On superstable CSA-groups

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Abstract

We prove that a non-abelian superstable CSA-group has an infinite definable simple subgroup all of whose proper definable subgroups are abelian. This imply in particular that the existence of non-abelian CSA-group of finite Morley rank is equivalent to the existence of a simple bad group all whose definable proper subgroups are abelian. We give a new proof of a result of E. Mustafin and B. Poizat [8] which states that a superstable model of the universal theory of non-abelian free groups is abelian. We deduce also that a superstable torsion-free hyperbolic group is cyclic. We close the paper by showing that an existentially closed CSA*-group is not superstable.

1 Introduction

This paper can be seen as a continuation of [6]. A subgroup H of a group G is *conjugately separated* in G, or *malnormal* in G, if $H \cap H^x = 1$ for every $x \in G \setminus H$. A *CSA-group* ("Conjugately Separated Abelian") is a group in which every maximal abelian subgroup is malnormal. An equivalent definition is that a group G is a CSA-group if and only if the centralizer of every nontrivial element of G is abelian and selfnormalizing. According to [9], a CSA-group having an involution is abelian. We denote by CSA^* the class of CSA-groups without involutions.

A *bad group* is a non-solvable connected group of finite Morley rank all of whose proper connected definable subgroups are nilpotent. Simple bad groups are hypothetical counterexamples to a well-known conjecture in model theory by G. Cherlin and B. Zil'ber which states that an infinite simple group of finite Morley rank is a linear algebraic group over an algebraically closed field.

The class of CSA-groups contains several interesting groups, but the point which interests us is that this class contains also simple bad groups of Morley rank 3 if they exist (this can be extracted from [3, Chapter 13], see also Proposition 3.3). Moreover, the existence of a non-abelian CSA-group of finite Morley rank is equivalent to the existence of a simple bad group all of whose definable proper subgroups are abelian (Proposition 3.4). So the class of CSA-groups is a good source for potential bad groups. In addition, the advantage of working in this class is that this one is axiomatizable and universal.

The work of E. Jaligot and the author in [6] was motivated by B. Poizat who asked whether existentially closed groups in the class of (torsion-free, say)

CSA*-groups were good candidates for being simple bad groups. The author showed in [6] that existentially closed CSA*-groups are not ω-stable and thus they are certainly not of finite Morley rank.

C. Berline has conjectured in [1] that an infinite superstable simple group is a linear algebraic group over an algebraically closed field and has defined a bad group to be a superstable non-solvable connected group all of whose proper connected definable subgroups are nilpotent. In view of this conjecture, the next natural step is to see if existentially closed CSA*-groups are at least superstable and this was also asked by F. Wagner. We prove in Subsection 3.2 that those groups are not superstable. The following theorem gives a relation between non-abelian CSA-groups and simple bad groups in the superstable case.

Theorem 1.1 *A non-abelian superstable CSA-group has an infinite simple definable subgroup all of whose proper definable subgroups are abelian.*

The proof, in [6], that an existentially closed CSA*-group is not ω-stable is based on the fact that its universal theory has $2^{\aleph_0}$ nonisomorphic finitely generated models [6, Corollary 8.1]. Since the universal theory of non-abelian free groups has $\aleph_0$ nonisomorphic finitely generated models and since its models are CSA-groups, it became natural to ask if it has a superstable non-abelian model. This was suggested by E. Jaligot and the author. The question was answered negatively by E. Mustafin and B. Poizat in [8]. Thanks to the preceding theorem, we obtain a new proof.

Corollary 1.2 *A superstable model of the universal theory of non-abelian free groups is abelian.*

Proof

By Theorem 1.1 it is sufficient to show that a non-abelian model of the universal theory of non-abelian free groups is not simple. By [7, Proposition 5.17, p.107], if a is a nontrivial element in a non-abelian free group, then a^2 is not in the normal closure of a. Since this last property is expressed by a collection of universal sentences, it is steal true in any nontrivial model of the universal theory of non-abelian free groups. □

Let G be a non-abelian superstable CSA-group which contains the free group F_2 of rank 2 (that seems to be a very reasonable condition) and let K be an existentially closed CSA*-group containing G. Then $Th_\forall(K) \subseteq Th_\forall(G) \subseteq Th_\forall(F_2)$. Corollary 1.2 excludes the possibility that $Th_\forall(G) = Th_\forall(F_2)$. In Theorem 3.5 we show that the case $Th_\forall(G) = Th_\forall(K)$ is excluded as well.

Since a torsion-free hyperbolic group is a CSA-group, it is natural to ask if it can be superstable. In addition, the question is also natural in the context of stability of torsion-free hyperbolic groups which is an open problem.

Corollary 1.3 *A superstable torsion-free hyperbolic group is cyclic.*

Proof

By Theorem 1.1, and since an abelian torsion-free hyperbolic group is cyclic, it is enough to show that a non-abelian torsion-free hyperbolic group G does not

contain an infinite simple group. By [10], if G is not cyclic then $\bigcap_{k\geq 1} G^k = 1$, where G^k denotes the subgroup of G generated by all kth powers of elements of G. From that one deduces easily that G is without infinite simple subgroup. □

Let us note that the proof of E. Mustafin and B. Poizat of Corollary 1.2 is strongly based on the fact that any model of the universal theory of non-abelian free groups is embeddable in $SL_2(K)$ for some algebraically closed field K. Thus it can not be adapted to the case of torsion-free hyperbolic groups as it is known that torsion-free hyperbolic groups without faithul 2-dimensional representation exist [4].

The paper is organized as follows. In the next section we record the material that we require and we prove some propositions needed in the sequel. In Section 3 we give some properties of superstable CSA-groups and we prove Theroem 1.1. In Subsection 3.2, we show that an existentially closed CSA*-group is not superstable.

2 Preliminaries

2.1 Model Theory

We assume that the reader is familiar with U-rank (Lascar rank). Our reference for superstability and U-rank questions are [2, 11, 12]. We will use exclusively the notation and conventions of [2] and thus we assume the reader familiar with that paper. The next lemma is a version of Corollary 3.6.12 in [12] which is corrected in [13]. For the reader's convenience we provide a proof. As in [2], a group $H \leq G$ is said $\bigwedge$-definable if it is the intersection of definable subgroups of G.

Lemma 2.1 [12, Corollary 3.6.12] *Let G be a superstable $|T|^+$-saturated group where $T = Th(G)$. Then every $\bigwedge$-definable simple group in G is definable.*

Proof

Let S be a simple $\bigwedge$-definable subgroup of G. We may assume that S is infinite. Since G is superstable and S is $\bigwedge$-definable, there exists a definable subgroup H containing S such that $U(S) = U(H)$ ([2, III, Lemma 4.7]). We claim that $S = H^0$ and thus S is normal in H. Since $U(S) = U(H)$, by [2, IV, Proposition 3.7] S has bounded index in H. By [2, IV, Lemma 3.10] $S^0 = H^0$ and, since S is connected as it is simple, we have $S^0 = S$ and thus $S = H^0$ as claimed.

Since S is simple, $U(S) = \omega^\alpha n$ for some ordinal α and $n \in \mathbb{N}^*$ ([2, IV, Corollary 2.8]). Since S is connected, S is α-connected. Thus, by [2, VI, Lemma 2.3], for every nontrivial $g \in S$, $g^{-1}g^S$ is α-indecomposable. By the Indecomposability Theorem [2, V, Theorem 3.1], there exist $m \in \mathbb{N}^*$ and nontrivial elements $g_1, \ldots, g_m$ in S such that $[S, S] = g_1^{-1}g_1^S \cdots g_m^{-1}g_m^S$.

Finally since S is infinite simple, $S = [S, S]$ and, since S is normal in H, S is clearly defined by the formula $\exists y_1 \cdots \exists y_m (x = g_1^{-1}g_1^{y_1} \cdots g_m^{-1}g_m^{y_m} \wedge_{1\leq i\leq m} \psi(y_i))$, where $\psi(y)$ is a formula defining H. □

Theorem 2.2 *Let G be a superstable $|T|^+$-saturated group where $T = Th(G)$. Let H be an infinite $\bigwedge$-definable subgroup without nontrivial abelian normal subgroups. Then H contains an infinite simple definable subgroup of G.*

Proof

Let H of minimal Lascar rank for which the theorem does not holds.

Set $U(H) = \omega^{\alpha_1}n_1 + \cdots + \omega^{\alpha_p}n_p$ with $n_1 > 0$. By [2, IV, Proposition 2.6], G has a definable subgroup H_0 such that $H_0 \cap H$ is normal in H and $U(H \cap H_0) = \omega^{\alpha_1}n_1$.

By [12, Lemma 3.4.2], $H_0 \cap H$ is without nontrivial abelian normal subgroups. Now $H_0 \cap H$ is an infinite $\bigwedge$-definable subgroup without nontrivial abelian normal subgroups. Suppose that $U(H_0 \cap H) < U(H)$. By the minimality of the rank of H, $H_0 \cap H$ has an infinite simple definable subgroup of G, and so H, a contradiction.

Thus, we may assume that $U(H) = \omega^{\alpha}n$ for $n > 0$. Let Γ be the α-connected component of H. Then $U(\Gamma) = \omega^{\alpha}n$ ([2, IV, Proposition 4.5]), and thus Γ is infinite. Since Γ is normal in H, by [12, Lemma 3.4.2], Γ is without nontrivial abelian normal subgroups.

We claim that Γ is not simple. Suppose towards a contradiction that Γ is simple. Then, by Lemma 2.1 and since Γ is $\bigwedge$-definable, Γ is definable. Therefore $\Gamma = \Gamma \cap H$ is simple with Γ definable, a contradiction with the choice of H. Thus Γ is not simple as claimed.

Let N be a normal nontrivial proper subgroup of Γ. Since N is normal in Γ, $[\Gamma, N] \leq N$. We also have $[\Gamma, N]$ is a normal subgroup of Γ. We claim that $[\Gamma, N]$ is an infinite $\bigwedge$-definable subgroup without nontrivial abelian normal subgroups and $U([\Gamma, N]) < U(\Gamma) = U(H)$.

Let us prove first that $[\Gamma, N]$ is $\bigwedge$-definable. Since Γ is α-connected $\bigwedge$-definable subgroup of G, for every $h \in N$, $h^{-1}h^{\Gamma}$ is α-indecomposable. By the Indecomposability Theorem [2, V, Theorem 3.1], the subgroup $[\Gamma, N]$ is $\bigwedge$-definable and α-connected.

We prove that $U([\Gamma, N]) < U(\Gamma) = U(H)$. Suppose towards a contradiction that $U([\Gamma, N]) = U(\Gamma) = U(H)$. Since $[\Gamma, N]$ is α-connected, by [2, IV, Corollary 4.6], $[\Gamma, N]$ is connected. By [2, IV, Proposition 3.7], $[\Gamma, N]$ has a bounded index in Γ. Therefore by [2, IV, Lemma 3.10], $\Gamma^{\circ} = [\Gamma, N]^{\circ}$. But since Γ and $[\Gamma, N]$ are connected, $\Gamma = [\Gamma, N] \leq N$, a contradiction with the fact that N is a proper subgroup of Γ. Therefore $U([\Gamma, N]) < U(\Gamma) = U(H)$ as claimed.

We prove now that $[\Gamma, N]$ is infinite. Suppose towards a contradiction that $[\Gamma, N]$ is finite. Since $[\Gamma, N]$ is connected, $[\Gamma, N] = 1$. Then $N \leq Z(\Gamma)$ and, since $N \neq 1$, N is an abelian nontrivial normal subgroup of Γ. A contradiction. Therefore $[\Gamma, N]$ is infinite.

Since $[\Gamma, N]$ is normal in Γ, by [12, Lemma 3.4.2], $[\Gamma, N]$ is without nontrivial abelian normal subgroups.

Thus $[\Gamma, N]$ is an infinite $\bigwedge$-definable subgroup without nontrivial abelian normal subgroups and $U([\Gamma, N]) < U(\Gamma) = U(H)$ as claimed. By the minimality of the rank of H, there exists an infinite definable simple subgroup K such that $K \leq [\Gamma, N]$. Contradiction with the choice of H. □

Corollary 2.3 *A superstable group without nontrivial abelian normal subgroups has a definable non-abelian simple subgroup.*

Proof

Let G be a superstable group and let $G \prec G_1$ be a $|T|^+$-saturated elementary extension of G, where $T = Th(G)$. If G_1 has a nontrivial abelian normal subgroup, then G_1 has a maximal one which will be definable. By elementary equivalence, G contains also a notrivial normal abelian subgroup. Therefore G_1 is without nontrivial abelian normal subgroups. Thus, by the above theorem, it contains a definable infinite simple subgroup. Again, by elementary equivalence, G contains a non-abelian simple definable subgroup, as simplicity is conserved by elementary equivalence for superstable groups. □

2.2 CSA-groups

Here are some simples facts about CSA-groups which will be used frequently.

Lemma 2.4
(1) [9, Proposition 10] *The class of CSA-groups is a universal class.*
(2) *A CSA-group containing a nontrivial normal abelian subgroup is abelian.*
(3) *A solvable-by-finite CSA-group is abelian. Thus a finite CSA-group is abelian.*
(4) *Let G be a simple infinite group all of whose proper definable subgroups are abelian. Then G is a CSA-group.*

Proof
(2) Let K be a nontrivial abelian normal subgroup of G. Then $K \leq C_G(a)$, for some nontrivial element $a \in K$. Since K is normal, for every $g \in G$, $g^{-1}Kg \leq C_G(a)$. Since $C_G(a)$ is malnormal and K nontrivial, for every $g \in G$, $g \in C_G(a)$. Thus G is abelian.
(3) By induction on the derived length of G and using (1), a solvable CSA-group G is abelian. Now we prove that a finite CSA-group is abelian. Suppose towards a contradiction that there exists a finite CSA-group which is not abelian and let G be such a group of minimal cardinality. Then every proper subgroup of G is abelian. But a finite group in which every proper subgroup is abelian, is solvable. By the above, G is abelian, a contradiction. Thus, if G is solvable-by-finite, then either G is finite or G contains a nontrivial normal abelian subgroup, hence G is abelian.
(4) Let a be a nontrivial element of G. Then $C_G(a)$ is a proper definable subgroup of G and thus it is abelian. Now $N_G(C_G(a))$ is also a definable subgroup of G which is proper as $C_G(a)$ is proper and normal in $N_G(C_G(a))$ and G is simple. Hence $N_G(C_G(a))$ is abelian and thus $C_G(a)$ is selfnormalizing. □

Corollary 2.5 *A locally-finite CSA-group is abelian. In particular an $\aleph_0$-categorical CSA-group is abelian.* □

3 Superstable CSA-groups

In this section we prove Theorem 1.1. We begin first with the following proposition.

Proposition 3.1 *Let α be an ordinal with $\alpha \geq 1$.*
(1) *A superstable CSA-group of Lascar rank ω^α or $\omega^\alpha 2$ is abelian.*
(2) *A connected non-abelian superstable CSA-group of Lascar rank $\omega^\alpha 3$ is simple.*

Proof
(1) If $U(G) = \omega^\alpha$, by [2, Corollary 1.4, p. 37] G has a definable abelian subgroup of finite index. Thus G is abelian-by-finite and by Lemma 2.4(2) it is abelian. If $U(G) = \omega^\alpha 2$, by [1, Lemma 3.1, p.53] G is solvable and by Lemma 2.4(2), G is abelian.
(2) If G contains a definable normal subgroup H with $\omega^\alpha \leq U(H) < \omega^\alpha 3$, then by [2, IV, Proposition 2.6], H contains a normal definable subgroup K such that $U(K) = \omega^\alpha$ or $U(K) = \omega^\alpha 2$. Therefore, by [1, Lemma 3.1, p.53] H/K is solvable and thus H is solvable-by-finite and by Lemma 2.4(3), H is abelian. Therefore every normal proper definable subgroup K of G satisfies $U(K) < \omega^\alpha$. By [2, VI, Proposition 2.6] G is simple. □

Proof of Theorem 1.1

Recall that we want to show that a superstable non-abelian CSA-group contains an infinite definable simple group all of whose definable proper subgroups are abelian. Let G be a non-abelian superstable CSA-group. By Lemma 2.4, G is infinite and without abelian normal subgroups. Thus by Corollary 2.3, G has a non-abelian definable simple subgroup wich is infinite by Lemma 2.4(3). Let K be of minimal Lascar rank such that K is infinite simple and definable. We claim that every proper definable subgroup of K is abelian. Indeed if $H \leq K$ is definable and non-abelian and proper then $U(H) < U(K)$ and H is infinite and without abelian normal subgroups. Thus, again by Corollary 2.3 and Lemma 2.4(3), H contains an infinite definable simple subgroup whose rank is less than $U(K)$, a contradiction. Therefore every proper definable subgroup of K is abelian as claimed. □

A property $\mathcal{P}$ of groups is said *good* if it is conserved by definable subgroups and if there is no infinite simple group satisfying it. For example 'being residually-finite', 'being abelian', 'being solvable', 'being free' are good properties. As we have seen in the proof of Corollaries 1.2 and 1.3, 'being a model of the universal theory of non-abelian free groups' and 'being a subgroup of a torsion-free hyperbolic group' are also a good properties. The following proposition is a simple consequence of Theorem 1.1.

Proposition 3.2 *A superstable CSA-group which satisfies a good property is abelian. In particular a residually-finite superstable CSA-group is abelian.* □

Question 1. Can we replace "residually-finite" by "locally-residually-finite" in Proposition 3.2 ?

3.1 CSA-groups of finite Morley rank

We begin with the following proposition.

Proposition 3.3 *A simple bad group of Morley rank 3 is a CSA-group.*

Proof

Let G be a simple bad group of Morley rank 3. As G is connected, every definable proper subgroup of G is of rank ≤ 2. By [3, Corollary 11.99], G is without definable proper subgroup of rank 2; as otherwise $G = PSL_2(K)$ for some algebraically closed field K and thus G will contains an involution contradicting [3, Theorem 13.3(iv)]. Let B be a Borel subgroup. Since G is infinite, by [3, Theorem 13.3(vi)], B is infinite. Then B is abelian as it is of rank 1 and connected. We claim that a maximal abelian subgroup of G is a Borel subgroup. Let A be a maximal abelian subgroup of G. By [3, Theorem 13.3(iv)], A is infinite. We also have $A = Z(C_G(A))$ and thus A is definable. Then $A \leq C_G(a)$ for any nontrivial $a \in A$. Therefore $A^\circ = C_G(a)^\circ$. Let B a Borel subgroup containing A°. Then $A^\circ = B$. Now if $c \in A \setminus B$, then we let C to be a Borel subgroup containing c. Then $C = C_G(c)^\circ$ and since $A = C_G(c)^\circ$ we find $A^\circ = C_G(c)^\circ = C = B$, a contradiction. Therefore $A = B$, and thus A is a Borel subgroup as claimed. Now since A^g is also a maximal abelian subgroup and thus a Borel subgroup, if $A^g \cap A \neq 1$, by [3, Theorem 13.3(ii)(v)] we get $g \in A$. Thus any maximal abelian subgroup is malnormal. □

We make a few further remarks about CSA-groups of finite Morley rank. Since a connected group of Morley rank ≤ 2 is solvable, as in Proposition 3.1, we get that a CSA-group of Morley rank ≤ 2 is abelian. Furthermore a connected non-abelian CSA-group of Morley rank 3 is without definable normal subgroups and thus it is definissably simple and therefore it is simple (and bad). By Theorem 1.1, a non-abelian CSA-group of finite Morley rank has a definable subgroup which is a simple (bad) subgroup all of whose proper definable subgroups are abelian. Thus if there exists a non-abelian CSA-group of finite Morley rank then there exists a bad group. More generally, by Theorem 1.1 and Lemma 2.4(4), we have:

Proposition 3.4 *The existence of a non-abelian CSA-group of finite Morley rank is equivalent to the existence of a simple bad group all of whose proper definable subgroups are abelian.* □

In the finite Morley rank case, the proof of the fact that a non-abelian CSA-group of finite Morley rank has a definable simple bad group is easier. Indeed, if G is a non-abelian CSA-group of finite Morley rank then by the descending chain condition it contains a non-abelian definable subgroup H all of whose proper definable subgroups are abelian. Since H is without normal abelian subgroups, H is connected and definissably simple and thus it is a simple bad group. The above proposition leaves the following question open.

Question 2. Is every simple bad group contains a simple bad CSA-group ?

3.2 Existentially closed CSA-groups

We assume the reader familiar with [6] and we use the same convention as in [6]. We restrict our attention to the subclass of CSA-groups without involutions. We debote by π the set of all prime numbers and we fix a function f from π to $\mathbb{N} \cup \{\infty\}$. We denote by CSA_f the class of CSA-groups in which $(\mathbb{Z}_p)^{f(p)+1}$ does not embed for every prime p such that $f(p) \in \mathbb{N}$. If $f(2) = 0$, we are dealing with a class of CSA-groups without involutions. Since we are mainly concerned with CSA-groups without involutions, we assume that $f(2) = 0$, i.e. we restrict our attention to subcasses of CSA^*-groups. The goal of this section is to show the following theorem.

Theorem 3.5 *A group having the same universal theory as an existentially closed CSA_f-group is not superstable.*

We collect in the next lemma some simple facts. We omit the proofs of the statements that must be part of the set theory folklore or that are easy to prove. For a group G and a nonempty subset A of G, we denote by A^{-1} the set of the inverses of elements of A.

Lemma 3.6
(1) *For every infinite cardinal μ there exists a cardinal λ such that $\lambda > \mu$ and $\lambda^{\aleph_0} > \lambda$.*
(2) *Let G be a group without involutions and A a subset of G of infinite cardinality λ and satisfying $A = A^{-1}$. Let*

$$\flat A = \{X \subseteq A : |X| = \aleph_0 \text{ and } (x, y \in X \Rightarrow x \neq y^{-1})\}.$$

Then $|\flat A| = \lambda^{\aleph_0}$.
(3) [6, Theorem 4.2] and [5, Corollary 8.2.]. *The class of CSA_f-groups is an universal class and has the joint embedding property. Furthermore for every existentially closed CSA_f-group G the universal theory of G is true in every CSA_f-group.* □

The following proposition can be extracted from the proof of [6, Theorem 1.3].

Proposition 3.7 *Let G be a CSA_f-group and $Y \subseteq G$ such that $|Y| = \aleph_0$ and $x \neq y^{-1}$ whenever $x, y \in Y$. Then G embeds into a CSA_f-group H having 4 elements a, b, c, t such that $Y = \{t^{-1}c^i b^{-i} a^i b^i t a^{-i} b^{-i} a^i \mid i \in 2\mathbb{N} + 1\}$.* □

Let G be a group, $\overline{a}$ a finite tuple of G and $A \subseteq G$. We define $\mathrm{tp}^G_{at}(\overline{a}/A)$ to be the set of atomic or negation of atomic formula $\psi(\bar{x})$ with parameters from A such that $G \models \psi(\overline{a})$. Notice that we have $\mathrm{tp}^G_{at}(\overline{a}/A \cup A^{-1}) = \mathrm{tp}^G_{at}(\overline{a}/A)$.

Proof of Theorem 3.5

Let G be a group having the same universal theory as an existentially closed CSA_f-group and let $A \subseteq G$ of infinite cardinality λ. We may assume $A = A^{-1}$. Using Proposition 3.7 and Lemma 3.6(3), G embeds into a CSA_f-group $\mathcal{K}$ having a sequence $(\overline{a}_x \mid x \in \flat A)$, $|\overline{a}_x| = 4$, such that $\mathrm{tp}^{\mathcal{K}}_{at}(\overline{a}_x/A) \neq \mathrm{tp}^{\mathcal{K}}_{at}(\overline{a}_y/A)$ for every $x \neq y$. By Lemma 3.6(4), every CSA_f-group is a model of $Th_\forall(G)$. Therefore $\mathcal{K}$ embeds in a model $\mathcal{N}$ of $Th(G)$. Since $\mathrm{tp}^{\mathcal{K}}_{at}(\overline{a}_x/A) = \mathrm{tp}^{\mathcal{N}}_{at}(\overline{a}_x/A)$ for every $x \in \flat A$ and since $\mathrm{tp}^{\mathcal{K}}_{at}(\overline{a}_x/A) \neq \mathrm{tp}^{\mathcal{K}}_{at}(\overline{a}_y/A)$ for every $x \neq y$, $\mathrm{tp}^{\mathcal{N}}_{at}(\overline{a}_x/A) \neq \mathrm{tp}^{\mathcal{N}}_{at}(\overline{a}_y/A)$ for $x \neq y$. Therefore $\mathrm{tp}^{\mathcal{N}}(\overline{a}_x/A) \neq \mathrm{tp}^{\mathcal{N}}(\overline{a}_y/A)$ for $x \neq y$ and thus $|S^{\mathcal{N}}_4(A)| \geq |\flat A|$. Now the theorem follows from the conclusion $|S^{\mathcal{N}}_4(A)| \geq \lambda^{\aleph_0} = |\flat A|$ and the appropriate choice of λ using Lemma 3.6(1). □

The above theorem leaves the following question open.

Question 3. Is an existentially closed CSA^*-group stable ?

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