIFOLDS WITH A HOMOLOGICAL VECTOR FIELD IS SO GREAT?

Why, indeed? The answer is that in the super-language it becomes natural and easy to define the differential calculus on Lie algebroids (or Q-manifolds). In particular:

- morphisms of Lie algebroids
- modules over Lie algebroids
- deformations of Lie algebroids

See for example [9] and you may compare the ease with which a super-mathematician treats the issues with the heavy machinery of traditional approach [4].

5. Lie algebroid via Gerstenhaber algebra

Definition 5.1. A triple $(g, [\cdot,\cdot], \mu)$ is called a Gerstenhaber algebra when

- $\mathfrak{g} = \oplus \mathfrak{g}^k$ is a graded vector space;
- $[\cdot,\cdot] : \mathfrak{g}^{k+1} \wedge \mathfrak{g}^{l+1} \to \mathfrak{g}^{k+l+1}$ defines a Lie bracket;
- $\mu : \mathfrak{g}^k \wedge \mathfrak{g}^l \to \mathfrak{g}^{k+l}$ defines an associative commutative product;
- there is the Leibniz rule: $[\cdot,\cdot]$ is a derivation of μ.

Attention: in most of standard sources the bracket is of degree 0 and the product of degree 1, it is consistent with our definition if instead of $\mathfrak{g}$ we consider $\mathfrak{g}[1]$, where $\mathfrak{g}[1]^k = \mathfrak{g}^{k+1}$. Consider for example the Schouten algebra (the algebra of polyvector fields on a manifold: ΛTM). The Schouten algebra is an example of a Gerstenhaber algebra with the product μ being the exterior product. The Lie algebra structure on the polyvector fields is obtained by the Leibniz rule from the Lie algebra structure on the sub-algebra of vector fields and functions. The Lie algebra structure on vector fields is given by the commutator, the Lie bracket of a vector field and a function is a vector field applied to the function.

The standard practice is to place the vector fields in the degree 0, bivector fields in degree 1, etc. The functions then are in degree -1.

We don't move the degree – maybe we will regret it later.

Theorem 5.2. *A vector bundle $A \to M$ is a Lie algebroid iff $\mathfrak{g} = \Gamma(M, \Lambda A)$ is a Gerstenhaber algebra.*

In fact, we see that it is nothing but the odd linear Poisson structure on the supermanifold ΠA.

6. Appendix

For completeness, we give here a definition in algebraic geometric terms taken from Beilinson, Berstein [2]:

Definition 6.1. For a scheme X a Lie algebroid L on X is a (quasi-coherent) $\mathcal{O}_X$-module equipped with a morphism of $\mathcal{O}_X$-modules $\sigma : L \to \mathcal{T}_X$(where $\mathcal{T}_X = Der\mathcal{O}_X$, the tangent sheaf of X) and a $\mathbb{C}$-linear pairing $[\cdot,\cdot] : L \otimes_{\mathbb{C}} L \to L$ such that

- $[\cdot,\cdot]$ is a Lie algebra bracket and σ commutes with brackets,
- For $l_1, l_2 \in L, f \in \mathcal{O}_X$ one has $[l_1, fl_2] = f[l_1, l_2] + \sigma(l_1)(f)l_2$.

Examples:

1. The tangent sheaf $\mathcal{T}_X$, the map $\sigma = Id_{\mathcal{T}_X}$.
2. Assume that a Lie algebra $\mathfrak{g}$ acts on X, i.e. we have a morphism of Lie algebras $\alpha : \mathfrak{g} \to \mathcal{T}_X$. Then $\mathfrak{g}_X = \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$ becomes a Lie algebroid: $\sigma(f \otimes \gamma) = f\alpha(\gamma)$, the bracket is defined by $[f_1 \otimes \gamma_1, f_2 \otimes \gamma_2] = f_1 f_2 \otimes [\gamma_1, \gamma_2] + f_1\alpha(\gamma_1)(f_2) \otimes \gamma_2 - f_2\alpha(\gamma_2)(f_1) \otimes \gamma_1$

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