Forcing in Model Theory revisited

Abderezak OULD HOUCINE
Équipe de Logique Mathématique, UFR de Mathématiques,
Université Denis-Diderot Paris 7,
2 place Jussieu 75251, Paris Cedex 05 France.
ould@logique.jussieu.fr

Abstract

We present a generalization of forcing in model theory which allows the construction of finitely generated models. This is used to prove a particular compactness theorem, and an omitting type theorem, for models generated by n elements. We study also some properties of finitely generated models of a countable theory having, up to isomorphism, at most $\aleph_0$ finitely generated models.

1 Introduction

Let L be a countable first order langauge. A model $\mathcal{M}$ is said to be n-generated if it can be generated by n elements, i.e. there exists $a_1, \cdots, a_n$ in $\mathcal{M}$ such that for every $a \in \mathcal{M}$ we have $a = \tau(a_1, \cdots, a_n)$ in $\mathcal{M}$ for some term $\tau(x_1, \cdots, x_n)$ of L.

In the study of some classes of finitely generated models, in particular in group theory, we are confronted to construct finitely generated models. The absence of a compactness theorem for such classes make their study difficult. One can see in general that one can not have a compactness theorem for finitely generated models analogue to the usual one of first order languages. Here an example in group theory which show this: suppose towards a contradiction that the compactness theorem for finitely generated models is true, i.e. *for every theory T, if every finite subset of T has a finitely generated model then T has a finitely generated model*. Supposing this we show the following property (*): *for every theory T, if $\mathcal{M}$ is a finitely generated model of $T^0_\forall$ then $\mathcal{M}$ is embeddable in a finitely generated model of T*, where $T^0_\forall$ is the set of universal sentences which are true in every finitely generated model of T. Let $\mathcal{M}$ be a finitely generated model of $T^0_\forall$, generated by the finite tuple $\overline{a}$, and let $\Gamma = T \cup \{\phi(\overline{a}) : \phi$ is free-quantifier such that $\mathcal{M} \models \phi(\overline{a})\}$ in the language $L(\overline{a})$. Since $\mathcal{M} \models T^0_\forall$, every finite subset of Γ has a finitely generated model. Thus, by our supposition, Γ has a finitely generated model. Hence there exists a finitely generated model of T which embeds $\mathcal{M}$. But the property (*) is false because $\mathbb{Z}^2$ satisfies the universal theory of $\mathbb{Z}$ but $\mathbb{Z}^2$ is not embeddable in a finitely generated model of the complete theory $Th(\mathbb{Z})$ of $\mathbb{Z}$, as every finitely generated model of $Th(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and $\mathbb{Z}^2$ is not a subgroup of $\mathbb{Z}$. Of course the above argument works if we restrict our attention to n-generated models.

The present work takes his origin from a question of G.Sabbagh who asked whenever there exists a complete theory of groups having $2^{\aleph_0}$ finitely generated nonisomorphic groups. This question is natural from the context of Theorem 4.10, who sate that the number $\alpha(T)$ of nonisomorphic finitely generated models of T is less than $\aleph_0$ or equal to $2^{\aleph_0}$ (Theorem 4.10 was also proved by A.Khelif). One can extracted from the papers of F.Oger [5, 6, 7] that for every natural number n there exists a complete theory T of groups such that $\alpha(T) = n$. Thus the question when $\alpha(T) = 2^{\aleph_0}$ is interesting to complete the possible cases.

G.Sabbagh and A.Khelif asked also whenever a finitely generated *pseudo-finite* group is finite. Recall that a group G is said to be pseudo-finite if it is a model of the theory of all finite groups.

In both cases the beginning idea is to use forcing to construct potential groups who answering above questions and to see if there is a "*reasonable*" version of compactness theorem for finitely generated models.

Forcing is a method of construction of models satisfying some properties forced by some conditions. In general one can not force a model to be finitely generated. Thus we must construct a form of forcing in which every "*generic*" model is finitely generated.

We give in this paper a generalization of forcing in model theory, which include the usual one of Robinson-Barwise-Kiesler (cf. [2]), and which allows to prove the following compactness theorem for n-generated models (for the definition of τ-closeness see the beginning of section 4):

Theorem 1.1 *Let T be τ-close theory in a countable language L such that every finite subset of T has an n-generated model. Then T has a n-generated model.*

We will also use this forcing to prove the following omitting type theorem for n-generated models (for the notation see the beginning of section 4):

Theorem 1.2 *Let T be an n-consistent theory in a countable language L. Let $(\triangle_i(\overline{x}_i))_{i\in\mathbb{N}}$ be a sequence of set of formula of L. If for every $i \in \mathbb{N}$ and for every tuple $\overline{\tau}$ of terms in $L(C)$ there is no sentence $\phi(\overline{c})$ in $L(C)$ which is n-consistent with T such that:*

$$T \vdash_n (\phi(\overline{c}) \Rightarrow \varphi(\overline{\tau})), \text{ for every } \varphi(\overline{x}_i) \in \triangle_i,$$

then there is a n-generated model of T which omit $\triangle_i$ for every $i \in \mathbb{N}$.

We have also the following result which give a necessary and sufficient condition for a n-generated model to be prime of his theory:

Theorem 1.3 *Let T be a complete theory in a countable language L having at most $\aleph_0$ n-generated models. Let $\mathcal{M}$ be a n-generated model of T. Then the following properties are equivalent:*

(1) $\mathcal{M}$ *is prime model of* T.

(2) *There is a formula* $\theta(x_1, \cdots, x_n)$ *consistent in* $\mathcal{M}$ *such that if* $\mathcal{M} \models \theta(b_1, \cdots, b_n)$ *then* $b_1, \cdots, b_n$ *generate* $\mathcal{M}$.

A theory T having a prime model has at most $\aleph_0$ types, thus the condition on T to have at most $\aleph_0$ n-generated models is a necessary condition for the exitance of prime model, as the complete diagram of a n-generated model of T is a n-type.

The paper is organized as follows. In the next section we recall the basic notions needed in the sequel. In section 3 we give a generalization of forcing and we prove a Generic Model Theorem analogue to the one in [2]. Finally, in section 4 we use the forcing to prove Theorem 1.1, Theorem 1.2 and Theorem 1.3. In that section we study also some properties of theory having at most $\aleph_0$ n-generated models.

Throughout this paper every language considered in the sequel will be *countable*. Thus we let L be a fixed countable first order language. L is arbitrary but is held fixed to simplify notation.

2 Background and Prerequisites

Our references books are [1],[3]. The majority of definitions given here are also in [2], but for the reader's convenience we recall them. The reader which is familiar with [2] can go directly to the next section.

The set of formulas of L is the smallest set of words build in L containing atomics formulas and closed by $\vee$, $\neg$ and the quantifier $\exists x$. The conjunction $\phi \wedge \psi$ (respectively universal quantifier $\forall x \phi(x)$) is regarded as abbreviation for $\neg(\neg\phi \vee \neg\psi)$ (respectively for $\neg\exists x \neg\phi$). This convention is necessary to give a *complete* definition of forcing in the next section, i.e. a definition which is valid for all sentences.

The infinitary logic $L_{\omega_1\omega}$ is built from L by allowing the infinite disjunction $\bigvee \Phi = \bigvee_{\phi \in \Phi} \phi$ for any countable set Φ of formulas. As before we abbreviate $\bigwedge \Phi = \bigwedge_{\phi\in\Phi} \phi$ for $\neg \bigvee_{\phi\in\Phi} \neg\phi$.

Let ϕ be a formula in $L_{\omega_1\omega}$. The set $\mathrm{sub}(\phi)$ of ***subformulas*** of ϕ is defined by induction on ϕ as follows.

If ϕ is atomic then $\mathrm{sub}(\phi) = \{\phi\}$,
$\mathrm{sub}(\phi \vee \psi) = \mathrm{sub}(\phi) \cup \ \mathrm{sub}(\psi) \cup \{\phi \vee \psi\}$,
$\mathrm{sub}(\neg\phi) = \mathrm{sub}(\phi) \cup \{\neg\phi\}$,
$\mathrm{sub}(\exists x \phi) = \mathrm{sub}(\phi) \cup \{\exists x \phi\}$,
$\mathrm{sub}(\bigvee \Phi) = \bigcup_{\phi\in\Phi} \mathrm{sub}(\phi) \cup \{\bigvee \Phi\}$.

By a ***fragment*** we mean a set L_A of formulas of $L_{\omega_1\omega}$ such that:

(1) Every formula of L belongs to L_A.
(2) L_A is closed under $\neg$, $\exists x$, and finite disjunction.
(3) If $\phi(x) \in L_A$ and τ is a term then $\phi(\tau) \in L_A$.

(4) If $\phi \in L_A$ then every subformula of ϕ is in $\L_A$.

Note that for every set Φ of formulas of $L_{\omega_1\omega}$, there is a least fragment L_A such that $\Phi \subseteq L_A$. Furthermore if Φ is countable then so is the least fragment containing Φ. We will us the above property freely and without any reference to it.

3 Generalized Forcing and generic models

Let C be a countable or **finite** non empty set of new constant symbols. We note $L(C)$ the first order language obtained by adding to L the constants $c \in C$. If L_A is a fragment we let $L(C)_A$ denote the set of all formulas obtained from formula $\phi \in L_A$ by replacing finitely many free variables occurring in ϕ by constants $c \in C$. Thus $L(C)_A$ is the least fragment of $L(C)_{\omega_1\omega}$ wich contains L_A. We start with the following definition.

Definition 3.1 *A* ***forcing property*** *for the langauge L is a quadruple $\mathbb{P} = \langle P, \leq, \Delta, f\rangle$ such that:*

(i) $\langle P, \leq\rangle$ is a partially ordered set with a least element 0.

(ii) Δ is a set of terms without variables of the language $L(C)$.

(iii) f is a function which associates with each $p \in P$ a set $f(p)$ of atomic sentences of $L(C)$.

(iv) Whenever $p \leq q$, $f(p) \subseteq f(q)$

(v) Let σ and τ be terms of $L(C)$ without variables and $p \in P$. Then:

(1) *If $(\tau = \sigma) \in f(p)$, then $(\sigma = \tau) \in f(q)$ for some $q \geq p$.*

(2) *If $(\tau = \sigma) \in f(p)$, $\phi(\tau) \in f(p)$, where $\phi(x)$ is an atomic formula, then $\phi(\sigma) \in f(q)$ for some $q \geq p$.*

(3) *For some $q \geq p$ and for some term $\tau' \in \Delta$, $(\tau' = \tau) \in f(q)$.*

The elements of P are called **conditions** of $\mathbb{P}$. Since the set of terms of $L(C)$ contains at least C, the condition $(v)(3)$ implies that Δ is not empty.

This definition generalize the usual one in [2]. Indeed by taking $\Delta = C$ we find the notion of forcing developed in [2]. The principal difference is that we introduce the set Δ which allows us to better control the properties of the generating set of generic models and we replace the condition: *for some c, $(\tau = c) \in f(q)$ for some $q \geq p$*, which appear in the definition of forcing in [2], by the condition $(v)(3)$ above.

Definition 3.2 *Let L_A be a fragment and $\mathbb{P}$ a forcing property. The relation $p \Vdash \phi$ in $\mathbb{P}$, (defined relatively to the fragment L_A), read p* ***forces*** *ϕ, is defined, by induction on ϕ, for $p \in P$ and $\phi \in L(C)_A$, as follows:*

If ϕ is an atomic sentence, then $p \Vdash \phi$ iff $\phi \in f(p)$.
$p \Vdash \neg\phi$ iff there is no $q \geq p$ such that $q \Vdash \phi$.
$p \Vdash \bigvee \Phi$ iff $p \Vdash \phi$ for some $\phi \in \Phi$.
$p \Vdash \exists x\phi(x)$ iff $p \Vdash \phi(\tau)$ for some term $\tau \in \Delta$.

We see also here that the latter condition is modified and replace the condition: $p \Vdash \exists x\phi(x)$ *iff* $p \Vdash \phi(c)$ *for some* $c \in C$, which appear in [2]. Thus by taking $\Delta = C$ we find the notion of $p \Vdash \phi$ developed in [2]. We say that p ***weakly forces*** ϕ, in symbols $p \Vdash^w \phi$, iff p forces $\neg\neg\phi$. We write $p \nVdash \phi$ (resp. $p \nVdash^w \phi$) to mean that p does *not forces* (resp. *not weakly forces*) ϕ.

From now on shall assume that L_A is a **countable** fragment of $L_{\omega_1\omega}$ and $\mathbb{P}$ a forcing property of L. The sets L_A and $\mathbb{P}$ are arbitraries but are fixed to simplify notation. Note that $L(C)_A$ is also countable.

Definition 3.3 *A subset* $G \subseteq P$ *is said to be* **generic** *iff*

(i) $p \in G$ *and* $q \leq p$ *implies* $q \in G$.

(ii) $p, q \in G$ *implies that there exists* $r \in G$ *with* $p \leq r$ *and* $q \leq r$.

(iii) For each sentences ϕ *in* $L(C)_A$ *there exists* $p \in G$ *such that either* $p \Vdash \phi$ *or* $p \Vdash \neg\phi$

Here there is no difference with the corresponding definition of generic sets in [2].

Definition 3.4 *A model* $\mathcal{M}$ *is called a* **generic model** (*relatively to the forcing property* $\mathbb{P}$ *and to the fragment* L_A) *iff*

(i) $\mathcal{M}$ *is an* $L(C)$*-structure generated by the interpretations of the constants symbols of* C,

(ii) there is a generic set G *which satisfies: every sentence* ϕ *in* $L(C)_A$ *which is forced by some* $p \in G$ *holds in* $\mathcal{M}$.

We say in that case that G **generically-generate** $\mathcal{M}$. If p is a condition we say that $\mathcal{M}$ is a ***generic model for*** p if there is a generic set G such that $p \in G$ and G generically-generate $\mathcal{M}$. We will show in the end of this section that by taking $\Delta = C$ we find the generic models developed in [2].

Note that when C is *finite*, every generic model is **finitely generated**. Remark also that since the notion of forcing depends of some countable fragment, generic models depends also of some fragment.

We begin by the following lemma which unites some properties about forcing:

Lemma 3.5 *We have the following properties:*

(1) *If* $p \leq q$ *and* $p \Vdash \phi$ *then* $q \Vdash \phi$.

(2) *We can not have both* $p \Vdash \phi$ *and* $p \Vdash \neg\phi$.

(3) $p \Vdash^w \phi$ *iff for every* $q \geq p$ *there is a condition* $r \geq q$ *such that* $r \Vdash \phi$.

Proof

(1) By induction on ϕ.

For ϕ atomic. The conclusion follows from $f(p) \subseteq f(q)$.

For $\neg\phi$. We have $p \Vdash \neg\phi$. Suppose towards a contradiction that $q \nVdash \neg\phi$. Then by definition of forcing: there is $q' \geq q$ such that $q' \Vdash \phi$. Therefore there is $q' \geq p$ such that $q' \Vdash \phi$, thus $p \nVdash \neg\phi$, which is a contradiction.

For $\bigvee \Phi$. Since $p \Vdash \bigvee \Phi$ then $p \Vdash \phi$ for some $\phi \in \Phi$. By induction $q \Vdash \phi$, and by definition $q \Vdash \bigvee \Phi$.

For $\exists x\phi(x)$. Since $p \Vdash \exists x\phi(x)$ then $p \Vdash \phi(\tau)$ for some term $\tau \in \Delta$. By induction $q \Vdash \phi(\tau)$, and by definition $q \Vdash \exists x\phi(x)$.

(2) Obvious.

(3) $p \Vdash^w \phi$ iff $p \Vdash \neg\neg\phi$ iff for every $q \geq p$, $q \nVdash \neg\phi$
iff for every $q \geq p$, there exists $r \geq q$ such that $r \Vdash \phi$. □

Now we prove the principal theorem of this section:

Theorem 3.6 (Generic Model Theorem) *For every $p \in P$ there is a generic model for p.*

Theorem 3.6 is a direct consequence of the two following lemmas:

Lemma 3.7 *Every $p \in P$ belongs to a generic set.*

Proof

The proof of this lemma is analogue to the one in [2], but for the reader's convenience we give the same proof here. Since $L(C)_A$ is countable, we can enumerate $L(C)_A$. Thus let $(\phi_n : n \in \mathbb{N})$ be an enumeration of $L(C)_A$. Form a chain of conditions $p_0 \leq p_1 \leq \cdots$ in P as follows. Let $p_0 = p$. If $p_n \Vdash \neg\phi_n$, let $p_{n+1} = p_n$, otherwise choose $p_{n+1} \geq p_n$ such that $p_{n+1} \Vdash \phi_n$. The set $G = \{q \in P : q \leq p_n \text{ for some } n \in \mathbb{N}\}$ is generic and contains p. □

Lemma 3.8 *Every generic set G generically-generate a model $\mathcal{M}$.*

Proof

The proof is sensibly different of the proof in [2]. Let T be the set of all sentences of $L(C)_A$ which are forced by some $p \in G$. Then T has the following properties for all sentences and all terms without variables in $L(C)_A$.

(1) Exactly one of ϕ, $\neg\phi$ is in T.

Proof. Since G is generic there is $p \in G$ such that $p \Vdash \phi$ or $p \Vdash \neg\phi$. Therefore $\phi \in T$ or $\neg\phi \in T$. Suppose towards a contradiction that $\phi, \neg\phi \in T$. Then there exists $p, q \in G$ such that $p \Vdash \phi$ and $q \Vdash \neg\phi$. By the property (3) of generic set, and by Lemma 3.5, we see that there is $r \in G$ such that $r \Vdash \phi$ and $r \Vdash \neg\phi$. This contradict the property (2) of Lemma 3.5. □

(2) $\bigvee \Phi \in T$ iff $\phi \in T$ for some $\phi \in \Phi$. This is obvious.

(3) $\exists x\phi(x) \in T$ iff $\phi(\tau) \in T$ for some term $\tau \in \Delta$. This is also obvious.

(4) (i) For every term τ, $(\tau = \tau) \in T$.
(ii) For every term τ, there exists a term $\tau' \in \Delta$ such that $(\tau' = \tau) \in T$.
(iii) If $(\tau = \sigma) \in T$ then $(\sigma = \tau) \in T$.
(iv) If $(\tau = \sigma), \phi(\tau) \in T$ where $\phi(x)$ is an atomic formula, then $\phi(\sigma) \in T$.

Proof.

(i) Let τ be a term and let $p \in P$. We claim that there exists $q \geq p$ such that $q \Vdash (\tau = \tau)$. By $(v)(3)$ of definition of forcing there exists $q_1 \geq p$ and $\tau' \in \Delta$ such that $(\tau' = \tau) \in f(q_1)$. Now by putting $\phi(x) := (x = \tau)$ we have

$\phi(\tau') \in f(q_1)$. Thus, by $(v)(2)$ of definition of forcing, we have $\phi(\tau) \in f(q)$ for some $q \geq q_1 \geq p$. Therefore there exists $q \geq p$ such that $q \Vdash (\tau = \tau)$ and this complete the proof of our claim.

Therefore every $p \in P$, $p \nVdash \neg(\tau = \tau)$. Since G is generic there is $r \in G$ such that $r \Vdash (\tau = \tau)$, thus $(\tau = \tau) \in T$.

(ii) Let τ be a term and $p \in P$. By the property $(v)(3)$ of the definition of forcing there exist $\tau' \in \Delta$ and $q \geq p$ such that $q \Vdash (\tau' = \tau)$. Thus $q \Vdash \exists x(x = \tau)$. Therefore every $p \in P$, $p \nVdash \neg\exists x(x = \tau)$. Since G is generic there is $r \in G$ such that $r \Vdash \exists x(x = \tau)$, and thus $\exists x(x = \tau) \in T$. Therefore, by (3), $(\tau' = \tau) \in T$ for some term $\tau' \in \Delta$.

(iii) If $(\tau = \sigma) \in T$ then there is $p \in G$ such that $p \Vdash (\tau = \sigma)$. Suppose towards a contradiction that $(\neg(\sigma = \tau)) \in T$. Then there is $q \in G$ such that $q \Vdash \neg(\sigma = \tau)$. Since G is generic there is $r \in G$ such that: $r \Vdash (\tau = \sigma)$ and $r \Vdash \neg(\sigma = \tau)$. Now by the property of forcing $(v)(1)$ there is $r' \geq r$ such that $r' \Vdash (\sigma = \tau)$, and this a contradiction with $r \Vdash \neg(\sigma = \tau)$. Therefore $(\sigma = \tau) \in T$.

(iv) Can be proved similarly using the property $(v)(2)$ of the definition of forcing. $\square$

Let Γ be the set of all terms without variables of $L(C)$. We define, on Γ, the following relation:

$$\tau \sim \sigma \text{ iff } (\tau = \sigma) \in T.$$

Then one can check, using $(4)(i)$-(iv), that $\sim$ is an equivalence relation. Let $\mathcal{M} = \Gamma/\sim$. We denote by $\hat{\tau}$ the class of τ. The function and relation symbols and constants symbols of L are interpreted in $\mathcal{M}$ in such a way that:

$F(\hat{\tau}_1, \cdots, \hat{\tau}_n) = \hat{\tau}_p$ iff $(F(\tau_1, \cdots, \tau_n) = \tau_p) \in T$.
$R(\hat{\tau}_1, \cdots, \hat{\tau}_n)$ iff $R(\tau_1, \cdots, \tau_n) \in T$.
If d a constant symbol of L then we interpret d in $\mathcal{M}$ by $\hat{d}$.

By $(4)(i)$-(iv) we see that this definitions are unambiguous. This make $\mathcal{M}$ an L-structure, generated by the interpretations of the constants symbols of C.

Let us show that: for each sentence ϕ in $L(C)_A$, $\mathcal{M} \models \phi$ iff $\phi \in T$.

We do this by induction on ϕ.

For ϕ atomic. The results follows from the definition of the L-structure $\mathcal{M}$.

For $\neg\phi$. Let $\neg\phi \in T$. Suppose that $\mathcal{M} \models \phi$, then by induction we have $\phi \in T$, which is a contradiction. Therefore $\mathcal{M} \models \neg\phi$. Now if $\mathcal{M} \models \neg\phi$, then ϕ is not in T, therefore $\neg\phi \in T$.

For $\bigvee \Phi$. If $\mathcal{M} \models \bigvee \Phi$ then $\mathcal{M} \models \phi$ for some $\phi \in \Phi$. By induction $\phi \in T$, thus by (2) $\bigvee \Phi \in T$. Now if $\bigvee \Phi \in T$ then by (2) $\phi \in T$ for some $\phi \in \Phi$. Therefore, by induction, $\mathcal{M} \models \phi$ and thus $\mathcal{M} \models \bigvee \Phi$.

For $\exists x \phi(x)$.

If $\mathcal{M} \models \exists x \phi(x)$, then $\mathcal{M} \models \phi(\tau)$ for some term τ. By $(4)(ii)$, there is some term $\tau' \in \Delta$ such that $(\tau' = \tau) \in T$. By induction $\mathcal{M} \models (\tau' = \tau)$. Since

$\mathcal{M} \models \phi(\tau)$ then $\mathcal{M} \models \phi(\tau')$. By induction $\phi(\tau') \in T$. By (3), $\exists x\phi(x) \in T$. The converse is obvious.

This completes the proof that $\mathcal{M}$ is a generic model generically-generated by G. $\square$

Now we prove the following Lemma.

Lemma 3.9 *We have the followings properties:*

(1) *$p \Vdash \forall x\phi(x)$ iff for every term τ and every $q \geq p$ there is $r \geq q$ such that $r \Vdash \phi(\tau)$.*

(2) *$p \Vdash^w \phi$ iff ϕ holds in every generic model for p.*

(3) *If $\mathcal{M}$ is a generic model then for every $x \in \mathcal{M}$ there exists $\tau \in \Delta$ such that $\mathcal{M} \models (x = \tau)$.*

Proof

(1) By definition $\forall x\phi(x)$ is the sentence $\neg\exists x\neg\phi(x)$.

$p \Vdash \forall x\phi(x)$ iff $p \Vdash \neg\exists x\neg\phi(x)$ iff

there is no $q \geq p$ such that $q \Vdash \exists x\neg\phi(x)$ iff

for every $q \geq p$, for every term $\tau \in \Delta$, $q \nVdash \neg\phi(\tau)$ iff

for every $q \geq p$, for every term $\tau \in \Delta$, there is $r \geq q$ such that $r \Vdash \phi(\tau)$.

Therefore if for every term τ and every $q \geq p$ there is $r \geq q$ such that $r \Vdash \phi(\tau)$, then $p \Vdash \forall x\phi(x)$.

We prove the inverse implication. Let $p \Vdash \forall x\phi(x)$, let $q \geq p$ and let τ be a term. Let $\mathcal{M}$ be a generic model for q, generically-generated by G. Then $\mathcal{M} \models \forall x\phi(x)$, and thus $\mathcal{M} \models \phi(\tau)$. Therefore, since G is generic, there exists $r \geq q$ such that $r \Vdash \phi(\tau)$.

(2) If $p \Vdash^w \phi$ then by Lemma 3.5(1) for every $q \geq p$ there is a condition $r \geq q$ such that $r \Vdash \phi$. Therefore if G is a generic set containing p there is no $r \in G$ such that $r \Vdash \neg\phi$. Thus ϕ is true in all generic models for p.

Suppose now that ϕ is true in all generic models of p. Let $q \geq p$. Since ϕ is true in all generic models of q there exists $r \geq q$ such that $r \Vdash \phi$. By Lemma 3.5 (1) $p \Vdash^w \phi$.

(3) Let $\mathcal{M}$ be generic model generically-generated by G and let $x \in \mathcal{M}$. Then there exists a term τ such that $x = \tau$ in $\mathcal{M}$. Therefore $\mathcal{M} \models \exists y(y = \tau)$ and thus there exists $r \in G$ such that $r \Vdash \exists y(y = \tau)$. Therefore there exists a term $\tau' \in \Delta$ such that $r \Vdash (\tau' = \tau)$. Therefore $\mathcal{M} \models (\tau' = \tau)$ and thus $\mathcal{M} \models (x = \tau')$. $\square$

We finish this section by showing how to obtain the usual forcing from our version. We have seen earlier that by taking $\Delta = C$ we find the usual notion of forcing property and the usual notion of the relation $p \Vdash \phi$. It is sufficient to show that, in case $\Delta = C$, the generic models in our sense coincide with those given in [2]. Obviously, in case $\Delta = C$, every generic model in sense of [2] is a generic model in our sense. Thus we show that, always in case $\Delta = C$, that every generic model in our sense is a generic model in sense of [2], i.e. we show that if $\mathcal{M}$ is a generic model in our sense then $\mathcal{M}$ is exactly the set of all interpretations of the constants symbols of C.

Let $\mathcal{M}$ be a generic model. By Lemma 3.9 for every $x \in \mathcal{M}$ there exists a term $\tau \in \Delta$ such that $\mathcal{M} \models (x = \tau)$. Since we have $\Delta = C$ then for every $x \in \mathcal{M}$ there exists $c \in C$ such that $\mathcal{M} \models (x = c)$. Therefore $\mathcal{M}$ is the set of interpretations of the constant symbols of C, and thus $\mathcal{M}$ is a generic model in the sense of [2].

4 Application to n-generated models

We let $C = \overline{c} = \{c_1, \cdots, c_n\}$ finite. We begin by useful definitions which are the adaptation to the context of n-generated models of the usual one in first order logic.

Definition 4.1 *Let $T(\overline{c})$ be a theory in $L(C)$ and $\phi(\overline{c})$ a sentence in $L(C)$.*

(1) *Let $\mathcal{M}$ be a model and $\overline{a} \in \mathcal{M}^n$. We say that $(\mathcal{M}, \overline{a})$ is a n-generated model of $T(\overline{c})$ and we write $\mathcal{M} \models T(\overline{a})$ if $\overline{a}$ generate $\mathcal{M}$ and $\mathcal{M} \models \varphi(\overline{a})$ for every sentence $\varphi(\overline{c}) \in T(\overline{c})$. The theory $T(\overline{c})$ is said to be* ***n-consistent*** *if it has a n-generated model, i.e. there is a n-generated model $\mathcal{M}$ generated by some n-tuple $\overline{a}$ such that $\mathcal{M} \models T(\overline{a})$. We say that $\phi(\overline{c})$ is n-consistent with $T(\overline{c})$ if $T(\overline{c}) \cup \{\phi(\overline{c})\}$ is n-consistent.*

(2) *We write $T(\overline{c}) \vdash_n \phi(\overline{c})$ to mean that for every n-generated model, generated by a n-tuple $\overline{a}$ we have $\mathcal{M} \models \phi(\overline{a})$ whenever $\mathcal{M} \models T(\overline{a})$.*

(3) *We say that $T(\overline{c})$ is* ***τ-close*** *if the following property holds: for every formula $\phi(x, \overline{c})$ in $L(C)$, if for every term τ there is a finite subset $S \subseteq T(\overline{c})$ such that $S \vdash_n \phi(\tau, \overline{c})$ then there exists a finite subset $S' \subseteq T(\overline{c})$ such that $S' \vdash_n \forall x \phi(x, \overline{c})$.*

(4) *Let $\triangle(\overline{c}, \overline{x})$ be a set of formulas of $L(C)$ whose free variables are among $\overline{x}$ where the length of $\overline{x}$ is finite and equal to m. We say that $\triangle(\overline{c}, \overline{x})$ is a* ***principal set relatively to*** *$T(\overline{c})$ (or simply principal set if there is no ambiguity) if there exists a sentence $\phi(\overline{c})$ of $L(C)$ and a finite m-tuple of terms $\overline{\tau}(c)$ of $L(C)$ such that:*

$$T(\overline{c}) \cup \{\phi(\overline{c})\} \quad \textit{is } n\textit{-consistent and}$$

$$T(\overline{c}) \vdash_n (\phi(\overline{c}) \Rightarrow \varphi(\overline{c}, \overline{\tau}(c))), \textit{ for every } \varphi(\overline{c}, \overline{x}) \in \triangle(\overline{c}, \overline{x}).$$

Note that the tuple $\overline{c}$ play a particular role in our definition, as it is always regarded like a generating tuple.

4.1 Compactness and Omitting types

In this subsection we prove the following two theorems which are more general version of Theorem 1.1 and Theorem 1.2 enounced in the introduction:

Theorem 4.2 (Compactness) *Let $T(\overline{c})$ be a τ-close theory in $L(C)$ such that every finite subset of $T(\overline{c})$ is n-consistent. Then $T(\overline{c})$ is n-consistent.*

Theorem 4.3 (Omitting types) *Let $T(\overline{c})$ be a n-consistent theory in $L(C)$. Let $(\triangle_i(\overline{c},\overline{x}_i) : i \in \mathbb{N})$ be a sequence of (countable) sets of formulas of $L(C)$ where the length of $\overline{x}_i$ is finite. If for every $i \in \mathbb{N}$ the set $\triangle_i(\overline{c},\overline{x}_i)$ is not a principal set relatively to $T(\overline{c})$ then there is a n-generated model $\mathcal{M}$ of $T(\overline{c})$ which* ***omit*** *$\triangle_i$ for every $i \in \mathbb{N}$, i.e. $\mathcal{M}$ is generated by some n-tuple $\overline{a}$ such that $\mathcal{M} \models T(\overline{a})$ and $\mathcal{M} \models \forall \overline{x}_i \bigvee_{\varphi\in\triangle_i} \neg\varphi(\overline{a},\overline{x}_i)$, for every $i \in \mathbb{N}$.*

First we prove the following lemma:

Lemma 4.4 *Let $T(\overline{c})$ be a theory in $L(C)$.*

(1) *Let $\Gamma(\overline{c}) = \{\phi(\overline{c}) : T(\overline{c}) \vdash_n \phi(\overline{c})\}$. If $T(\overline{c})$ is n-consistent then $\Gamma(\overline{c})$ is τ-close.*

(2) *Suppose that every finite subset of $T(\overline{c})$ is n-consistent and let:*

$$P = \{p : p \textit{ is a finite set of sentences in } L(C) \textit{ and } S \cup p \textit{ is } n\textit{-consistent}$$

$$\textit{for every finite subset } S \subseteq T(\overline{c})\}.$$

For every $p \in P$ define $f(p) = \{\varphi(\overline{c}) : \varphi$ atomic and $\varphi(\overline{c}) \in p\}$. Then the quadruple $\mathbb{P} = (P, \leq, \Delta, f)$ with $\leq$ is the inclusion relation $\subseteq$ and Δ is the set of all terms of $L(C)$ is a forcing property. We call $\mathbb{P}$ the ***canonical forcing property*** *of $T(\overline{c})$.*

Remark 4.5 **(1)** means that in "*some sense*" the condition of τ-closeness is a necessary condition for a theory $T(\overline{c})$ to be n-consistent.

Proof

(1) Let $\phi(x,\overline{c})$ be a formula of $L(C)$ such that for every term τ there is a finite subset $S \subseteq \Gamma(\overline{c})$ such that $S \vdash_n \phi(\tau,\overline{c})$ and prove that there exists a finite subset $S' \subseteq \Gamma(\overline{c})$ such that $S' \vdash_n \forall x\phi(x,\overline{c})$. We claim in fact that we have $\forall x\phi(x,\overline{c}) \in \Gamma(\overline{c})$. Let $(\mathcal{M},\overline{a})$ be a n-generated model of $T(\overline{c})$. Then $\mathcal{M} \models \Gamma(\overline{a})$ and therefore for every term $\tau(\overline{c})$ we have $\mathcal{M} \models \phi(\tau(\overline{a}),\overline{a})$. Since $\mathcal{M}$ is finitely generated by $\overline{a}$ we have $\mathcal{M} \models \forall x\phi(x,\overline{a})$. Thus we have proved that if $\mathcal{M}$ is generated by $\overline{a}$ such that $\mathcal{M} \models T(\overline{a})$ then $\mathcal{M} \models \forall x\phi(x,\overline{a})$. By definition of $\Gamma(\overline{c})$ this implies that $\forall x\phi(x,\overline{c}) \in \Gamma(\overline{c})$. This terminate the proof of our claim and on the same time the proof of (1).

(2) First notice that $(P,\leq)$ is partially ordered with a least element $\emptyset$. Now if $p \subseteq q$ then we notice also that $f(p) \subset f(q)$. Thus it is sufficient to prove the properties (v)(1)-(3) of the definition of forcing property.

Let τ,σ be a terms of $L(C)$ without variables and $p \in P$.

(v)(1) Let $(\tau = \sigma) \in f(p)$. By putting $q = p \cup \{(\sigma = \tau)\}$ we see that $q \in P$ and that q has the required properties.

(v)(2) Let $(\tau = \sigma) \in f(p)$, $\phi(\tau) \in f(p)$ where $\phi(x)$ is an atomic formula. Then as before by putting $q = p \cup \{\phi(\sigma)\}$ we see that $q \in P$ and $\phi(\sigma) \in f(q)$ as required.

$(v)(3)$ By taking $\tau' = \tau$ and $q = p \cup \{\tau = \tau\}$ we see as before that q has the desired properties.

Thus $\mathbb{P} = (P, \leq, \Delta, f)$ is a forcing property as claimed. $\square$

To prove Theorem 4.2 we need the following lemma:

Lemma 4.6 *Let $T(\bar{c})$ be a τ-close theory such that every finite subset of $T(\bar{c})$ is n-consistent. Let L_A be a countable fragment in $L_{\omega_1\omega}$ and let $\mathbb{P}$ be the canonical forcing property of $T(\bar{c})$. Then for every sentence ϕ of $L(C)$ we have:*

(1) *For every p in P, if $p \Vdash \phi$ then $p \cup \{\phi\}$ is n-consistent.*

(2) *For every p in P, if $p \vdash_n \phi$ then $p \Vdash^w \phi$.*

Proof

The proof is by induction on ϕ and we prove (1) and (2) simultaneously.

(i) ϕ is atomic.

(1) Since $p \Vdash \phi$ then $\phi \in f(p)$ therefore $p \cup \{\phi\}$ is n-consistent.

(2) Let $q \geq p$ and let $r = q \cup \{\phi\}$ then $\phi \in f(r)$ and $r \in P$, as $p \vdash_n \phi$. Therefore by Lemma 3.5 $p \Vdash^w \phi$ as required.

(ii) $\phi = \neg\varphi$.

(1) We have $p \Vdash \neg\varphi$. Suppose towards a contradiction that $p \cup \{\neg\varphi\}$ is not n-consistent, and thus $p \vdash_n \phi$. By induction hypothesis (2), we have $p \Vdash^w \varphi$. So there exists $q \geq p$ such that $q \Vdash \varphi$, a contradiction with $p \Vdash \neg\varphi$. Thus $p \cup \{\neg\varphi\}$ is n-consistent.

(2) We have $p \vdash_n \neg\varphi$. Suppose towards a contradiction that $p \nVdash^w \neg\varphi$. Then by Lemma 3.5 there is $q \geq p$ such that for every $q' \geq q$, $q' \nVdash \neg\varphi$. Therefore $q \nVdash \neg\varphi$ and thus there exists $r \geq q$ such that $r \Vdash \varphi$. By induction $r \cup \{\varphi\}$ is n-consistent. But since $r \geq q \geq p$ and $p \vdash_n \neg\varphi$, we have $\neg\varphi \wedge \varphi$ is n-consistent which is a contradiction. Thus $p \Vdash^w \neg\varphi$ as required.

(iii) $\phi = \varphi_1 \vee \varphi_2$.

(1) Since $p \Vdash \varphi_1 \vee \varphi_2$ we have $p \Vdash \varphi_1$ or $p \Vdash \varphi_2$. The conclusion follows by induction.

(2) We have $p \vdash_n \varphi_1 \vee \varphi_2$. Suppose towards a contradiction that $p \nVdash^w \varphi_1 \vee \varphi_2$. Then as before by Lemma 3.5 there exists $q \geq p$ such that $q \Vdash \neg(\varphi_1 \vee \varphi_2)$. Let $\mathcal{M}$ be a generic model for q. Then $\mathcal{M} \models \neg(\varphi_1 \vee \varphi_2)$, therefore $\mathcal{M} \models (\neg\varphi_1 \wedge \neg\varphi_2)$ and thus $\mathcal{M} \models \neg\varphi_1$ and $\mathcal{M} \models \neg\varphi_2$. Hence there exists $r \geq q$ such that $r \Vdash \neg\varphi_1$ and $r \Vdash \neg\varphi_2$.

Now suppose that for every finite subset $S \subseteq T(\bar{c})$, $S \cup r \cup \{\varphi_1\}$ is n-consistent. Then $r \cup \{\varphi_1\} \in P$, and thus by induction $r \Vdash^w \varphi_1$, a contradiction with $r \Vdash \neg\varphi_1$.

Now suppose that there exists a finite subset $S_0 \subseteq T(\bar{c})$ such that $S_0 \cup r \cup \{\varphi_1\}$ is not n-consistent. Then for every finite subset $S \subseteq T(\bar{c})$, $S \cup r \cup \{\neg\varphi_1\}$ is n-consistent. Since $p \vdash_n \varphi_1 \vee \varphi_2$ and $r \geq p$ we have for every finite subset $S \subseteq T(\bar{c})$, $S \cup r \cup \{\varphi_2\}$ is n-consistent. Then $r \cup \{\varphi_2\} \in P$, and thus by induction $r \Vdash^w \varphi_2$, and this is a contradiction with $r \Vdash \neg\varphi_2$.

Hence our initial supposition is false, thus $p \Vdash^w \varphi_1 \vee \varphi_2$.

(iv) $\phi = \exists x\varphi(x)$.

(1) Since $p \Vdash \exists x\varphi(x)$, there is some term $\tau \in \Delta$ such that $p \Vdash \varphi(\tau)$. By induction $p \cup \{\varphi(\tau)\}$ is n-consistent, therefore $p \cup \{\exists x\varphi(x)\}$ is n-consistent.

(2) $p \vdash_n \exists x\varphi(x)$. Suppose towards a contradiction that $p \nVdash^w \exists x\varphi(x)$, thus there is some $q \geq p$ such that $q \Vdash \neg\exists x\varphi(x)$. Let $\mathcal{M}$ be a generic model for q. Then $\mathcal{M} \models \forall x\neg\varphi(x)$. Hence there exists $r \geq q$ such that $r \Vdash \forall x\neg\varphi(x)$.

Suppose towards a contradiction that there is some term τ_0 such that for every finite subset $S \subseteq T(\overline{c})$, $S \cup r \cup \{\varphi(\tau_0)\}$ is n-consistent. Then $r \cup \{\varphi(\tau_0)\} \in P$, and by induction we have $r \Vdash^w \varphi(\tau_0)$. Since $r \Vdash \forall x\neg\varphi(x)$, by Lemma 3.9 for every term τ there exists $r' \geq r$ such that $r' \Vdash \neg\phi(\tau)$ and thus $r' \Vdash \neg\phi(\tau_0)$ for some $r' \geq r$. A contradiction with $r \Vdash^w \varphi(\tau_0)$.

Therefore for every term τ, there is a finite subset $S_\tau \subseteq T(\overline{c})$ such that $S_\tau \cup r \vdash_n \neg\varphi(\tau)$. Therefore we have: for every term τ, there is a finite subset $S_\tau \subseteq T(\overline{c})$ such that $S_\tau \vdash_n (\bigwedge_{\phi \in r} \phi \Rightarrow \neg\varphi(\tau))$. Since $T(\overline{c})$ is τ-close there is some finite subset $S \subseteq T(\overline{c})$ such that $S \vdash_n \forall y(\bigwedge_{\phi \in r} \phi \Rightarrow \neg\varphi(y))$, thus $S \vdash_n (\bigwedge_{\phi \in r} \phi \Rightarrow \forall y(\neg\varphi(y))$ and thus $S \cup r \vdash_n \forall y(\neg\varphi(y))$. But, since $p \vdash_n \exists x\varphi(x)$ and $r \geq p$ we have a final contradiction.

Therefore our initial hypothesis is false, thus for every $q \geq p$ there exists $r \geq q$ such that $r \Vdash \exists x\phi(x)$. Hence $p \Vdash^w \exists x\phi(x)$.

This complete the proof of the lemma. $\square$

Lemma 4.7 *Let $T(\overline{c})$ be a τ-close theory such that every finite subset of $T(\overline{c})$ is n-consistent. Let L_A be a countable fragment in $L_{\omega_1\omega}$ and let $\mathbb{P}$ be the canonical forcing property of $T(\overline{c})$. Let $\mathcal{M}$ be a generic model relative to $\mathbb{P}$ and L_A, and let $\overline{a}$ be the interpretation of $\overline{c}$ in $\mathcal{M}$. Then $\mathcal{M} \models T(\overline{a})$. Furthermore for every $p \in P$ there exists a generic model $\mathcal{M}$ such that $\mathcal{M} \models (\bigwedge_{\theta \in p} \theta)$.*

Proof

Let $\phi(\overline{c}) \in T(\overline{c})$. Suppose towards a contradiction that there exists a generic model $\mathcal{M}$ generically-generated by G such that $\mathcal{M} \models \neg\phi(\overline{a})$ where $\overline{a}$ is the interpretations of the constants symbols $\overline{c}$. Then there is $p \in G$ such that $p \Vdash \neg\phi(\overline{c})$. Let $q = p \cup \{\phi(\overline{c})\}$. Then $q \in P$ and by lemma 4.6 we have $q \Vdash^w \phi(\overline{c})$, this contradict the fact that $p \Vdash \neg\phi(\overline{c})$.

Now let $p \in P$ and let $\mathcal{M}$ be a generic model generically-generated by G such that $p \in G$. Since $p \vdash_n (\bigwedge_{\theta \in p} \theta)$, by Lemma 4.6, $p \Vdash^w (\bigwedge_{\theta \in p} \theta)$. Therefore there is no $q \geq p$ such that $q \Vdash \neg(\bigwedge_{\theta \in p} \theta)$. Therefore, since G is a generic set, there exists $r \in G$ such that $r \Vdash (\bigwedge_{\theta \in p} \theta)$, and thus $\mathcal{M} \models (\bigwedge_{\theta \in p} \theta)$. $\square$

Proof of Theorem 4.2 The theorem is a mere consequence of the Lemma 4.7.

$\square$

Now we are going to prove Theorem 4.3. To do this we need the following lemma:

Lemma 4.8 *Let:*

$\Phi = \forall x_1 \cdots \forall x_m \bigvee_{i\in\mathbb{N}} \varphi_i(\overline{c}, x_1, \cdots, x_m)$, *where* $\varphi_i(\overline{c}, x_1, \cdots, x_m)$ *is in* $L(C)$.

Let $T(\overline{c})$ *be* τ*-close theory such that every finite subset of* $T(\overline{c})$ *is* n*-consistent. Let* L_A *be a fragment such that* $\Phi \in L(C)_A$*, and let* $\mathbb{P}$ *be the canonical forcing property of* $T(\overline{c})$*. Then the following properties are equivalent:*

(I) Φ *holds in all generic models (relatively to* L_A *and to* $\mathbb{P}$*).*

(II) *For every finite tuple* $\tau_1, \cdots, \tau_m$ *of terms and for every* $p \in P$ *there is* $i \in \mathbb{N}$ *such that* $T(\overline{c}) \cup p \cup \{\varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)\}$ *is* n*-consistent.*

Proof

By Lemma 3.5 and Lemma 3.9, each of the following statements is equivalent:

(1) Φ holds in all generic models for 0.

(2) For every finite tuple $\tau_1, \cdots, \tau_m$ of terms, $\bigvee_{i\in\mathbb{N}} \varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)$ holds in all generic models for 0.

(3) For every finite tuple $\tau_1, \cdots, \tau_m$ of terms, $0 \Vdash^w \bigvee_{i\in\mathbb{N}} \varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)$.

(4) For every finite tuple $\tau_1, \cdots, \tau_m$ of terms, and for every $p \in P$ there exists $q \geq p$ and there exists $i \in \mathbb{N}$ such that $q \Vdash \varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)$.

Now we prove that (4) is equivalent to the following:

(5) For every finite tuple $\tau_1, \cdots, \tau_m$ of terms, and for every $p \in P$ there exists $q \geq p$ and there exists $i \in \mathbb{N}$ such that: for every finite subset $S \subseteq T(\overline{c})$, $S \cup q \cup \{\varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)\}$ is n-consistent.

Let us show that (4)$\Rightarrow$(5). Since $q \Vdash \varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)$, then for every finite subset $S \subseteq T(\overline{c})$, $S \cup q \Vdash \varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)$, as $q' = S \cup q \in P$ and $q' \geq q$. Therefore by Lemma 4.6, we have: for every finite subset $S \subseteq T(\overline{c})$, $S \cup q \cup \{\varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)\}$ is n-consistent. Thus we have (5).

We show now that (5)$\Rightarrow$(4). Since for every finite subset $S \subseteq T(\overline{c})$, $S \cup q \cup \{\varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)\}$ is n-consistent, then $q \cup \{\varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)\} \in P$. By Lemma 4.6 we have $q \Vdash^w \varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)$. Therefore there is $q' \geq q \geq p$ such that $q' \Vdash \varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)$, thus we have (4).

Now (5) is clearly equivalent to the following:

(6) For every finite tuple $\tau_1, \cdots, \tau_m$ of terms, and for every $p \in P$ there exists $i \in \mathbb{N}$ such that: for every finite subset $S \subseteq T(\overline{c})$, $S \cup p \cup \{\varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)\}$ is n-consistent.

Now let us show that (6) is equivalent to the following:

(7) for every finite tuple $\tau_1, \cdots, \tau_m$ of terms, for every $p \in P$, there is $i \in \mathbb{N}$ such that: $T(\overline{c}) \cup p \cup \{\varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)\}$ is n-consistent.

Obviously we have (7)$\Rightarrow$(6).

We show (6)$\Rightarrow$(7). By (6) $p \cup \{\varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)\} \in P$. Therefore, by Lemma 4.7, there exists a generic model $\mathcal{M}$ such that $\mathcal{M} \models (\bigwedge_{\theta\in p} \theta) \wedge \varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)$. Since, by the same Lemma, $\mathcal{M} \models T(\overline{c})$ the theory $T(\overline{c}) \cup p \cup \{\varphi_i(\overline{c}, \tau_1, \cdots, \tau_m)\}$ is n-consistent.

Therefore (1)$\Leftrightarrow$(7) and this complete the proof of the lemma. $\square$

Proof of Theorem 4.3 For every $i \in \mathbb{N}$ put:

$$\triangle_i(\overline{c}, \overline{x}_i) = \{\varphi_j(\overline{c}, x_1, \cdots, x_{m_i}) : j \in \mathbb{N}\},$$

where m_i is the length of $\overline{x}_i$. For every $i \in \mathbb{N}$ let:

$$\Phi_i = \forall x_1 \cdots \forall x_{m_i} \bigvee_{j \in \mathbb{N}} \neg\varphi_j(\overline{c}, x_1, \cdots, x_{m_i}), \text{ where } \varphi_j(\overline{c}, x_1, \cdots, x_{m_i}) \in \triangle_i.$$

Let L_A be a fragment such that $L(C)_A$ contains Φ_i for every $i \in \mathbb{N}$.
Since $T(\overline{c})$ is n-consistent by Lemma 4.4 the theory

$$\Gamma(\overline{c}) = \{\phi(\overline{c}) : T(\overline{c}) \vdash_n \phi(\overline{c})\}$$

is τ-close (and $T(\overline{c}) \subseteq \Gamma(\overline{c})$).

Since for every $i \in \mathbb{N}$ the set $\triangle_i$ is not a principal set then for every i and for every m_i-tuple $\overline{\tau}$ of terms there is no sentence $\phi(\overline{c})$ of $L(C)$ which is n-consistent with $T(\overline{c})$ such that:

$$T(\overline{c}) \vdash_n (\phi(\overline{c}) \Rightarrow \varphi(\overline{c}, \overline{\tau})), \text{ for every } \varphi(\overline{c}, \overline{x}_i) \in \triangle_i.$$

Therefore for every $i \in \mathbb{N}$ and for every tuple $\overline{\tau}$ of terms there is no sentence $\phi(\overline{c})$ of $L(C)$ which is n-consistent with $\Gamma(\overline{c})$ such that:

$$\Gamma(\overline{c}) \vdash_n (\phi(\overline{c}) \Rightarrow \varphi(\overline{c}, \overline{\tau})), \text{ for every } \varphi(\overline{c}, \overline{x}) \in \triangle_i.$$

Therefore for every $i \in \mathbb{N}$ and for every finite tuple $\tau_1, \cdots, \tau_m$ of terms, and for every $p \in P$ there exists $j \in \mathbb{N}$ such that $\Gamma(\overline{c}) \cup p \cup \{\neg\varphi_j(\overline{c}, \tau_1, \cdots, \tau_m)\}$ is n-consistent where $\varphi_j \in \triangle_i$, and $\mathbb{P}$ is the canonical forcing property of $\Gamma(\overline{c})$ (note that, by Lemma 4.7, for every $p \in P$ the theory $\Gamma(\overline{c}) \cup p$ is n-consistent).

Therefore by Lemma 4.8 every generic model, relatively to L_A and $\mathbb{P}$, satisfies Φ_i for every $i \in \mathbb{N}$. Thus every generic model, relatively to L_A and $\mathbb{P}$, omit $\triangle_i$ for every i. By Lemma 4.7 every generic model is a n-generated model of $\Gamma(\overline{c})$, and thus a n-generated model of $T(\overline{c})$. Thus $T(\overline{c})$ has a n-generated model which omit $\triangle_i$ for every $i \in \mathbb{N}$. $\square$

4.2 The number of n-generated models

Definition 4.9 *Let $T(\overline{c})$ be a theory in $L(C)$. We denote by $\alpha_n(T(\overline{c}))$ the number of nonisomorphic n-generated models of $T(\overline{c})$. If $\mathcal{M}$ is a model we denote $\alpha_n(\mathcal{M})$ the number $\alpha_n(Th(\mathcal{M}))$. By $\alpha(T(\overline{c}))$ we denote the number of finitely generated models of $T(\overline{c})$ and by $\alpha(\mathcal{M})$ the number $\alpha(Th(\mathcal{M}))$.*

We will prove the following:

Theorem 4.10 *Let $T(\overline{c})$ be a n-consistent theory. Then either $\alpha_n(T(\overline{c})) \leq \aleph_0$ or $\alpha_n(T(\overline{c})) = 2^{\aleph_0}$. Furthermore $\alpha(T(\overline{c})) \leq \aleph_0$ or $\alpha(T(\overline{c})) = 2^{\aleph_0}$.*

To do this we need some notions and a proposition from [4]. Let Φ be sentence of $L_{\omega_1\omega}$ and L_A be a countable fragment of $L_{\omega_1\omega}$. We define an L_A-n-type to be a set p such that:

$$p = \{\varphi(x_1, \cdots, x_n) \in L_A : \mathcal{M} \models \varphi(a_1, \cdots, a_n)\},$$

for some model $\mathcal{M}$ of Φ and some tuple $\overline{a} \in \mathcal{M}^n$. The set of all L_A-n-type of Φ is denoted by $S_n(L_A, \Phi)$. We need the following proposition:

Proposition 4.11 [4, Corollary 2.4] *For every countable fragment L_A the set $S_n(L_A, \Phi)$ is either countable or of power $2^{\aleph_0}$.*

The reader can found some help about this also in [3].

Proof of Theorem 4.10 Let

$$\Phi = \bigwedge_{\varphi \in T(\overline{c})} \varphi(\overline{c}) \wedge \forall y(\bigvee_{\tau \in Ter} y = \tau(\overline{c})),$$

where Ter is the set of all terms of $L(C)$. Then $\Phi \in L(C)_{\omega_1\omega}$ and every model of Φ is a n-generated model of $T(\overline{c})$. Let L_A be the set of all formulas of $L(C)$. Then L_A is a countable fragment and by Proposition 4.11 $S_n(L_A, \Phi)$ is countable or of power $2^{\aleph_0}$.

Let $\mathcal{M}$ be a model of Φ and let $\overline{a}$ the interpretation of the constants symbols $\overline{c}$. Then the set

$$\triangle(\mathcal{M}, \overline{a}) = \{\varphi(x_1, \cdots, x_n) \in L_A : \mathcal{M} \models \varphi(a_1, \cdots, a_n)\}$$

is a L_A-n-type.

Therefore if $S_n(L_A, \Phi)$ is countable Φ has at most $\aleph_0$ models, and if $S_n(L_A, \Phi)$ has a power $2^{\aleph_0}$ then Φ must have $2^{\aleph_0}$ models (and thus $2^{\aleph_0}$ n-generated models), as any model of Φ realize at most a countable number of L_A-n-type.

Now $\alpha(T(\overline{c})) \leq \aleph_0$ whenever $\alpha_n(T(\overline{c})) \leq \aleph_0$ for every $n \in \mathbb{N}$, and $\alpha(T(\overline{c})) = 2^{\aleph_0}$ whenever $\alpha_n(T(\overline{c})) = 2^{\aleph_0}$ for some $n \in \mathbb{N}$. $\square$

We are going to see some consequence of the fact that $\alpha_n(T) \leq \aleph_0$.

Theorem 4.12 *Let $T(\overline{c})$ be a n-consistent theory in $L(C)$ such that $\alpha_n(T(\overline{c})) \leq \aleph_0$. Then there exist a n-generated model $(\mathcal{N}, \overline{a})$ of $T(\overline{c})$ and a sentence $\phi(\overline{c})$ in $L(C)$ such that $\mathcal{N} \models \phi(\overline{a})$ and for every n-generated model $(\mathcal{K}, \overline{b})$ of $T(\overline{c})$ we have $(\mathcal{N}, \overline{a}) \cong (\mathcal{K}, \overline{b})$ if and only if $\mathcal{K} \models \phi(\overline{b})$.*
In other words we have:

(1) $\mathcal{N} \models \phi(\overline{a})$,

(2) *if $\mathcal{K}$ is a n-generated model, generated by the n-tuple $\overline{b}$, such that $\mathcal{K} \models T(\overline{b})$ and $\mathcal{K} \models \phi(\overline{b})$ then the function defined by $f(a_i) = b_i$ (for $1 \leq i \leq n$) extends to an isomorphism.*

Definition 4.13 *A model $(\mathcal{N}, \overline{a})$ of $T(\overline{c})$ for which there exists a sentence ϕ satisfying (1) and (2) above is called a* ***pseudo-prime*** *model of $T(\overline{c})$.*

Proof

Let $\alpha = \alpha_n(T(\overline{c}))$ and let $(\mathcal{M}_i)_{i\in\alpha}$ be a representative list of all n-generated models of $T(\overline{c})$, up to isomorphism. For every $i \in \alpha$ let

$$\Omega(\mathcal{M}_i) = \{\overline{a} \in \mathcal{M}_i^n : \overline{a} \text{ generate } \mathcal{M}_i,\ \mathcal{M}_i \models T(\overline{a})\},$$

and let

$$I = \{(i, \overline{a}) : i \in \alpha,\ \overline{a} \in \Omega(\mathcal{M}_i)\}.$$

For every $(i, \overline{a}) \in I$ let

$$\triangle_{(i,\overline{a})} = \{\varphi(\overline{c}) : \varphi(\overline{c}) \text{ is in } L(C) \text{ such that } \mathcal{M}_i \models \varphi(\overline{a})\}.$$

Suppose towards a contradiction that for every $(i, \overline{a}) \in I$, $\triangle_{(i,\overline{a})}$ is not a principal set. Since $\alpha \leq \aleph_0$ the set I is at most countable. Then by the omitting type theorem there is a n-generated model $\mathcal{M}$ of $T(\overline{c})$ which omit $\triangle_{(i,\overline{a})}$ for every $(i, \overline{a}) \in I$, i.e. $\mathcal{M}$ has a generating n-tuple $\overline{b}$ such that:

$$(*) \qquad \mathcal{M} \models T(\overline{b}) \text{ and } \mathcal{M} \models \bigvee_{\varphi\in\triangle_{(i,\overline{a})}} \neg\varphi(\overline{b}), \text{ for every } (i, \overline{a}) \in I.$$

Since $\mathcal{M}$ is a n-generated model of $T(\overline{c})$ there is $i_0 \in \alpha$ such that $(\mathcal{M}, \overline{b}) \cong (\mathcal{M}_{i_0}, \overline{d})$ for some generating n-tuple $\overline{d}$ of $\mathcal{M}_{i_0}$. Now since $\mathcal{M} \models T(\overline{b})$ we have $\mathcal{M}_{i_0} \models T(\overline{d})$ and thus $(i_0, \overline{d}) \in I$. Therefore

$$\mathcal{M}_{i_0} \models \varphi(\overline{d}) \text{ for every } \varphi \in \triangle_{(i_0,\overline{d})},$$

and by (*)

$$\mathcal{M}_{i_0} \models \bigvee_{\varphi\in\triangle_{(i_0,\overline{d})}} \neg\varphi(\overline{d}) \quad \text{as } (\mathcal{M}, \overline{b}) \cong (\mathcal{M}_{i_0}, \overline{d}),$$

which is a contradiction.

Therefore there is $(k, \overline{a}) \in I$ such that $\triangle_{(k,\overline{a})}$ is a principal set, i.e. there exists a formula $\phi(\overline{c})$ such that:

$$T(\overline{c}) \cup \{\phi(\overline{c})\} \text{ is } n\text{-consistent and } T(\overline{c}) \vdash_n (\phi(\overline{c}) \Rightarrow \varphi(\overline{c})), \text{ for every } \varphi(\overline{c}) \in \triangle_{(k,\overline{a})}.$$

We claim that $\mathcal{M}_k$ has the required properties (1) and (2). First of all we have $\mathcal{M}_k \models \phi(\overline{a})$. Indeed if $\mathcal{M}_k \models \neg\phi(\overline{a})$ then $\neg\phi(\overline{c}) \in \triangle_{(k,\overline{a})}$, thus $T(\overline{c}) \vdash_n (\phi(\overline{c}) \Rightarrow \neg\phi(\overline{c}))$ and thus $T(\overline{c}) \vdash_n \neg\phi(\overline{c})$ a contradiction with the n-consistency of $T(\overline{c}) \cup \{\phi(\overline{c})\}$. Therefore we have (1).

Now let $\mathcal{K}$ be a n-generated model, generated by a n-tuple $\overline{b}$, such that $\mathcal{K} \models T(\overline{b})$ and $\mathcal{K} \models \phi(\overline{b})$. Then

$$\mathcal{K} \models \varphi(\overline{b}) \text{ for every } \varphi \in \triangle_{(k,\overline{a})}, \text{ i.e. for every } \varphi \text{ such that } \mathcal{M}_k \models \varphi(\overline{a}),$$

and thus the function defined by $f(a_i) = b_i$ (for $1 \leq i \leq n$) extends to an isomorphism. This complete the proof of the theorem. $\square$

Corollary 4.14 *Let $\mathcal{M}$ be a n-generated model such that $\alpha_n(\mathcal{M}) \leq \aleph_0$. Then there exist a n-generated model $\mathcal{N} = \langle a_1, \cdots, a_n \rangle$ such that $\mathcal{M} \equiv \mathcal{N}$ and a sentence $\phi(c_1, \cdots, c_n)$ in $L(C)$ such that $\mathcal{N} \models \phi(a_1, \cdots, a_n)$, and: if $\mathcal{K}$ is a n-generated model, generated by the tuple $b_1, \cdots, b_n$, such that $\mathcal{K} \equiv \mathcal{M}$ and $\mathcal{K} \models \phi(b_1, \cdots, b_n)$ then the function defined by $f(a_i) = b_i$ (for $1 \leq i \leq n$) extends to an isomorphism.* $\square$

We give now a sufficient condition to have $\alpha_n(T) = 2^{\aleph_0}$.

Definition 4.15 *Let $T(\overline{c})$ be a n-consistent theory and $\phi(\overline{c})$ a sentence of $L(C)$. Then ϕ is said to be n-**complete** if for every sentence $\varphi(\overline{c})$ we have:*

$$T(\overline{c}) \vdash_n (\phi(\overline{c}) \Rightarrow \varphi(\overline{c})) \text{ or } T(\overline{c}) \vdash_n (\phi(\overline{c}) \Rightarrow \neg\varphi(\overline{c})).$$

Then we have:

Theorem 4.16 *Let T be a n-consistent theory in L and suppose that there exists a sentence $\theta(\overline{c})$ n-consistent with T such that $T \nvdash_n (\phi(\overline{c}) \Rightarrow \theta(\overline{c}))$ whenever ϕ is a n-complete sentence. Then $\alpha_n(T) = 2^{\aleph_0}$.*

We begin by the following.

Lemma 4.17 *Let T be a n-consistent theory in L and $\mathcal{N}$ a n-finitely generated model of T such that there exists a n-tuple $\overline{a}$ which generate $\mathcal{N}$ and n-complete sentence $\phi(\overline{c})$ such that $\mathcal{N} \models \phi(\overline{a})$. Then for every generating n-tuple $\overline{b}$ of $\mathcal{N}$ there is a n-complete sentence $\phi_0(\overline{c})$ such that $\mathcal{N} \models \phi_0(\overline{b})$.*

Proof

Since $\phi(\overline{c})$ is a n-complete sentence and $\mathcal{N} \models \phi(\overline{a})$, we have:

$$T \vdash_n (\phi(\overline{c}) \Rightarrow \varphi(\overline{c})) \text{ for every } \varphi \text{ such that } \mathcal{N} \models \varphi(\overline{a}).$$

Let $\overline{b}$ be a generating n-tuple of $\mathcal{N}$. Then there is a finite tuple of terms $\tau_1(\overline{a}), \cdots, \tau_n(\overline{a})$ such that $\mathcal{N} \models \bigwedge_{i=1}^{i=n}(b_i = \tau_i(\overline{a}))$. We claim that the sentence

$$\phi_0(\overline{c}) = \exists \overline{x}(\textstyle\bigwedge_{i=1}^{i=n}(c_i = \tau_i(\overline{x})) \wedge \phi(\overline{x}))$$

is a n-complete one. We prove that:

$$(*) \qquad T \vdash_n (\phi_0(\overline{c}) \Rightarrow \varphi(\overline{c})) \text{ for avery } \varphi \text{ such that } \mathcal{N} \models \varphi(\overline{b}).$$

Let $\mathcal{K}$ be a n-generated model of T generated by the n-tuple $\overline{d}$ such that $\mathcal{K} \models \phi_0(\overline{d})$. Then there is a n-tuple $\overline{k}$ in $\mathcal{K}$ such that

$$\mathcal{K} \models (\bigwedge_{i=1}^{i=n}(d_i = \tau_i(\overline{k})) \wedge \phi(\overline{k})).$$

Since $\mathcal{K}$ is generated by $\overline{d}$ and $\mathcal{K} \models (\bigwedge_{i=1}^{i=n}(d_i = \tau_i(\overline{k}))$, $\mathcal{K}$ is generated by $\overline{k}$. Since $\mathcal{K} \models \phi(\overline{k})$ and $\overline{k}$ generate $\mathcal{K}$ the function defined by $f(a_i) = k_i$ (for $1 \leq$

$i \leq n$) extends to an isomorphism. Therefore for every sentence $\varphi(\overline{c})$, $\mathcal{K} \models \varphi(\overline{d})$ whenever $\mathcal{N} \models \varphi(\overline{b})$, as $\mathcal{K} \models (\bigwedge_{i=1}^{i=n}(d_i = \tau_i(\overline{k}))$ and $\mathcal{N} \models \bigwedge_{i=1}^{i=n}(b_i = \tau_i(\overline{a}))$. Thus we have $(*)$.

Let $\varphi(\overline{c})$ be a sentence in $L(C)$. Then either $\mathcal{N} \models \varphi(\overline{b})$ or $\mathcal{N} \models \neg\varphi(\overline{b})$. Therefore by (*) we have

$$T \vdash_n (\phi_0(\overline{c}) \Rightarrow \varphi(\overline{c})) \text{ or } T \vdash_n (\phi_0(\overline{c}) \Rightarrow \neg\varphi(\overline{c})),$$

and thus $\phi_0(\overline{c})$ is a n-complete sentence as claimed. $\square$

Remark 4.18

(1) In Lemma 4.17 one can not take T in $L(C)$ because one can not ensure, in the proof of that Lemma, that if $\mathcal{K} \models (\bigwedge_{i=1}^{i=n}(d_i = \tau_i(\overline{k})) \wedge \phi(\overline{k}))$ then $\mathcal{K} \models T(\overline{k})$.

(2) A model $\mathcal{M}$ is a pseudo-prime model of T if and only if there exist a n-generating tuple $\overline{a}$ and a n-complete sentence $\phi(\overline{c})$ such that $\mathcal{M} \models \phi(\overline{a})$.

Proof of Theorem 4.16 Suppose towards a contradiction that $\alpha_n(T) \leq \aleph_0$. Let $(\mathcal{N}_i)_{i\in\gamma}$ be a representative list of n-generated models of T, up to isomorphism, such that: for every generating n-tuple $\overline{a}$ of $\mathcal{N}_i$ and for every n-complete sentence $\phi(\overline{c})$, $\mathcal{N}_i \models \neg\phi(\overline{a})$.

As in the proof of Theorem 4.12 for every $i \in \gamma$ let

$$\Omega(\mathcal{N}_i) = \{\overline{a} \in \mathcal{N}_i^n : \overline{a} \text{ generate } \mathcal{N}_i\}, \quad I = \{(i, \overline{a}) : i \in \gamma,\ \overline{a} \in \Omega(\mathcal{N}_i)\}.$$

For every $(i, \overline{a}) \in I$ let

$$\triangle_{(i,\overline{a})} = \{\varphi(\overline{c}) : \varphi(\overline{c}) \text{ is in } L(C) \text{ such that } \mathcal{N}_i \models \varphi(\overline{a})\}.$$

Now let $T' = T \cup \{\theta(\overline{c})\}$. We claim that, for every $(i, \overline{a}) \in I$, $\triangle_{(i,\overline{a})}$ is not a principal set relatively to T', i.e. there is no sentence $\psi(\overline{c})$ which is n-consistent with T' such that:

$$T' \vdash_n (\psi(\overline{c}) \Rightarrow \varphi(\overline{c})), \text{ for every } \varphi(\overline{c}) \in \triangle_{(i,\overline{a})}.$$

Indeed suppose that the opposite is true for some $(i, \overline{a}) \in I$ and for some sentence $\psi(\overline{c})$. Then

$$T \vdash_n (\theta(\overline{c}) \wedge \psi(\overline{c}) \Rightarrow \varphi(\overline{c})), \text{ for every } \varphi(\overline{c}) \in \triangle_{(i,\overline{a})},$$

and thus the sentence $\xi(\overline{c}) := \theta(\overline{c}) \wedge \psi(\overline{c})$ will be a n-complete sentence and $T \vdash_n \xi(\overline{c}) \Rightarrow \theta(\overline{c})$, which is a contradiction. Therefore for every $(i, \overline{a}) \in I$, $\triangle_{(i,\overline{a})}$ is not a principal set relatively to T' as claimed.

Since γ is at most countable, I is also at most countable and by the omitting type theorem there is a n-finitely generated model $\mathcal{M}$ of T' (and thus of T) generated by some n-tuple $\overline{d}$ such that $\mathcal{M} \models \theta(\overline{d})$ and $\mathcal{M}$ omit every $\triangle_{(i,\overline{a})}$, i.e.

$$(*) \qquad \mathcal{M} \models \bigvee_{\varphi \in \triangle_{(i,\overline{a})}} \neg\varphi(\overline{d}), \text{ for every } (i, \overline{a}) \in I.$$

By $(*)$ we have $(\mathcal{M}, \overline{d}) \not\cong (\mathcal{N}_i, \overline{a})$ for every $(i, \overline{a}) \in I$. But we have also:

$$T \vdash_n \theta(\overline{c}) \Rightarrow \neg\phi(\overline{c}) \text{ for every } n\text{-complete sentence } \phi.$$

Therefore $\mathcal{M} \models \neg\phi(\overline{d})$ for every n-complete sentence ϕ. By Lemma 4.17 we see that for every generating n-tuple $\overline{m}$ of $\mathcal{M}$ and for every n-complete sentence $\phi(\overline{c})$, $\mathcal{M} \models \neg\phi(\overline{m})$. Therefore $(\mathcal{M}, \overline{d}) \cong (\mathcal{N}_i, \overline{a})$ for some $(i, \overline{a}) \in I$, which is a contradiction.

Thus our initial supposition is false and by Theorem 4.10 we have $\alpha_n(T) = 2^{\aleph_0}$. □

Corollary 4.19 *Let T be a n-consistent theory in L such that $\alpha_n(T) \leq \aleph_0$. Then for every sentence $\varphi(\overline{c})$ n-consistent with T there exists a n-complete sentence $\phi(\overline{c})$ such that $T \vdash_n (\phi(\overline{c}) \Rightarrow \varphi(\overline{c}))$.* □

4.3 Pseudo-prime finitely generated models

Definition 4.20 *Let T be a theory in L.*
(1). *We denote by $\mathcal{M}od_n(T)$ a representative list of all n-generated models of T, up to isomorphism.*
(2). *Let $\mathcal{E}$ be a class of n-generated models of T and let $\mathcal{M}$ be a n-generated model of T. We say that $\mathcal{M}$ is* ***pseudo-prime over $\mathcal{E}$*** *if there exist a generating n-tuple $\overline{a}$ of $\mathcal{M}$ and a sentence $\phi(\overline{c})$ in $L(C)$ such that $\mathcal{M} \models \phi(\overline{a})$ and for every n-generated model $(\mathcal{N}, \overline{b})$ of T which is not isomorphic to any model of $\mathcal{E}$ we have $(\mathcal{M}, \overline{a}) \cong (\mathcal{N}, \overline{b})$ if and only if $\mathcal{N} \models \phi(\overline{b})$.*
In other words we have:

(1). $\mathcal{M} \models \phi(\overline{a})$,

(2). *if $\mathcal{N}$ is a n-generated model of T, generated by the n-tuple $\overline{b}$, and $\mathcal{N} \models \phi(\overline{b})$ and $\mathcal{N}$ is not isomorphic to any model of $\mathcal{E}$ then the function defined by $f(a_i) = b_i$(for $1 \leq i \leq n$) extends to an isomorphism.*

Thus the pseudo-prime models of T are juste the pseudo-prime models of T over the class $\mathcal{E} = \emptyset$.
(3). *We define a sequence $(\mathcal{K}_\gamma(T) \subseteq \mathcal{M}od_n(T) : \gamma \in Ord)$ of classes of n-generated model of T, where Ord is the class of all ordinals, as follows:*
- $\mathcal{K}_0(T) = \emptyset$,
- $\mathcal{K}_{\gamma+1}(T) = \mathcal{K}_\gamma(T) \cup \{\mathcal{M} \in \mathcal{M}od_n(T) \ : \ \mathcal{M}$ *is pseudo-prime over* $\mathcal{K}_\gamma(T)\}$,
- $\mathcal{K}_\beta(T) = \bigcup_{\gamma<\beta} \mathcal{K}_\gamma(T)$ *if β is a limit ordinal.*

We let $\mathcal{K}(T) = \bigcup_{\gamma \in Ord} \mathcal{K}_\gamma(T)$.

We want to show the following theorem which translate another consequence of $\alpha_n(T) \leq \aleph_0$.

Theorem 4.21 *Let T be a n-consistent theory in L such that $\alpha_n(T) \leq \aleph_0$. Let $\mathcal{M}$ be a n-generated model of T. Then for every generating n-tuple $\overline{a}$ of $\mathcal{M}$ there exists a formula $\phi(\overline{x})$ such that $\mathcal{M} \models \phi(\overline{a})$ and such that for every generating n-tuple $\overline{b}$ of $\mathcal{M}$ if $\mathcal{M} \models \phi(\overline{b})$ then the function defined by $f(a_i) = b_i$ (for $1 \leq i \leq n$) extends to an automorphism of $\mathcal{M}$.*

The above theorem translate an *"internal"* property of all n-generated models of a theory T having at most $\aleph_0$ n-generated models. To prove it we begin by proving the following two lemmas. From now we assume that T is a n-consistent theory in L.

Lemma 4.22 *Let $\mathcal{E}$ be a class of n-generated model of T and $\mathcal{M}$ a n-generated model of T which is pseudo-prime over $\mathcal{E}$. Then for every generating n-tuple $\overline{b}$ of $\mathcal{M}$ there is a sentence $\psi(\overline{c})$ of $L(C)$ such that:*

(1). $\mathcal{M} \models \psi(\overline{b})$,

(2). *if $\mathcal{N}$ is a n-generated model of T, generated by the n-tuple $\overline{d}$, and $\mathcal{N} \models \psi(\overline{d})$ and $\mathcal{N}$ is not isomorphic to any model of $\mathcal{E}$ then the function defined by $f(b_i) = d_i$(for $1 \leq i \leq n$) extends to an isomorphism.*

Proof

The proof is analogue to the one of Lemma 4.17. Since $\mathcal{M}$ is pseudo-prime over $\mathcal{E}$ there exists a generating n-tuple $\overline{a}$ of $\mathcal{M}$ and a sentence $\phi(\overline{c})$ of $L(C)$ such that:

(1). $\mathcal{M} \models \phi(\overline{a})$,

(2). if $\mathcal{N}$ is a n-generated model of T, generated by the n-tuple $\overline{h}$, and $\mathcal{N} \models \phi(\overline{h})$ and $\mathcal{N}$ is not isomorphic to any model of $\mathcal{E}$ then the function defined by $f(a_i) = h_i$(for $1 \leq i \leq n$) extends to an isomorphism.

Let $\overline{b}$ be a generating n-tuple of $\mathcal{M}$. Then there is a finite tuple of terms $\tau_1(\overline{a}), \cdots, \tau_n(\overline{a})$ such that $\mathcal{M} \models \bigwedge_{i=1}^{i=n}(b_i = \tau_i(\overline{a}))$. Let:

$$\psi(\overline{c}) = \exists\overline{x}(\bigwedge_{i=1}^{i=n}(c_i = \tau_i(\overline{x})) \wedge \phi(\overline{x}))$$

then $\mathcal{M} \models \psi(\overline{b})$. We claim that $\psi(\overline{c})$ has the required properties.

Let $\mathcal{N}$ be a n-generated model of T, generated by the n-tuple $\overline{p}$, and $\mathcal{N} \models \psi(\overline{p})$ and $\mathcal{N}$ is not isomorphic to any model of $\mathcal{E}$. Since $\mathcal{N} \models \exists\overline{x}(\bigwedge_{i=1}^{i=n}(p_i = \tau_i(\overline{x})) \wedge \phi(\overline{x}))$, there exists $d_1, \cdots, d_n$ such that:

$$\mathcal{N} \models \phi(\overline{d}), \quad \mathcal{N} \models \bigwedge_{i=1}^{i=n}(p_i = \tau_i(\overline{d})).$$

Since $\mathcal{N}$ is generated by $\overline{p}$ and $\mathcal{N} \models (p_i = \tau_i(\overline{d}))$(for $1 \leq i \leq n$) then $\mathcal{N}$ is generated by $\overline{d}$. Since $\mathcal{N} \models \phi(\overline{d})$ the function defined by $f(a_i) = d_i$(for $1 \leq i \leq n$) extends to an isomorphism. Thus the function defined by $f(b_i) = h_i$(for $1 \leq i \leq n$) extends also to an isomorphism. □

Lemma 4.23 *There is a least ordinal γ such that $\mathcal{K}_\beta(T) = \mathcal{K}_\gamma(T)$ for every $\beta \geq \gamma$(and thus $\mathcal{K}(T) = \mathcal{K}_\gamma(T)$). If $\mathcal{K}(T) \neq \emptyset$ then for every $\mathcal{M} \in \mathcal{K}(T)$ there is an ordinal β such that $\mathcal{M} \in \mathcal{K}_{\beta+1}(T) \setminus \mathcal{K}_\beta(T)$. If $\alpha_n(T) \leq \aleph_0$ then $\mathcal{K}(T) = \mathcal{M}od_n(T)$.*

Proof

Since $\mathcal{M}od_n(T)$ has a cardinal at most $2^{\aleph_0}$, there is a least ordinal γ such that $\mathcal{K}_{\gamma+1} = \mathcal{K}_\gamma$. Then we have also, by definition of $\mathcal{K}(T)$, that $\mathcal{K}_\beta(T) = \mathcal{K}_\gamma(T)$ for every $\beta \geq \gamma$ and thus $\mathcal{K}(T) = \mathcal{K}_\gamma(T)$.

Suppose that $\mathcal{K}(T) \neq \emptyset$. If $\mathcal{M} \in \mathcal{K}(T)$ then $\mathcal{M} \in \mathcal{K}_\gamma(T)$ for some ordinal γ. Let δ be the least ordinal such that $\mathcal{M} \in \mathcal{K}_\delta(T)$. Then we see, by definition of $\mathcal{K}(T)$, that δ is not a limit ordinal and that $\delta \geq 1$. Therefore by putting $\delta = \beta + 1$ we have $\mathcal{M} \in \mathcal{K}_{\beta+1}(T) \setminus \mathcal{K}_\beta(T)$, as δ is the least ordinal such that $\mathcal{M} \in \mathcal{K}_\delta(T)$.

Now suppose that $\alpha_n(T) \leq \aleph_0$ and let γ be the least ordinal such that $\mathcal{K}_{\gamma+1}(T) = \mathcal{K}_\gamma(T) = \mathcal{K}(T)$. By Theorem 4.12 T has a pseudo-prime model over $\emptyset$ thus $\mathcal{K}_1(T) \neq \emptyset$ and therefore $\mathcal{K}(T) \neq \emptyset$. Therefore for every $\mathcal{M} \in \mathcal{K}_\gamma(T)$ there exist β such that $\mathcal{M} \in \mathcal{K}_{\beta+1}(T) \setminus \mathcal{K}_\beta(T)$ and thus $\mathcal{M}$ is pseudo-prime over $\mathcal{K}_\beta(T)$.

Suppose towards a contradiction that $\mathcal{K}_\gamma(T) \neq \mathcal{M}od_n(T)$. Since $\mathcal{K}_\gamma(T) \subseteq \mathcal{M}od_n(T)$ we have $\mathcal{M}od_n(T) \setminus \mathcal{K}_\gamma(T) \neq \emptyset$. Let $\mathcal{K}_\gamma(T) = (\mathcal{M}_i : i \in \rho)$ where $|\mathcal{K}_\gamma(T)| = \rho$. For every $i \in \rho$ let

$$\Omega(\mathcal{M}_i) = \{\overline{a} \in \mathcal{M}_i^n : \overline{a} \text{ generate } \mathcal{M}_i\},$$

and let

$$I = \{(i, \overline{a}) : i \in \rho,\ \overline{a} \in \Omega(\mathcal{M}_i)\}.$$

Then by Lemma 4.22, for every $(i, \overline{a}) \in I$ there exists a sentence $\phi_{(i,\overline{a})}(\overline{c})$ in $L(C)$ satisfying the conditions of the definition of the fact that $\mathcal{M}_i$ is pseudo-prime over the class $\mathcal{K}_\beta(T)$ where $\mathcal{M}_i \in \mathcal{K}_{\beta+1}(T) \setminus \mathcal{K}_\beta(T)$, i.e. $\mathcal{M}_i \models \phi_{(i,\overline{a})}(\overline{a})$ and for every n-generated model $(\mathcal{N}, \overline{b})$ of T which not isomorphic to any model of $\mathcal{K}_\beta(T)$ we have $(\mathcal{M}_i, \overline{a}) \cong (\mathcal{N}, \overline{b})$ if and only if $\mathcal{N} \models \phi_{(i,\overline{a})}(\overline{b})$.

Let:

$$\Gamma(\overline{c}) = T \cup \{\neg\phi_{(i,\overline{a})}(\overline{c}) \mid (i, \overline{a}) \in I\}$$

Since $\mathcal{M}od_n(T) \setminus \mathcal{K}_\gamma(T) \neq \emptyset$, $\Gamma(\overline{c})$ is n-consistent (in fact every n-generated model $\mathcal{Q}$ in $\mathcal{M}od_n(T) \setminus \mathcal{K}_\gamma(T)$ generated by a n-tuple $\overline{d}$ satisfies $\Gamma(\overline{d})$).

Since every n-generated model of $\Gamma(\overline{c})$ is a n-generated model of T and $\alpha_n(T) \leq \aleph_0$ we have $\alpha_n(\Gamma(\overline{c})) \leq \aleph_0$. Thus by Theorem 4.12, $\Gamma(\overline{c})$ has a pseudo-prime model $\mathcal{N}$. Thus $\mathcal{N}$ is a n-generated model of T and without lost generality we may assume that $\mathcal{N} \in \mathcal{M}od_n(T)$.

We claim that $\mathcal{N}$ is a pseudo-prime model over $\mathcal{K}_\gamma(T)$. Now there is a generating n-tuple $\overline{b}$ of $\mathcal{N}$ such that $\mathcal{N} \models \Gamma(\overline{b})$ and a sentence $\phi(\overline{c})$ in $L(C)$ such that:

(1) $\mathcal{N} \models \phi(\overline{b})$,

(2) if $\mathcal{Q}$ is a n-generated model, generated by the tuple $\overline{d}$, such that $\mathcal{Q} \models \Gamma(\overline{d})$ and $\mathcal{Q} \models \phi(\overline{d})$ then the function defined by $f(b_i) = d_i$(for $1 \leq i \leq n$) extends to an isomorphism.

First of all, since $\mathcal{N} \models \Gamma(\overline{a})$, $\mathcal{N} \notin \mathcal{K}_\gamma(T)$.

Now let $\mathcal{Q}$ be a n-generated model of T generated by the n-tuple $\overline{d}$ such that $\mathcal{Q}$ is not isomorphic to any model of $\mathcal{K}_\gamma(T)$ and $\mathcal{Q} \models \phi(\overline{d})$. Since $\mathcal{Q}$ is not isomorphic to any model of $\mathcal{K}_\gamma(T)$ we have $\mathcal{Q} \models \Gamma(\overline{d})$ and by the property (2) above the function defined by $f(b_i) = d_i$(for $1 \leq i \leq n$) extends to an isomorphism.

Therefore $\mathcal{N}$ is a pseudo-prime model over $\mathcal{K}_\gamma(T)$ and thus $\mathcal{N} \in \mathcal{K}_{\gamma+1}(T) \setminus \mathcal{K}_\gamma(T)$, a contradiction with $\mathcal{K}_{\gamma+1}(T) = \mathcal{K}_\gamma(T)$.

Therefore our initial supposition is false and thus we have $\mathcal{M}od_n(T) = \mathcal{K}_\gamma(T) = \mathcal{K}(T)$. □

Definition 4.24 *Let T such that $\alpha_n(T) \leq \aleph_0$. Then by Lemma 4.23, there is a least γ such that $\mathcal{K}_\gamma(T) = \mathcal{K}_{\gamma+1}(T)$. We call that γ the **hight** of T. Since, by Lemma 4.23, $\mathcal{M}od_n(T) = \mathcal{K}_\gamma(T) = \bigcup_{\beta \leq \gamma} \mathcal{K}_\beta(T)$, we define the **hight** of a n-generated model $\mathcal{M}$ of T to be the least ordinal β such that $\mathcal{N} \in \mathcal{K}_{\beta+1}(T) \backslash \mathcal{K}_\beta(T)$ where $\mathcal{N} \cong \mathcal{M}$ and $\mathcal{N} \in \mathcal{M}od_n(T)$.*

Proof of Theorem 4.21 Let $\mathcal{M}$ be a n-generated model of T and let β be the hight of $\mathcal{M}$. Without lost generality we may assume that $\mathcal{M} \in \mathcal{M}od_n(T)$. Thus $\mathcal{M}$ is pseudo-prime model over $\mathcal{K}_\beta(T)$. By Lemma 4.22 for every generating n-tuple $\overline{a}$ of $\mathcal{M}$ there is a sentence $\phi(\overline{c})$ such that:

(1) $\mathcal{M} \models \phi(\overline{a})$,

(2) if $\mathcal{N}$ is a n-generated model of T, generated by the n-tuple $\overline{b}$, and $\mathcal{N} \models \phi(\overline{b})$ and $\mathcal{N}$ is not isomorphic to any model of $\mathcal{K}_\beta(T)$ then the function defined by $f(a_i) = d_i$(for $1 \leq i \leq n$) extends to an isomorphism.

By replacing in (2) $\mathcal{N}$ by $\mathcal{M}$ we find the required conclusions. □

Now we give a proof of the following theorem enounced in the introduction.

Theorem 4.25 *Let T be a complete theory such that $\alpha_n(T) \leq \aleph_0$. Let $\mathcal{M}$ be a n-generated model of T. Then the following properties are equivalent:*

(1) *$\mathcal{M}$ is prime model of T.*

(2) *There is a formula $\theta(x_1, \cdots, x_n)$ consistent in $\mathcal{M}$ such that if $\mathcal{M} \models \theta(b_1, \cdots, b_n)$ then $b_1, \cdots, b_n$ generate $\mathcal{M}$.*

We use the following classical result:

Proposition 4.26 [1] *Let $\mathcal{M}$ be a countable model. Then $\mathcal{M}$ is a prime model of its theory iff for every $m \in \mathbb{N}$, each orbit under the action of $Aut(\mathcal{M})$ on $\mathcal{M}^m$ is first-order definable without parameters.*

Proof of Theorem 4.25

(1)⇒(2). Suppose that $\mathcal{M}$ is a prime model of its theory and let $a_1, \cdots, a_n$ generate $\mathcal{M}$. Then there is some orbit $\mathcal{O}_n$ such that $(a_1, \cdots, a_n) \in \mathcal{O}_n$. By Proposition 4.26 $\mathcal{O}_n$ is definable by a first order formula $\theta(x_1, \cdots, x_n)$. Now if $\mathcal{M} \models \theta(b_1, \cdots, b_n)$ then there is an automorphism f such that $f(a_i) = b_i$ (for $1 \leq i \leq n$) and therefore we see that $b_1, \cdots, b_n$ generate $\mathcal{M}$, thus we have (2).

(2)⇒(1). Let $a_1, \cdots, a_n$ in $\mathcal{M}$ such that $\mathcal{M} \models \theta(a_1, \cdots, a_n)$. Then by Theorem 4.21 there exists a sentence $\phi(c_1, \cdots, c_n)$ in $L(C)$ such that $\mathcal{M} \models \phi(a_1, \cdots, a_n)$, and if $b_1, \cdots, b_n$ generate $\mathcal{M}$ such that $\mathcal{M} \models \phi(b_1, \cdots, b_n)$ then the function defined by $f(a_i) = b_i$ (for $1 \leq i \leq n$) extends to an automorphism.

Let $\mathcal{O}_m$ be an orbit and $(t_1(\overline{a}), \cdots, t_m(\overline{a})) \in \mathcal{O}_m$. Let us show that $\mathcal{O}_m$ is defined by the following formula:

$$\psi(y_1, \cdots, y_m) = \exists z_1, \cdots, \exists z_n(\phi(z_1, \cdots, z_n) \wedge \theta(z_1, \cdots, z_n) \wedge \bigwedge_{i=1}^{i=m} y_i = t_i(\overline{z}))$$

Let $b_1, \cdots, b_m \in \mathcal{O}_m$, then there is an automorphism f such that $f(t_i(\overline{a})) = b_i$ for $1 \leq i \leq m$. Therefore we have $t_i(f(a_1), \cdots, f(a_n)) = b_i$ (for $1 \leq i \leq n$), and $\mathcal{M} \models \phi(f(a_1), \cdots, f(a_n)) \wedge \theta(f(a_1), \cdots, f(a_n))$, thus $\mathcal{M} \models \psi(b_1, \cdots, b_m)$.

Now let $b_1, \cdots, b_m \in \mathcal{M}$ such that $\mathcal{M} \models \psi(b_1, \cdots, b_m)$. Therefore there is tuple $d_1, \cdots, d_n$ in $\mathcal{M}$ such that $\mathcal{M} \models \phi(d_1, \cdots, d_n) \wedge \theta(d_1, \cdots, d_n) \wedge \bigwedge_{i=1}^{i=m} b_i = t_i(\overline{d}))$. Hence $d_1, \cdots, d_m$ generate $\mathcal{M}$ and since $\mathcal{M} \models \phi(d_1, \cdots, d_n)$ there is an automorphism f such that $f(a_i) = d_i$ (for $1 \leq i \leq n$). Therefore $f(t_i(\overline{a})) = t_i(\overline{d}) = b_i$ (for $1 \leq i \leq n$), thus $(b_1, \cdots, b_m) \in \mathcal{O}_m$. □

Definition 4.27 *A theory T is said to be n-**categorical** if $\alpha_n(T) = 1$.*

Examples

(1) Let F_2 be the free non-abelian group on two generator. Then it is not difficult to see that $Th_{\forall\exists}(F_2)$ is 2-categorical.

(2) For every n, $Th(\mathbb{Z}^n)$ is m-categorical for every m.

Corollary 4.28 *Let T be a complete theory such that $\alpha_n(T) \leq \aleph_0$. Suppose that there exists a formula $\theta(x_1, \cdots, x_n)$ such that: for every n-generated model $\mathcal{M}$ of T and for every $b_1, \cdots, b_n$ in $\mathcal{M}$ if $\mathcal{M} \models \theta(b_1, \cdots, b_n)$ then $b_1, \cdots, b_n$ generate $\mathcal{M}$. Then T is n-categorical and his unique n-generated model is prime.* □

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