CSA Existentially closed groups
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Abstract
We study the CSA existentially closed groups. We prove in particular that every CSA e.c. group is simple. Also that every countable CSA∗ group G is embeddable in finitely generated CSA∗ group K such that G and K have the same maximal abelian subgroups which are not infinite cyclic. This imply in particular that there exists $2^{\aleph_0}$ CSA∗ e.c. groups.

1 Introduction

We define a subgroup $H$ of a group $G$ to be malnormal if for every $x$ in $G$, $H \cap H^x \neq e$ implies that $x \in H$. We call a group $G$ a CSA group if every maximal abelian subgroup of $G$ is malnormal. A group $G$ is called CSA∗ if it is CSA group without element of order two. Being CSA group is equivalent to the fact that the centralizer $C_G(g)$ of every non trivial element is abelian and malnormal.

Proposition 1.1 Let $G$ be a CSA∗ group and $G^* = < G, t \mid A^t = B >$ an HNN-extension of $G$. Then $G^*$ is a CSA∗ group if and only if for every $a \in A$, $C_{G^*}(a)$ is abelian and malnormal.

We will use the following theorem

Theorem 1.2 [1] Let $C$ be a subgroup of the HNN extension $G^* = < G, t \mid A^t = B >$ and let $g$ be a non trivial element of $Z(C)$. If $g$ is not in some conjugate of $A$ then $C$ is conjugate to some HNN extension of the form $< A', t \mid A'^n = A' >$ where $A'$ is a subgroup of $A$, or $C$ is in some conjugate of $G$.

Proof It is clear that the condition enounced in the proposition is necessary. Let us show that it is sufficient. We are going to prove that for every non trivial element $g$ of $G^*$, $C_{G^*}(g)$ is abelian and malnormal. By the above theorem there are three cases to consider:

(i) $g$ is in some conjugate of $A$.
(ii) $C_{G^*}(g)$ is in some conjugate of $G$.
(iii) $C_{G^*}(g)$ is conjugate to some HNN extension of the form $< A', s \mid A'^n = A' >$ where $A'$ is a subgroup of $A$.

In the case (i) we have $g^r \in A$ and hence we have $C_{G^*}(g) = C_{G^*}(g^r)^{r^{-1}}$. By hypothesis $C_{G^*}(g^r)$ is abelian and malnormal. Hence $C_{G^*}(g)$ is abelian and malnormal.

In the case (ii), $C_{G^*}(g)^x$ is in $G$. But $C_{G^*}(g)^x = C_{G^*}(g^x)$. Hence it is sufficient to prove that if $g \in G$ then $C_{G^*}(g)$ is abelian and malnormal. It is also sufficient to suppose that $g$ is not in some conjugate of $A$. Let us prove that $C_{G^*}(g) \subseteq G$. Let $x \in G^*$ such that $g^x = g$ and $x = g_0 t^{\varepsilon_1}...t^{\varepsilon_n} g_n$ in normal form where $n \geq 1$. Then we have:

$$g_n^{-1} t^{-\varepsilon_n}...t^{-\varepsilon_1} g_0^{-1} g g_0 t^{\varepsilon_1}...t^{\varepsilon_n} g_n g^{-1} = e$$
Hence $g_0^{-1}g_0 \in A$ and $\varepsilon_1 = 1$ or $g_0^{-1}g_0 \in B$ and $\varepsilon_1 = -1$. If $g_0^{-1}g_0 \in A$ then $g$ is in some conjugate of $A$ which is a contradiction. If $g_0^{-1}g_0 \in B$ then $g_0^{-1}g_0 = t^{-1}at$ for some $a \in A$. Hence $g$ is in some conjugate of $A$, which is a contradiction. Therefore $C_G(-g) \subseteq G$. Since $G$ is CSA group $C_G(-g)$ is abelian. Let us show that $C_G(-g)$ is a normal subgroup of $G$. Choose $x \in G$ such that $g^x = g$ where $g',g'' \in C_G(-g)$ and $x = g_0t^\varepsilon_1...t^\varepsilon_n g_n$ in normal form where $n \geq 1$. Then we have:

$$g_n^{-1}t^{-\varepsilon_n}...t^{-\varepsilon_1}g_0^{-1}g_0t^\varepsilon_1...t^\varepsilon_n g_n g''^{-1} = e$$

Hence $g_0^{-1}g_0 \in A$ and $\varepsilon_1 = 1$ or $g_0^{-1}g_0 \in B$ and $\varepsilon_1 = -1$. By the same method as the one used before we conclude that $g'$ is in some conjugate of $A$. Hence by (i) $C_G(-g')$ is abelian and normal. Since $g',g'' \in C_G(-g)$ and $C_G(-g)$ is abelian, $g'' \in C_G(-g')$. But $C_G(-g')$ is normal hence $x \in C_G(-g')$. Since $C_G(-g')$ is abelian and $g \in C_G(-g')$ we have $x \in C_G(-g) \subseteq G$ which is a contradiction. Therefore if $x \in G^*$ such that $g^x = g'$ where $g',g'' \in C_G(-g)$, then $x \in G$. But in this case we see that $x \in C_G(-g)$ because $G$ is a CSA group.

In the case (iii), after conjugation we can suppose that $C_G(-g) = < A', s | A'^n = A' >$. Since $g \in Z(G)$ then $g \in A'$ or $g = s^{\pm 1}$. Hence we can suppose that $g = s^{\pm 1}$ and in this case we have for every $a \in A'$, $a^g = a$. Suppose first that $A' \neq e$ and let $a \in A'$ non trivial. Therefore $g \in C_G(-a)$. But $C_G(-a)$ is normal, therefore $C_G(-g) \subseteq C_G(-a)$. Since $C_G(-a)$ is abelian we have $C_G(-g) = C_G(-a)$. Therefore $C_G(-g)$ is abelian and normal. Now suppose that $C_G(-g) = < g >$. Let us show that $C_G(-g)$ is normal. We can write $g = a^{-1}h^m$ where $h$ cyclically reduced. Now if $h$ is in $G$ then by the above method we have the result. Hence we can suppose that $|h| \geq 1$. Let $x$ such that $x^{-1}h^nx = h^m$. Since $h$ is cyclically reduced then $h^n$ and $h^m$ are cyclically reduced and therefore by the Conjugacy Theorem of HNN Extensions $n = \pm m$. Hence $x^2 \in C_G(-h^n)$. Let $A = < h^n, x >$ then $x^2 \in Z(A)$.

**Lemma 1.3** Let $G^* = < G, t | A' = B >$ be CSA* HNN-extension of $G$. Then for every $g^* \in G^*$ $C_G(-g^*)$ is infinite cyclic or it is conjugate to $C_G(-g)$ for some $g$ in $G$.

**Proof.** The proof is analogue to the proof of the previous proposition. By the same theorem we have three cases to consider:

(i) $g^*$ is in some conjugate of $A$.

(ii) $C_G(-g^*)$ is in some conjugate of $G$.

(iii) $C_G(-g^*)$ is conjugate to some HNN extension of the form $< A', s | A'^n = A' >$ where $A'$ is a subgroup of $A$.

The cases (i) and (ii) are clear and included in the proof of the above proposition. In the case (iii) we have seen in the proof of the proposition, that if $C_G(-g^*) = < A', s | A'^n = A' >^x$, then $C_G(-g^*) = C_G(-a)^x$, for some $a \in A$, or $C_G(-g^*) = < s >^x$. This complete the proof.

**Definition** An HNN extension $G^* = < G, t | A' = B >$ is called separated if for every $x \in G$ $A \cap B^x = e$.

We have the following theorem which can be found in [2] and which can be proved from proposition 1.1.
Theorem 1.4 Let $G^* = < G, t | A^t = B >$ be a separated HNN-extension of $G$ where $A$ and $B$ are malnormal and $G$ is CSA*. Then $G^*$ is CSA*.

Lemma 1.5 Let $G^* = < G, t | A^t = B >$ be a separated HNN-extension of $G$ where $A$ and $B$ are malnormal. Then for every $g \in G$, $C_{G^*}(g) \subseteq G$.

Proof. Begin by treating the case where $g \in A$. Let $x \in C_{G^*}(g)$ such that $|x| \geq 1$ and $x = g_0t^{\varepsilon_1}...t^{\varepsilon_n}g_n$ in normal form where $n \geq 1$. Then we have:

$$g_n^{-1}t^{\varepsilon_n}...t^{\varepsilon_1}g_0^{-1}g_0t^{\varepsilon_1}...t^{\varepsilon_n}g_ng^{-1} = e$$

Hence $g_0^{-1}g_0 \in A$ and $\varepsilon_1 = 1$ or $g_0^{-1}g_0 \in B$ and $\varepsilon_1 = -1$. If $g_0^{-1}g_0 \in A$ then $g_0 \in A$. Now if $n = 1$ then $g_1^{-1}\varphi(g_0^{-1}g_0)g_1 = g$, which is a contradiction. Now if $n \geq 2$ then $g_1^{-1}\varphi(g_0^{-1}g_0)g_1 \in A$ or $g_1^{-1}\varphi(g_0^{-1}g_0)g_1 \in B$. The first possibility is impossible. Now if $n \geq 2$ and $g_1^{-1}\varphi(g_0^{-1}g_0)g_1 \in B$ implies that $g_1 \in B$. Hence the sequence $(t, g_1, t^{-1})$ is not reduced which is a contradiction. Therefore $C_{G^*}(g) \subseteq G$. Now the case where $g \in B$ can be treated similarly. Treat the case where $g$ is not conjugate, in $G$, to some element in $A$ and is not conjugate, in $G$, to some element in $B$. We are going to prove that $C_{G^*}(g) \subseteq G^*$. Let $x \in G^*$ such that $g^x = g$ and $x = g_0t^{\varepsilon_1}...t^{\varepsilon_n}g_n$ in normal form where $n \geq 1$. Then we have:

$$g_n^{-1}t^{\varepsilon_n}...t^{\varepsilon_1}g_0^{-1}g_0t^{\varepsilon_1}...t^{\varepsilon_n}g_ng^{-1} = e$$

Hence $g_0^{-1}g_0 \in A$ and $\varepsilon_1 = 1$ or $g_0^{-1}g_0 \in B$ and $\varepsilon_1 = -1$. If $g_0^{-1}g_0 \in A$ then $g$ is conjugate, in $G$, to some element of $A$ which is a contradiction. If $g_0^{-1}g_0 \in B$ then $g$ is conjugate, in $G$, to some element of $B$ which is a contradiction. Now the case where $g$ is conjugate, in $G$, to some element of $A$ or $B$ the result is clear.

Corollary 1.6 Let $G^* = < G, t | A^t = B >$ be a separated HNN-extension of $G$ where $A$ and $B$ are malnormal. Then, up isomorphism and by excluding the infinite cyclic group, $G$ and $G^*$ have the same maximal abelian subgroups.

2 Malnormal subgroup in free product

Definition Let $F$ be a free product and $A, B$ elements of $F$. We say that $AB$ is in reduced form if $A = e$ or $B = e$ or the normal form of $AB$ can be obtained by adjusting the normal form of $A$ and the normal form of $B$. We say that $AB$ is in semi-reduced form if $A = e$ or $B = e$ or : if $(a_1, ..., a_n)$ is the normal form of $A$ and $(b_1, ..., b_n)$ is the normal form of $B$ then $(a_1, ..., (a_n, b_1), ..., b_n)$ is the normal form of $AB$. We abbreviate cyclically reduced by c.r and weakly cyclically reduced by w.c.r.

Theorem 2.1 Let $F$ be a free product. Let $x, y, z$ in $F$ such that $|y| \geq 2$, $|z| \geq 2$ and $y^z = z$. Suppose that $y$ and $z$ are c.r or $y$ and $z$ are w.c.r. Then there exists A, B in F such that $y = (AB)^n$, $z = (BA)^n$, $n \geq 1$ and $x \in \{ (AB)^p A, A(AB)^p, A \}$ or $x^{-1} \in \{ B, (BA)^p B, B(AB)^p \}$ where :

(i) If $y$ and $z$ are c.r then $AB$ and $BA$ are in reduced form.
(ii) If $y$ and $z$ are w.c.r then $AB$ and $BA$ are in semi-reduced form.

Lemma 2.2 Let $F$ be a free product and $g$ an element of $F$ such that $|g| \geq 2$. If $g$ is c.r or w.c.r then $C_F(g)$ is infinite cyclic.
Proof Let $x$ in $C_F(g)$. Then $g \in Z(C_F(g))$. By the Kurosh subgroup theorem $C_F(g)$ is a free product $*A_i$ where $A_i$ is infinite cyclic or $A_i = C_F(g) \cap G_i^*$ for some factor $G_i$. Since $g \in Z(C_F(g))$ we see that we must have $C_F(g) = A_i$ for some $i$. But $C_F(g)$ can not be of the form $A_i = C_F(g) \cap G_i^*$ because $G_i^*$ does not contains any w.c.r or c.r element $y$ such that $|y| \geq 2$. Therefore $C_F(g)$ is infinite cyclic.

Proof of the theorem We treat first the case when $y$ and $z$ are c.r. Let $x = x_1...x_p$, $y = y_1...y_n$, $z = z_1...z_m$ in normal form. We are going to prove the theorem by induction on $p$.

(1) For $p = 1$. Then $x^{-1}y_1...y_nx = z$. Now since $y$ and $z$ are c.r we must have $x^{-1}y_1 = e$ or $y_nx = e$. If $x^{-1}y_1 = e$ then $y_n$ and $x_1$ are in different factor and we have : $y_2...y_nx = z_1...z_m$ in reduced form. By putting $B = y_2...y_n$ and $x = A$ we have $y = AB$, $z = BA$ and $A = x$. Now if $y_nx = e$ then $x^{-1}y_1...y_{n-1} = z_1...z_m$. By putting $A = y_1...y_{n-1}$ and $B = y_n$ we have $y = AB$, $z = BA$ and $x^{-1} = B$. Hence we have the result.

(2) We go from $p$ to $p + 1$. We have $x_{p+1}^{-1}...x_1^{-1}y_1...y_nx_1...x_{p+1} = z$. Since $y$ and $z$ are c.r, we have $x_{p+1}^{-1}y_1 = e$ or $y_nx_1 = e$. We treat first the case $x_{p+1}^{-1}y_1 = e$.

Case (I) $x_1^{-1}y_1 = e$. Hence we have : $x_{p+1}^{-1}...x_2^{-1}y_2...y_nx_2...x_{p+1} = z$. Put $x' = x_2...x_{p+1}$, $y' = y_2...y_ny_1$, then $y'$ is c.r and by induction hypothesis there exists $A_1$ and $B_1$ such that $y' = (A_1B_1)^n$, $z = (B_1A_1)^n$.

subcase (i) $A_1 \neq e$ or $B_1 \neq e$. Then we have $y' = z$ and $x'$ commute with $y'$. By lemma , there exists a c.r. such that $y' = a^r$, $r > 0$ and $x' = a'$.If $r = 1$ then we put $A = y_1$ and $B = y_2...y_n$ then we have $y = AB$ and $z = BA$ and we can write $x = x_1x' = y_1(y_2...y_ny_1)^s = A(AB)^s$ and we have the result. Now if $r > 1$ we can write $a = a'$. Therefore we have : $y = y_1...y_n = y_1a^{r-1}a' = y_1(a'y_1)^{r-1}a' = (y_1a'y_1)^r$, $z = y_2...y_ny_1 = (a'y_1)^r$ and $x = x_1x' = y_1(a'y_1)^r$. We put $A = y_1$ and $B = a'$ then $y = (AB)^r$, $z = (BA)^r$ and $x = A(AB)^s$, hence we have the result.

subcase (ii) $A_1 \neq e$ and $B_1 \neq e$. Since $y' = (A_1B_1)^n$ we can write $B_1 = B'y_1$ in reduced form. Therefore $y = y_1(A_1B'y_1)^{n-1}A_1B' = (y_1A_1B')^n$, always in reduced form, and $z = (B_1A_1)^n = (B'y_1A_1)^n$. Put $A = A_1B'y_1$ and $B = B'$. Then $y = (AB)^s$ and $z = (BA)^s$. Now if $x = (A_1B_1)^pA_1$ and $p > 0$ then $x = x_1x' = y_1(A_1B_1)^pA_1 = y_1(A_1B_1)^pA_1 = (A_1B_1)^pA_1 = (AB)^pA$ and if $p < 0$ then $x = A_1^{-1}(A_1B_1)^{-p}y_1 = (A_1B_1)^{p-1}B_1y_1 = (B'y_1A_1)^{p-1}B_1 = (BA)^{-p}B$ and we have the result. Now if $x' = A_1(B_1A_1)^p$ and $p > 0$ then $x = x_1x' = y_1(A_1B_1)^p = y_1(A_1B')y_1A_1 = (AB)^pA$ and if $p < 0$ then $x = A_1^{-1}(A_1B_1)^{-p}y_1 = (A_1B_1)^{p-1}A_1^{-1}y_1 = (A_1B'y_1)^{p-1}A_1 = (BA)^{-p}B$ and we have the result. Now if $x' = A_1$ then $x = y_1x' = y_1A_1 = A$ and we have the result. If $x'^{-1} = B_1$ then $x'^{-1}x_1y_1 = B_1y_1^{-1} = B^+ = B'$ and we have the result. If $x'^{-1} = (A_1B_1)^pB_1$ and $p > 0$ then $x'^{-1}y_1^{-1} = (A_1B_1)^pB_1y_1^{-1} = (B'y_1A_1)^pB' = (BA)^pB$ and if $p < 0$ then $x = y_1x' = y_1B_1^{-1}(B_1A_1)^{-p} = y_1(A_1B')y_1^{-1}B_1^{-1} = (A_1B'y_1)^{-p-1}y_1A_1 = (AB)^{-p}A$ and we have the result. If $x'^{-1} = B_1(A_1B_1)^p$ and $p > 0$ then $x'^{-1}y_1^{-1} = (A_1B_1)^pB_1y_1^{-1} = (B'y_1A_1)^pB' = (BA)^pB$ and if $p < 0$ then $x = y_1x' = y_1(A_1B_1)^pB_1^{-1} = y_1A_1(B'y_1A_1)^{-p} = A(AB)^{-p}A$ we have the result.

Case (II) $y_nx_1 = e$. By taking the inverses elements we have $x_{p+1}^{-1}...x_2^{-1}y_n^{-1}...y_1^{-1}x_2...x_{p+1} = z^{-1}$. Therefore, by the case (i), there exists $A_1B_1$ such that $y_1^{-1} = (A_1B_1)^n$, $z^{-1} = (B_1A_1)^n$ and $x = (A_1B_1)^pA_1$ or $x = A_1(B_1A_1)^p$ or $x = A_1$ or $x^{-1} = B_1$ or $x^{-1} = (B_1A_1)^pB_1$ or $x^{-1} = B_1(A_1B_1)^p$. Now by taking $A = B_1^{-1}$ and $B = A_1^{-1}$ we have $y = (AB)^n$, $z = (BA)^n$ and we see that we have if $x = (A_1B_1)^pA_1$ then $x^{-1} = B(AB)^p$, if $x = A_1(B_1A_1)^p$ then $x^{-1} = (BA)^pB$, if $x = A_1$ then $x^{-1} = B(AB)^p$, if $x = A_1$ then $x^{-1} = B(AB)^p$, if $x = A_1$ then $x^{-1} = B(AB)^p$, if $x = A_1$ then $x^{-1} = B(AB)^p$, if $x = A_1$ then $x^{-1} = B(AB)^p$.
if $x = A_1$ then $x^{-1} = B$, if $x^{-1} = B_1$ then $x = A$, if $x^{-1} = (B_1 A_1)^p B_1$ then $x = A B A^p$, if $x^{-1} = B_1 (A_1 B_1)^p$ then $x = (A B)^p$. Therefore we have the result.

Now we are going to treat the case where $y$ and $z$ are w.c.r. Let $x = x_1 ... x_p$, $y = y_1 ... y_n$, $z = z_1 ... z_m$ in normal form. Then $y' = y_1^{-1} y_1 = y_2 ... (y_n y_1)$, $z' = z_1^{-1} z_1 = z_2 ... (z_m z_1)$ are c.r. and : $z_1^{-1} x_1^{-1} (y_1 y') z_1 = z'$. Therefore there exists $A$ and $B$ such that $y' = (A B)^n$ and $z' = (A B)^n$ which satisfies the conditions of the theorem. Therefore :

**subcase (i)** $A_1 = e$ or $B_1 = e$. Then we have $y' = z'$ and $x' = y_1^{-1} x z_1$ commute with $y'$.

By lemma , there exists a w.c.r. such that $y' = a^r$, $r > 0$ and $x' = a^s$. (Remark that we have $y_n y_1 = z_m z_1$ and $y_2 ... y_n - 1 = z_2 ... z_m - 1$)

If $r = 1$ then we put $A = y_1 y_2 ... y_n - 1 z_m$ and $B = z_1 y_1^{-1}$ then we have :

$$y = y_1 ... y_n - 1 z_m (z_1 y_1^{-1}) = AB$$ and $z = (z_1 y_1^{-1}) (y_1 (z_2 ... z_m - 1)) z_m = BA$

We see that $y = AB$ $z = BA$ are in semi-reduced form. We have $x = y_1 x' z_1^{-1} = y_1 (y_2 ... y_n y_1)^s z_1^{-1}$.

If $s > 1$ then we have :

$$x = y_1 x' z_1^{-1} = y_1 (y_2 ... y_n y_1)^s z_1^{-1} = y_1 ((y_2 ... y_n - 1 z_m z_1 y_1^{-1}) y_1)^s z_1^{-1}$$

$$= (y_1 y_2 ... y_n - 1 z_m z_1 y_1^{-1}) (y_1 y_2 ... y_n - 1 z_m) z_1^{-1} = (AB)^{s-1} A$$

(We see easily that if $s = 1$ then $y = A$). Therefore we have the result.

If $s < 0$ then we have :

$$x^{-1} = z_1 x'^{-1} y_1^{-1} = z_1 (y_2 ... y_n y_1)^{-s} y_1^{-1} = z_1 y_1^{-1} y_1 (y_2 ... y_n y_1)^{-s} y_1^{-1}$$

$$= z_1 y_1^{-1} (y_2 ... y_n)^{-s} y_1^{-1} = z_1 y_1^{-1} (y_1 y_2 ... y_n - 1 z_m z_1 y_1^{-1})^{-s} = B (A B)^{-s}$$

Therefore we have the result.

Now we treat the case $r > 1$. We can write $a = a' y_n y_1$. Put $A = y_1 a' z_m$ and $B = z_1 y_1^{-1}$ then we have :

$$y = y_1 a'^{-1} a' y_n = y_1 (a' y_n y_1)^{r-1} a' y_n = y_1 (a' z_m z_1 y_1^{-1} y_1)^{r-1} a' y_n = y_1 (a' z_m z_1 y_1^{-1} y_1)^{r-1} y_1 a' y_n$$

but $y_1 a' y_n = y_1 a' z_m z_1 y_1^{-1}$, hence $y = (A B)^r$

We have : $z = z_1 (z_2 ... z_m - 1) z_m = z_1 (a'^{-1} a') z_m = z_1 (a' y_n y_1)^{r-1} a' z_m = z_1 (a' z_m z_1)^{r-1} a' z_m = (z_1 a' z_m)^{r-1} z_1 a' z_m$

but $z_1 a' z_m = z_1 y_1^{-1} y_1 a' z_m = BA$, hence $z = (A B)^r$

We see that $y = AB$ $z = BA$ are in semi-reduced form. We have $x = y_1 x' z_1^{-1} = y_1 a^s z_1^{-1}$. If $s > 0$ then we have :

$$x = y_1 x' z_1^{-1} = y_1 (a' y_n y_1)^s z_1^{-1} = (y_1 a' y_n)^s y_1 z_1^{-1}$$

Hence $x^{-1} = z_1 y_1^{-1} (y_1 a' y_n)^{-s}$. but $y_1 a' y_n = y_1 a' z_m z_1 y_1^{-1} = AB$, therefore $x^{-1} = B (A B)^{-s}$, and we have the result.

If $s < 0$ then we have :

$$x^{-1} = z_1 x'^{-1} y_1^{-1} = z_1 y_1^{-1} y_1 (a' y_n y_1)^{-s} y_1^{-1} = z_1 y_1^{-1} (y_1 a' y_n)^{-s} y_1^{-1} = B (A B)^{-s}$$

Therefore we have the result.

**subcase (ii)** $A_1 \neq e$ and $B_1 \neq e$. Since $y' = (A_1 B_1)^n$ we can write $B_1 = B(1) y_n y_1$ in reduced form, and $A_1 = A' z_m z_1$. Put $A = y_1 A' z_m$ and $B = z_1 B' y_n$. Then
\[ y = y_{1}(A_{1}B'y_{1})^{n-1}A_{1}B'y_{n} = (y_{1}A_{1}B'y_{1})^{n-1}y_{1}A_{1}B'y_{n} = (y_{1}A'z_{m}z_{1}B'y_{n})^{n-1}A'z_{m}z_{1}B'y_{n} = (y_{1}A'z_{m}z_{1}B'y_{n})^{n} = (AB)^{n}. \]

always in semi-reduced form. We have also:

\[ z = z_{1}(B'y_{n}y_{1}A_{1})^{n-1}B'A'z_{m} = z_{1}(B'y_{n}y_{1}A'z_{m}z_{1})^{n-1}B'y_{n}y_{1}A'z_{m} = (z_{1}B'y_{n}y_{1}A'z_{m})^{n-1}z_{1}B'y_{n}y_{1}A'z_{m} = (z_{1}B'y_{n}y_{1}A'z_{m})^{n} = (BA)^{n}. \]

Now if \( x' = (A_{1}B_{1})^{p}A_{1} \) and \( p > 0 \) then:

\[ x = y_{1}x'z_{1}^{-1} = y_{1}(A_{1}B_{1})^{p}A_{1}z_{1}^{-1} = y_{1}(A'z_{m}z_{1}B'y_{n}y_{1})^{p}A'z_{m}z_{1}^{-1} = (y_{1}A'z_{m}z_{1}B'y_{n})^{p}y_{1}A'z_{m} = (AB)^{p}A. \]

And we have the result. The case \( x' = (A_{1}B_{1})^{p}A_{1} \) and \( p < 0 \) can be treated similarly. Now if \( x' = A_{1} \) then \( x = y_{1}x'z_{1}^{-1} = y_{1}A'z_{m} = A \) and we have the result. If \( x'^{-1} = B_{1} \) then \( x^{-1} = z_{1}x'^{-1}y_{1}^{-1} = z_{1}B'y_{n}y_{1}^{-1} = B \) and we have the result. If \( x'^{-1} = (B_{1}A_{1})^{p}B_{1} \) and \( p > 0 \) then:

\[ x^{-1} = z_{1}x'^{-1}y_{1}^{-1} = z_{1}(B'y_{n}y_{1}A'z_{m}z_{1})^{p}B'y_{n}y_{1}^{-1} = (z_{1}B'y_{n}y_{1}A'z_{m})^{p}z_{1}B'y_{n} = (BA)^{p}B \]

and we have the result. If \( x'^{-1} = B_{1}(A_{1}B_{1})^{p} \) and \( p > 0 \) then:

\[ x^{-1} = z_{1}x'^{-1}y_{1}^{-1} = z_{1}B'y_{n}y_{1}(A'z_{m}z_{1}B'y_{n}y_{1})^{p}y_{1}^{-1} = z_{1}B'y_{n}(y_{1}A'z_{m}z_{1}B'y_{n})^{p} = B(AB)^{p} \]

The cases where \( p < 0 \) can be treated similarly. Therefore we have the result.

**Theorem 2.3** Let \( G \) be a group. Let \( X \) be the set of odd number in \( \omega \). Let \( \{g_{i}| i \in X \} \) be an infinite sequence of elements of \( G \) such that \( g_{i}^{-1} \neq g_{j} \), \( g_{i}^{-1} \neq g_{k}^{-1} \) for every tuple \( i,j,k \) in \( X \). Let \( F = G < a, b | > \) and \( K \) be the subgroup generated by \( \{g_{i}b^{-a}b^{i}| i \in X \} \). Then \( K \) is a free group with basis the set of the indicate elements and \( K \) is malnormal in \( F \).

(Remark that \( g_{i}^{-1} \neq g_{i} \) implies that \( g_{i}^{2} \neq e \).)

**Proof** It is clear that \( K \) is a free group with basis the set of indicate elements. We view, in this proof, \( F \) as the free product \( G* < a | > * < b | > \) and we consider normal form relative to this free product decomposition. Let us show that \( K \) is malnormal. Before this let us prove the following claim:

**Claim 1.** Let \( y \) in \( K \) such that \( y = A^{n}, n > 1 \) then:

(i) If \( y \) is c.r and \( AA \) is in reduced form then \( A \in K \)

(ii) If \( y \) is w.c.r and \( AA \) is in semi-reduced form then \( A \in K \)

**Proof.** Let \( y = y_{1}...y_{n} \) in normal form, and \( y = (g_{1}]b^{-a}b^{i})^{a_{1}}...g_{p}b^{-a}b^{r})^{a_{p}}. \) Then we see that \( y_{1} = g_{i_{1}} \) or \( y_{1} = b^{-a_{1}} \), and \( y_{n} = g_{r}^{-1} \) or \( y_{n} = b^{a_{r}}. \)

Proof of (i).

If \( y_{1} = g_{i_{1}} \), then since \( y \) is c.r we must have \( y_{n} = b^{a_{r}}. \) Therefore the first element of the normal form of \( A \) is \( g_{i_{1}} \), and the last element of the normal of \( A \) is \( b^{a_{r}}. \) Since \( AA \) is in reduced
form there exist \( i \) such that \( i < n, y_i = b^{i^p} \). Now the only element of the normal form of \( y \), which are in \( < b > \) are : \( b^i, b^{-i}, b^i b^{-i+1} \). Since \( b \) is of infinite order and \( i_i - i_j \) is even and \( i_k \) is odd (because \( i_i - i_j \) is even and \( i_k \) is odd), we must have \( y_i = b^{i^p} = b^i \) for some \( k \). Therefore \( A = y_1 ... y_i = (g_{i_1} b^{i_1} a b^{i_1})^{\gamma_1} ... (g_{i_k} b^{i_k} a b^{i_k})^{\gamma_k} \) where \( \beta \leq \alpha_k \). Hence \( A \) is in \( K \).

If \( y_i = b^{-i_1} \), then since \( y \) is c.r. we must have \( y_n = g_{i_1}^{-1} \). Therefore the first element of the normal form of \( A \) is \( b^{-i_1} \), and the last element of the normal of \( A \) is \( g_{i_1}^{-1} \). Since \( AA \) is in reduced form there exist \( i \) such that \( i < n, y_i = g_{i_1}^{-1} \) and \( A = y_1 ... y_i \). Now the only element of the normal form of \( y \), which are in \( G \) are : \( g_{i_k}^{-1} g_{i_k}^{-1} g_{i_k+1} \). Now by the conditions imposed on \( g_i \) we must have \( y_i = g_{i_1}^{-1} = g_{i_k}^{-1} \) for some \( k \). Therefore \( A = y_1 ... y_i = (g_{i_1} b^{i_1} b^{i_1})^{\gamma_1} ... (g_{i_k} b^{i_k} b^{i_k})^{\gamma_k} \) where \( \beta \geq \alpha_k < 0 \). Hence \( A \) is in \( K \). This complete the proof of (i).

Proof of (ii).

If \( y_1 = g_{i_1} \), then since \( y \) is w.c.r. we must have \( y_n = g_{i_1}^{-1} \). Therefore the first element of the normal form of \( A \) is \( g_{i_1} \), and the last element of the normal of \( A \) is \( g_{i_1}^{-1} \). Since \( AA \) is in semi-reduced form there exist \( i \) such that \( i < n, y_i = g_{i_1}^{-1} \) and \( A = y_1 ... y_i b^{i_1} \). Now the only element of the normal form of \( y \), which are in \( G \) are : \( g_{i_1} b^{i_1} g_{i_1}^{-1} g_{i_1+1} \). Now by the conditions imposed on \( g_i \) we must have \( y_i = g_{i_1}^{-1} g_{i_1} g_{i_1+1} \) for some \( k \). We have \( A = y_1 ... y_i g_{i_1}^{-1} \) and \( y_{i-1} = b^{i_1} = b^{i_1} \) therefore \( i_k = i_p \) and \( A = y_1 ... y_i g_{i_1}^{-1} = (g_{i_1} b^{i_1} a b^{i_1})^{\gamma_1} ... (g_{i_k} b^{i_k} a b^{i_k})^{\gamma_k} \) where \( \beta \geq \alpha_k < 0 \). Hence \( A \) is in \( K \).

If \( y_1 = b^{-i_1} \), then since \( y \) is w.c.r. we must have \( y_n = b^{i_1} \). Therefore the first element of the normal form of \( A \) is \( b^{-i_1} \), and the last element of the normal of \( A \) is \( b^{i_1} \). Since \( AA \) is in semi-reduced form there exist \( i \) such that \( i < n, y_i = b^{i_1} = b^{i_1} \) and \( A = y_1 ... y_i = b^{i_1} \). Now the only element of the normal form of \( y \), which are in \( G \) are : \( b^{i_1}, b^{-i_1}, b^{i_1} b^{-i_1} \). Since \( b \) is of infinite order and \( i_i - i_j \) is even and \( i_k \) is odd (because \( i_i - i_j \) is even and \( i_k \) is odd), we must have \( y_i = b^{i_1} = b^{i_1} \) for some \( k \). Therefore \( A = y_1 ... y_i = (g_{i_1} b^{i_1} a b^{i_1})^{\gamma_1} ... (g_{i_k} b^{i_k} a b^{i_k})^{\gamma_k} \) where \( \beta \geq \alpha_k \). Hence \( A \) is in \( K \), and therefore \( B \) is in \( K \). Hence we have the result.

If \( z_1 = b^{-j_1} \), then since \( z \) is c.r. we must have \( z_n = g_{j_1}^{-1} \). Therefore the last element of the normal of \( A \) is \( g_{j_1}^{-1} \). Since \( AB \) is in reduced form there exist \( i \) such that \( i < n, y_i = g_{i_1}^{-1} \).
and $A = y_1...y_{i-1}y_i$. Now the only element of the normal form of $y$, which are in $G$ are : $g_i, g_i^{-1}, g_i^{-1}g_{i+1}$, Not by the conditions imposed on $g_i$ we must have $y_i = g_i^{-1} = g_i^{-1}$ for some $k$. Therefore $A = y_1...y_{i-1}g_i^{-1} = (g_i^{-1}b^{-i+1}ab_1^i)...(g_i^{-1}b^{-i+k}ab_k^i)^{\beta}$ where $\beta \geq \alpha_k < 0$. Hence $A$ is in $K$.

**Case(2)** $y_1 = b^{-i_1}$. Then since $y$ is c.r we must have $y_n = g_i^{-1}$. Put $y' = y^{-1}$, $z' = z^{-1}$, $A' = B^{-1}, B' = A^{-1}$ then $y', z', A', B'$ satisfies the conditions of (i) and we are in the case(1), therefore we have the result.

**Proof of (ii).**

**Case(1)** $y_1 = g_i$. Then since $y$ is w.c.r we must have $y_n = g_i^{-1}$. Therefore the first element of the normal form of $A$ is $g_i$.

If $z_1 = g_i$ then $z_n = g_i^{-1}$ and the last element of the normal of $A$ is $g_i^{-1}$. Since $AB$ is in semi-reduced form and the first element in the normal form of $B$ is $g_i$ there exist $i$ such that $i < n, y_i = g_i^{-1}g_i$ and $A = y_1...y_{i-1}g_i^{-1}$. Now the only element of the normal form of $y$, which are in $G$ are : $g_i, g_i^{-1}, g_i^{-1}g_{i+1}$. Now by the conditions imposed on $g_i$ we must have $y_i = g_i^{-1}g_i = g_i^{-1}g_{i+1}^{-1}$ for some $k$. We have $A = y_1...y_{i-1}g_i^{-1}$ and $y_{i-1} = b^{k_i} = b^q$ therefore $i_k = i_p$ and $A = y_1...y_{i-1}g_i^{-1} = (g_i^{-1}b^{-i_1}ab_i^1)...(g_i^{-1}b^{-i+k}ab_k^i)^\beta$ where $\beta \geq \alpha_k < 0$. Hence $A$ is in $K$ and therefore $B$ in $K$.

If $z_1 = b^{-i_1}$, then since $y$ is w.c.r we must have $z_n = b^{k_i}$. Therefore the last element of the normal of $A$ is $b^{k_i}$. Since $AB$ is in semi-reduced form and the first element of the normal form of $B$ is $b^{-i_1}$ there exist $i$ such that $i < n, y_i = b^{k_i}b^{-i_1}$ and $A = y_1...y_{i-1}b^{k_i}$. Now the only element of the normal form of $y$, which are in $G$ are in $B$ are : $b^{k_i}, b^{-k_i}, b^{-k_i+k+1}$. Since $b$ is of infinite order and $i_i - i_j \neq i_k$ (because $i_i - i_j$ is even and $i_k$ is odd), we must have $y_i = b^{k_i}b^{-i_1} = b^{k_i-k+1}$ for some $k$. We have $A = y_1...y_{i-1}b^{k_i}$ and $y_{i-1} = a^{k_i} = a^{k_i}$ therefore $i_k = j_q$ and $A = y_1...y_{i-1}b^{k_i} = (g_i^{-1}b^{-i_1}ab_i^1)...(g_i^{-1}b^{-i+k}ab_k^i)^\beta$ where $\beta \leq \alpha_k$. Hence $A$ is in $K$ and therefore $B$.

**Case(2)** $y_1 = b^{-i_1}$. Then since $y$ is w.c.r we must have $y_n = b^{k_i}$. Therefore the last element of the normal of $A$ is $b^{k_i}$. Since $AB$ is in semi-reduced form and the first element of the normal form of $B$ is $b^{-i_1}$ there exist $i$ such that $i < n, y_i = b^{k_i}b^{-i_1}$ and $A = y_1...y_{i-1}b^{k_i}$. Now the only element of the normal form of $y$, which are in $G$ are in $B$ are : $b^{k_i}, b^{-k_i}, b^{-k_i+k+1}$. Since $b$ is of infinite order and $i_i - i_j \neq i_k$ (because $i_i - i_j$ is even and $i_k$ is odd), we must have $y_i = b^{k_i}b^{-i_1} = b^{k_i-k+1}$ for some $k$. We have $A = y_1...y_{i-1}b^{k_i}$ and $y_{i-1} = a^{k_i} = a^{k_i}$ therefore $i_k = j_q$ and $A = y_1...y_{i-1}b^{k_i} = (g_i^{-1}b^{-i_1}ab_i^1)...(g_i^{-1}b^{-i+k}ab_k^i)^\beta$ where $\beta \leq \alpha_k$. Hence $A$ is in $K$ and therefore $B$.

This complete the proof of (ii).

Now let us show that $K$ is malnormal. Let $x \in F$, and $y, z \in K$ such that $y^x = z$. Now we can suppose that $y$ and $z$ are cyclically reduced in $K$ (relative to the basis of $K$). Indeed if $y = y^\alpha$ and $z = z^\beta$ where $y'$ and $z'$ are reduced in $K$, $\alpha \in K, \beta \in K$, then $\beta x^{-1} \alpha^{-1}y^\alpha x \beta^{-1} = z'$ and if we can prove that $\beta x^{-1} \alpha^{-1} \in A$ then $x$ is in $A$. But it is not difficult to see that if $y$ is c.r in $K$ (relative to the basis of $K$), then $y$ is c.r or w.c.r in $F$. Therefore we have only the two following cases to consider : $y$ and $z$ are c.r, and $y$ and $z$ are w.c.r. Now by the theorem , there exists $A, B$ in $F$ such that $y = (AB)^n, z = (BA)^n, n \geq 1$ and $x \in \{(AB)^p A, A(AB)^p, A\}$ or $x^{-1} \in \{B, (BA)^p B, B(AB)^p\}$ where :
(i) If $y$ and $z$ are c.r then $AB$ and $BA$ are in reduced form.
(ii) If $y$ and $z$ are w.c.r then $AB$ and $BA$ are in semi-reduced form.

Now it is sufficient to show that $A, B$ are in $K$. But by the Claim 1, $AB$ is in $K$ and $BA$ is in $K$. By Claim 2, $A, B$ are in $K$. Therefore we have the result.

### 3 Embedding in finitely generated CSA groups

**Theorem 3.1** Every countable CSA$^*$ group $C$ is embeddable in a finitely generated CSA$^*$ group $K$ such that, up isomorphism and by excluding the infinite cyclic group, $C$ and $K$ have the same maximal abelian subgroups.

**Proof.** Let $X$ be the set of odd number in $\omega$. Since $C$ is CSA$^*$ then $x^2 \neq e$ for every non trivial $x$ in $C$, therefore it is not difficult, by using axiom of choice, to find a set $\{c_i|i \in X\}$ which generate $C$ and such that $c_i^2 \neq c_j^2$, $c_i^2 \neq c_k^2$ for every tuple $i, j, k$ in $X$. Let $F = C^* < a, b, d >$. Let $D$ and $E$ be the subgroups of $F$ generated by the sets $\{d^{b^{-1}a'b'}|i \in X\}, \{c_i a^{-1}b'a'|i \in X\}$, respectifuly. Then it is clear that $D$ and $E$ are free over the indicate generator and that the function $\varphi(d^{b^{-1}a'b'}) = c_i a^{-1}b'a'$ define an isomorphism. Let $F^* = < F, t | t^{-1} d^{b^{-1}a'b'} = c_i a^{-1}b'a', i \in X >$. We are going to prove that $F^*$ has the desired properties. First we see that $F^*$ is finitely generated by $\{a, b, d, t\}$. Now it is sufficient to prove that $F^*$ is separated and that $D, E$ are malnormal, and to apply theorem and corollary.

Now by theorem $E$ is malnormal in $C^* < a, b >$, and by well-know properties of free product, $E$ is malnormal in $F$. Now it is clear that the sequence $\{d^i|i \in X\}$ satisfies the conditions of the theorem, and therefore $D$ is malnormal in $< a, b, d >$, and by well-know properties of free product, $D$ is malnormal in $F$. Let us show that $F^*$ is separated.

Let $x \in F$, and $y \in E, z \in D$ such that $y^2 = z$. Now we can suppose that $y$ is cyclically reduced in $E$(relative to the basis of $E$), and $z$ is cyclically reduced in $D$(relative to the basis of $D$). Indeed if $y = y^n$ and $z = z^m$ where $y'$ and $z'$ are reduced in $E, D$ respectifuly then $\beta x^{-1} y'^{-1} \alpha x \beta^{-1} = z'$ and if we can prove that this impossible of $y'$ and $z'$ then this prove that that is impossible for $y$ and $z$. Now it is not difficult, as in the proof of the theorem, to see that $y$ and $z$ are c.r in $F$ or that $y$ and $z$ are w.c.r in $F$. By theorem, there exist $A$ and $B$ such that $y = (AB)^n$ and $z = (BA)^n$. But we see that in normal form of $A$ or $B$ there exists at least one element which is in $C$, and which can not be cancellaed in the product $BA$. Therefore in the normal form $z$ appear an element from $C$ which is a contradiction. Therefore $F^*$ is separated and we have the desired conclusion.

**Theorem 3.2** There exists $2^{\omega}$ finitely generated CSA$^*$ groups. In particular this implies that there exists $2^{\omega}$ CSA$^*$ e.c. groups.

**Proof.** For every non empty set $X \subseteq \omega - \{1\}$ let $G(X) = \ast_{n \in X} \mathbb{Z}(n)$. Now the maximal abelian subgroups, which are not infinite cyclic, of $G(X)$ are $\mathbb{Z}(n)$, $n \in X$. By the theorem $G(X)$ is embeddable in finitely generated CSA$^*$ group such that $G(X)$ and $F(X)$ has the same maximal abelian subgroup which are not infinite cyclic. Hence the maximal abelian subgroup which are not infinite cyclic of $F(X)$ are $\mathbb{Z}(n)$, $n \in X$. Therefore if $X \neq Y$ then $F(X) \n\neq F(Y)$. Since $|\{X|X \subseteq \omega - \{1\}\}| = 2^{\omega}$ we have the result.
References


[3] Myasinkov A.G, Remeslennikov V.N., Exponential groups 2; Extensions of centralizers and tensor completion of CSA-groups.