MAKE IT YOURSELF STRONG HOMOTOPY STRUCTURE

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Abstract

This are notes for the talk at the conference GAP III, in Perugia. We talk about a general definition of a strong homotopy structure as a solution of a Maurer-Cartan equation on a corresponding governing Lie algebra. We apply this philosophy to construct strong homotopy Lie bialgebras.

1. Basic example: strong homotopy associative algebra

Why is it called homotopy? In fact, the following general definition of chain homotopic maps is used over and over again.

Definition 1. Two maps α and β between complexes $(\mathcal{C}, d) = \cdots \to A \to B \to C \to \cdots$ and $(\mathcal{C}', d') = \cdots \to A' \to B' \to C' \to \cdots$ are called chain homotopic if there is a map $h : \mathcal{C} \to \mathcal{C}[-1]$ such that

$$\alpha - \beta = hd + d'h.$$

We could draw it diagrammatically as follows:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & A & \xrightarrow{d} & B & \xrightarrow{d} & C & \xrightarrow{d} & \cdots \\ & & \beta\big\downarrow\big\downarrow\alpha & \overset{h}{\swarrow} & \beta\big\downarrow\big\downarrow\alpha & \overset{h}{\swarrow} & \beta\big\downarrow\big\downarrow\alpha & & \\ \cdots & \xrightarrow{d'} & A' & \xrightarrow{d'} & B' & \xrightarrow{d'} & C' & \xrightarrow{d'} & \cdots \end{array} \tag{1}$$

This is a useful definition in many instances: for example to show that a complex is acyclic one could look for a homotopy of the identity map to a zero map.

Let $A = (\oplus A_i, d)$ be a graded differential space, in other words a complex with a differential $d : A_i \to A_{i+1}$, (an element $a \in A_i$ if its degree $\overline{a} = i$.) Define a grading on the tensor powers of A : $A^{\otimes n}$ as follows

$$\overline{(a_1 + a_2 + \cdots + a_n)} = \sum_{j=1}^{n} \overline{a_j} - n + 1. \tag{2}$$

Consider a map μ which acts from $A_{k+1} \otimes A_{l+1} \to A_{k+l+1}$, this map is of degree 0 with respect to the chosen grading. Indeed, the degree of elements from $A_{k+1} \otimes A_{l+1}$ is $(k+1)+(l+1)-2+1 = k+l+1$, so $A_{k+1} \otimes A_{l+1} \subset (A^{\otimes 2})_{k+l+1}$. In particular, if $k = l = 0$, an example of such a map will be a usual multiplication on an ordinary ungraded vector space.

We say that the product $\mu : A \otimes A \to A$ is associative up-to homotopy if the maps $\mu(\mu \otimes \mathrm{Id}) : A^{\otimes 3} \to A$ and $\mu(\mathrm{Id} \otimes \mu) : A^{\otimes 3} \to A$ are homotopic in the sense of (1). That is, if there is a map μ_3 of degree -1 which provides a homotopy of $\mu(\mu \otimes \mathrm{Id}) - \mu(\mathrm{Id} \otimes \mu)$ to a zero map:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & (A^{\otimes 3})_{i-1} & \xrightarrow{d} & (A^{\otimes 3})_{i} & \xrightarrow{d} & (A^{\otimes 3})_{i+1} & \xrightarrow{d} & \cdots \\ & & {\scriptstyle \mu(\mu\otimes \mathrm{Id})-\mu(\mathrm{Id}\otimes\mu)}\big\downarrow & \overset{\mu_3}{\swarrow} & \big\downarrow & \overset{\mu_3}{\swarrow} & \big\downarrow & & \\ \cdots & \xrightarrow{d} & A_{i-1} & \xrightarrow{d} & A_i & \xrightarrow{d} & A_{i+1} & \xrightarrow{d} & . \end{array} \tag{3}$$

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The differential $d : A \to A$ we extend to act on the product $A \otimes A$ by the Leibniz rule. If we denote $\mu(a \otimes b)$ just by ab then $\mu(\mu \otimes \mathrm{Id})(a \otimes b \otimes c) = (ab)c$ and $\mu(\mathrm{Id} \otimes \mu)(a \otimes b \otimes c) = a(bc)$ and we have the following homotopy condition:

$$(ab)c - a(bc) = d\mu_3(a \otimes b \otimes c) + \mu_3(da \otimes b \otimes c) + \mu_3(a \otimes db \otimes c) + \mu_3(a \otimes b \otimes dc) \tag{4}$$

We could look at it in a more general way. Namely, we define a lift of any map $\mu_k : A^{\otimes k} \to A$ on any tensor power product of $A^{\otimes n}$ as follows:

$$\widehat{\mu_k} = \begin{cases} 0 & \text{if } n < k \\ \sum_{r=0}^{n-k-1} (\mathrm{Id}^{\otimes r} \otimes \mu \otimes \mathrm{Id}^{\otimes (n-k-r)}) & \text{otherwise} \end{cases} \tag{5}$$

Here Id denotes the identity map of the space A. In particular, for our map $\mu : A \otimes A \to A$ and $n = 3$ it becomes $\widehat{\mu}(a_1 \otimes a_2 \otimes a_3) = \mu(a_1 \otimes a_2) \otimes a_3 - a_1 \otimes \mu(a_2 \otimes a_3)$.

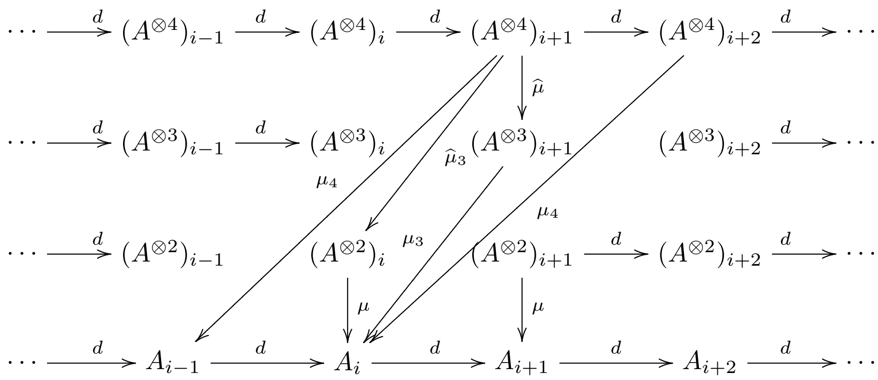
This way we could say that Equation (4) defines the homotopy of the map $\mu\widehat{\mu}$ to the zero map.

One could look further for a condition on the μ_3, by defining its lift to $A^{\otimes 4} : \widehat{\mu}_4(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = \mu_3(a_1 \otimes a_2 \otimes a_3) \otimes a_4 + a_1 \otimes \mu_3(a_2 \otimes a_3 \otimes a_4)$. Then in its turn there is a higher homotopy for it if there is another map μ_4 of degree -2 such that

$$\mu\widehat{\mu}_3 + \mu_3\widehat{\mu} = d\mu_4 + \mu_4 d$$

as shown on the diagram:

(6)



A condition on μ_4 will be given by a higher homotopy map $\mu_5 : A^{\otimes 5} \to A$ of degree -3. Similarly, one could consider all higher homotopies $\mu_n : A^{\otimes n} \to A$ of degree $2 - n$.

The lift (5) used to define the action of these maps on the any tensor power of A is the right one for the construction of the associative structure up-to homotopy. There is an explanation in terms of coderivations. On the other hand it is easier to see it in the dual picture which we will show in the next section.

Anyway, here is a definition of a strong associative algebra (also known as A_∞-algebra) in simple terms (here we follow closely [5]):

Definition 2. An A_∞-algebra over a field $\Bbbk$ is a $\mathbb{Z}$-graded space

$$A = \oplus_{i \in \mathbb{Z}} A_i$$

endowed with graded maps

$$\mu_n : A^{\otimes n} \to A,$$

of degree $2 - n$, satisfying the following equations.

- Firstly, $\mu_1\mu_1 = 0$, that is (A, μ_1) is a differential complex.
- Secondly,

$$\mu_1\mu_2 = \mu_2\mu_1.$$

  So μ_1 is a graded derivation with respect to multiplication μ_2.

- Finally, we have for any $n \geq 1$:

$$\sum_{s+t=n+1} \mu_t \mu_s = 0$$

the maps μ_n act on $A^{\otimes k}$ by their lifts given by formula (5), where by abuse of notation we omit the difference between an initial map and a one with a hat.

Remark 3. We could make several general remarks:

- Homology of A, $H^*A = H^*(A, \mu_1)$ has a structure of an associative algebra.
- If A is not graded, that is all $A_n = \emptyset$ for all $n \neq 0$ then $A = A_0$ is an ordinary associative algebra. Indeed, since μ_n is of degree $2-n$ all μ_n except μ_2 have to vanish.
- If all $\mu_n = 0$ for all $n \geq 3$ then A is a differential graded associative algebra.
- All signs are actually taken care of by the lift (5). They do appear however as soon as we write the action of μ_n on $A^{\otimes}n$ as we have seen already in Equation (4).

If we endow the set of maps $TA \to A$ by a commutator Lie bracket then the conditions on maps μ_n from Definition of the A_∞-structure could be expressed as a Maurer-Cartan equation for the operator $\sum \mu_n$, defined on TA using the lift (5).

To make sense out of this last idea let us pass to the dual picture.

2. Dual picture - (co)bar construction

We start here first by reproducing almost word by word Victor Ginzburg's unpublished lectures [3].

Let A be a finite dimensional space. Then we could consider its dual space $A^* = Hom_{\Bbbk}(A, \Bbbk)$ and form a non-unital tensor algebra

$$TA^* := A^* \oplus (A^* \otimes A^*) \oplus \cdots \oplus (A^*)^{\otimes m} \oplus \cdots .$$

This is a free associative algebra on A^* without a unit.

Proposition 4. *Giving an associative algebra structure on A is equivalent to giving a map $D : TA^* \to TA^*$ such that*

(1) $D^2 = 0$,
(2) *D is a super-derivation of degree 1, that is $D : (A^*)^k \to (A^*)^{k+1}$.*

Proof. Given a multiplication map $m : A \otimes A \to A$ and using canonical isomorphism of $(A \otimes A)^*$ and $A^* \otimes A^*$ for a finite dimensional A we get the D to be the transpose operator $m^\perp : A^* \to A^* \otimes A^*$. Then it could be extended to a super-derivation on TA^* by applying a super-Leibniz rule.

In the other direction: given a super-derivation on TA^* we transpose its restriction to A^* to get the multiplication on A using identifications A^{**} with A and $(A^* \otimes A^*)^*$ with $A \otimes A$.

We have to show now that the associativity of m is equivalent to $D^2 = 0$. Since TA^* is generated by A^* we need only to show that $D^2 : A^* \to A^* \otimes A^* \otimes A^*$ is the zero map, the super-Leibniz rule takes care of the rest. Consider some linear functional $\lambda \in A^*$. Then $d\lambda \in A^* \otimes A^*$ and we could find $\eta_1, \cdots \eta_n, \xi_1, \cdots \xi_n \in A^*$ so that

$$d\lambda(a \otimes b) = \left(\sum_{i=1}^n (\eta_i \otimes \xi_i) \right) (a \otimes b) = \sum_{i=1}^n \eta_i(a) \xi_i(b).$$

On the other hand,

$$d\lambda(a \otimes b) = m^\perp \lambda(a \otimes b) = \lambda(m(a, b)) = \lambda(ab).$$

Using the fact that d is a super-derivation we get:

$$\begin{aligned} d(d\lambda)(a\otimes b\otimes c) &= d\left(\sum_{i=1}^n(\eta_i\otimes\xi_i)\right)(a\otimes b\otimes c)\\ &= (\sum_{i=1}^n(d\eta_i\otimes\xi_i-\eta_i\otimes d\xi_i)(a\otimes b\otimes c)\\ &= \sum_{i=1}^n d\eta_i(a\otimes b)\xi_i(c)-\eta_i(a)(d\xi_i)(b\otimes c)\\ &= \sum_{i=1}^n(\eta_i(ab)\xi_i(c)-\eta_i(a)\xi_i(bc))\\ &= \lambda((ab)c)-\lambda(a(bc))=\lambda((ab)c-a(bc)). \end{aligned}$$

So if $D^2=0$ we see that $\lambda((ab)c-a(bc))=0$ for every functional λ, hence $(ab)c-a(bc)=0$, that is m is associative. Conversely, we see that if m is associative, then $D^2=0$. (We really copied here from [3].) □

This complex (TA^*,D) is a dual of a bar-complex on associative algebras, that is why this section is called how it is called.

Now, let the space A be a graded differential space $A=(\oplus A_i, A_i\xrightarrow{d}A_{i+1})$, that is, A is a complex:

$$\cdots\xrightarrow{d}A_i\xrightarrow{d}A_{i+1}\xrightarrow{d}\cdots \tag{7}$$

We take each A_i to be finite dimensional to be able to take dual spaces, and we would also want that there is a number $l\in\mathbb{Z}$ such that $A_i=\emptyset$ for $i\le l$. Consider now $TA^*=T(\oplus A_i^*)$. It inherits the grading and the differential in the following way. The degree of $\Bbbk$ should be zero, and we assume that the pairing $<,>: A^*\otimes A\to\Bbbk$ is of degree 0 as well, as a result $(A_i)^*$ is of degree $-i:\ (A_i)^*=A^*_{-i}$. The differential on A^* we define by the pairing as well: we want that if $b=da,\ a\in A_i, b\in A_{i+1}$, and $<a^*,a>=1$ and $<b^*,b>=1$

$$1=<b^*,da>=<d^\perp b^*,a>,$$

from which we conclude that $d^\perp b^*=a^*$. Hence, $d^\perp: A^*_{-(i+1)}\to A^*_{-i}$. We could lift it to act on TA^* by a graded Leibniz rule.

To define an up-to homotopy structure the only thing missing now is the grading of the whole TA^*. Of course we could just take the degree of the product of elements from A^* to be the sum of degrees. However for a concise definition it is useful to take into account also how many elements are there in the product. Hence as a result we put the degree of $(x_1\otimes x_2\otimes\cdots\otimes x_k)\in(A^*)^{\otimes k}$ to be $\sum\overline{x_i}+k-1$, which is consistent with the degree of (2). Finally, the definition of the strong homotopy associative structure becomes

Proposition 5. *Giving an strong homotopy associative algebra structure on A is equivalent to giving a map $Q: TA^*\to TA^*$ such that*

1. $Q^2=0$,
2. *Q is a graded derivation of degree 1, that is $Q:(TA^*)\to(TA^*)[1]$.*

Proof. We have to repeat step by step the proof of Proposition 4. Starting form the statement that any map $A^*\to TA^*$ could be extended to act on $(A^*)^{\otimes n}$ by the Leibniz rule and so on. □

We could consider this Lie algebra $\mathcal{A}$ of derivations of the tensor algebra TA^* as the **governing Lie algebra** of an associative structure on A (Lie bracket is given by the commutator of operators on TA^*). The A_∞-structure then is a solution of Maurer-Cartan equation on $\mathcal{A}$.

A good exercise is to do the same construction to get a strong homotopy Lie algebra (otherwise called an L_∞-algebra). The governing algebra will be then the derivations of the exterior algebra (see for example [7]).

This is the point where one could employ the machinery of operads. Namely, take an operad, such that an algebra over it is of the type we are looking for. Then one needs to construct a free resolution of it (this is sort of a bar-construction). Then algebras over this free resolution will be strong homotopy versions of the original algebras. Here we will not venture in this direction, we will just refer to [2, 4, 8] for further explanations.

3. Q-manifold

In fact, the case of an L_∞-algebra is quite special because it allows a geometric interpretation.

If we mimic what was just done for A_∞-algebras (the bar-complex and so on) we see that instead of the tensor algebra we need to take the algebra of exterior powers: after all, a Lie bracket on V is just a map from $V \wedge V \to V$.

Proposition 6. *Giving an strong homotopy Lie algebra structure on a differential graded space* $V = (\oplus V_i, d)$ *is equivalent to giving a map* $Q : \Lambda V^* \to \Lambda V^*$ *such that*

(1) $Q^2 = 0$,
(2) Q *is a graded derivation of degree* 1, *that is* $Q : (\Lambda V^*) \to (\Lambda V^*)[1]$.

Looks exactly the same as Proposition 5, but there is one thing different, namely the derivation we define by the Leibniz rule is now with respect to a product $\Lambda V^* \otimes \Lambda V^* \to \Lambda V^*$ which is not any more the concatenation product of the free tensor algebra, but the shuffle product on ΛV^*.

Now to get geometric, ΛV^* should be identified with symmetric powers of a suspended space $sV^* = V^*[1]$, which is defined component by component as $sV_i^* = V_{i+1}^*$. So $\Lambda V^* = Sym(sV^*)$ and hence could be regarded as polynomial functions on some formal space.

Then maps $Q : V^* \to \Lambda V^*$ could be viewed as vector fields on a formal space on which ΛV^* is an algebra of functions. Each term of such a map $\lambda_k : V^* \to \Lambda V^*$ becomes the k-coefficient in the Taylor decomposition of a vector field Q.

When Q is of degree 1 and homological (that is $Q^2 = 0$) Q defines an L_∞-algebra structure on V. At the same time it fits the definition of a Q-manifold from [1].

4. Self-made strong homotopy Lie bialgebra structure

To define a strong homotopy Lie bialgebra structure on V we need to find the corresponding differential graded governing Lie algebra and then write a Maurer-Cartan equation on it.

Such a Lie algebra is known thanks to Kosmann-Schwarzbach [6]. On the space $\Lambda(V^* \oplus V) = \Lambda V^* \otimes \Lambda V$ the natural pairing (symmetric form) could be considered as a Lie bracket in the suspended space $s(V^* \oplus V)$.

A Lie bialgebra structure thus becomes written as a solution of Maurer-Cartan equation on this graded Lie algebra

$$\mathcal{B} = \sum_{p+q=0}^{+\infty} \Lambda^{p+1} V^* \otimes \Lambda^{q+1} V = (V^* \otimes V) \oplus (V^* \otimes V \wedge V \oplus V^* \wedge V^* \otimes V) \oplus \cdots .$$

A **Lie bialgebra structure** on V, is given by $\theta \in V^* \otimes V \wedge V$ (generating a cobracket) and $\lambda \in V^* \wedge V^* \otimes V$ (generating a bracket). The commutator $[\lambda + \theta, \lambda + \theta]$ lies in B^2 : $\Lambda^3 V^* \otimes V$, $V^* \otimes \Lambda^3 V$, and $\Lambda^2 V^* \otimes \Lambda^2 V$. Then $\lambda + \theta$ defines a Lie bialgebra structure if the commutator is 0, that is all three components of it in B^2 must be equal to 0. This leads to three axioms of a Lie bialgebra:

- **a:** Jacobi identity follows from $[\lambda, \lambda] = 0$, for the derived bracket given by $\{x, y\} = [[\lambda, x]y]$.
- **b:** Co-Jacobi identity follows from $[\theta, \theta] = 0$.
- **c:** The cocycle condition translates as: $[\lambda, \theta] = 0$.

For a graded space V we could define grading on the complex $\mathcal{B}$ by setting the degree of $x_1 x_2 ... x_{k+2} \in B^k = \sum_{p+q=k} \Lambda^{p+1} V^* \otimes \Lambda^{q+1} V$ to be $\sum |x_i| - k + 1$ where all $x_i \in V^* \oplus V$ and degree of x_i is denoted $|x_i|$.

A strong homotopy Lie bialgebra structure is given by a solution of Maurer-Cartan equation on $\mathcal{B}$. Any element Q of degree 1 from $\mathcal{B}$, satisfying an equation $[Q, Q] = 0$ defines a strong homotopy Lie bialgebra structure. We could see Q as living in the double graded space $\mathcal{B}$:

	$\ldots$				
4	$\Lambda^4 V^* \otimes V$	$\ldots$			
3	$\Lambda^3 V^* \otimes V$	$\Lambda^3 V^* \otimes \Lambda^2 V$	$\ldots$		
2	$\Lambda^2 V^* \otimes V$	$\Lambda^2 V^* \otimes \Lambda^2 V$	$\Lambda^2 V^* \otimes \Lambda^3 V$	$\ldots$	
1	$V^* \otimes V$	$V^* \otimes \Lambda^2 V$	$V^* \otimes \Lambda^3 V$	$V^* \otimes \Lambda^4 V$	$\cdots$
$(p+1)$ / $(q+1)$	1	2	3	4	

That is $Q = \sum_{k \geq 1, l \geq 0} t_{kl}$, where $t_{kl} \in \Lambda^k V^* \otimes \Lambda^l V$.

This way the condition $[Q, Q] = 0$ gives a set of equation indexed by two numbers. For each $p \geq 1$, $q \geq 1$ we get an equation:

$$\sum_{k+k'=p+1} \sum_{l+l'=q+1} [t_{kl}, t_{k'l'}] = 0, \tag{8}$$

The first one is

$$[t_{11}, t_{11}] = 0 \tag{9}$$

providing the equation which defines a differential $d = ad_{t_{11}}$ on V. The next couple of equations for $p = 1, q = 2$ and $p = 2, q = 1$ gives respectively:

$$[t_{11}, t_{12}] = 0, \ [t_{11}, t_{21}] = 0, \tag{10}$$

which is a condition that d is a derivation of the cobracket and of the bracket. Next one $p = 2, q = 2$:

$$[t_{11}, t_{22}] + [t_{12}, t_{21}] = 0 \tag{11}$$

is the condition o the cocycle which is up-to homotopy given by an element $t_{22} \in \Lambda^2 V^* \otimes \Lambda^2 V$. The terms

The Jacobi identity holds also up-to homotopy only, this condition is given by the equation for $p = 3, q = 1$:

$$[t_{11}, t_{31}] + \frac{1}{2}[t_{21}, t_{21}] = 0, \tag{12}$$

as well as the coJacobi identity - equation with $p = 1, q = 3$:

$$[t_{11}, t_{13}] + \frac{1}{2}[t_{12}, t_{12}] = 0. \tag{13}$$

Remark 7. (1) The homology of $H^*V = H^*(V, d)$ is a Lie bialgebra.

(2) If V is not graded, that is all $V_n = \emptyset$ for all $n \neq 0$ then $V = V_0$ is an ordinary Lie bialgebra. Indeed, since t_{kl} is of degree $2 - (k + l - 1)$ all t_{kl} except for t_{12} and t_{21} have to vanish.

(3) If all $t_{kl} = 0$ for all $k + l \geq 3$ then V is a differential graded Lie bialgebra.

5. Final remarks

We have not given any motivations or examples but these are plenty in the literature - to mention a few just for A_∞ structures alone: topological origins in Stasheff's early work, a nice example of an A_∞ structure on a Kähler manifold of Merkulov [9], interpretation of higher homotopies as deformations by Penkva-Schwarz [10].

We have also left out a lot of useful things in the setup of strong homotopy structures. For example a morphism of such algebras is a morphism of governing differential graded Lie algebras, commuting with Q.

Another necessary thing to define is a notion of a module over, say, an A_∞ algebra. For it we could use a generalization of what Ginzburg calls a "square zero" construction ([3]): Given an algebraic structure P on A, define its action on M by saying that the projection $A \otimes M \to M$ is a P-map, and all operations restricted on M are 0 and M is an ideal in AxM. (For the case of an associative algebra it gives this square 0 condition $M^2 = 0$.)

We have not mentioned the deformation problems either though it is easy to talk about them in our setup...

To conclude: given a problem of defining a strong homotopy analogue of a certain (quadratic) structure, the general somewhat simplified scheme of its solution is

1: Take a differential graded space and consider the original structure on it making sure that the differential is a derivation with respect to the structure.

2: Find the governing graded Lie algebra, where the original structure has the first degree and satisfies the "square zero" condition: its Lie bracket with itself is 0.

2: Write the Maurer-Cartan equation on this dgLie algebra. Its solutions are what we are looking for.

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