NOTE ON FREE CONJUGACY PINCHED ONE-RELATOR GROUPS

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Abstract.

(*a*) Let L be a group having a presentation $L = \langle x_1, \cdots, x_n, y_1, \cdots, y_m | u = v\rangle$, where $u \in F_1 = \langle x_1, \cdots, x_n|\rangle$, $u \neq 1$, $v \in F_2 = \langle y_1, \cdots, y_m|\rangle$, $v \neq 1$. Then L is free if and only if either u is primitive in F_1 or v is primitive in F_2.

(*b*) Let L be a group having a presentation $L = \langle x_1, \cdots, x_n, t | u^t = v\rangle$, where both u and v are nontrivial in the free group F on $x_1, \cdots, x_n$. Then L is free if and only if one of the following cases holds:

(*i*) F has a basis $\{u, y_1, \cdots, y_{n-1}\}$ such that v is conjugates to $v' \in \langle y_1, \cdots, y_{n-1}\rangle$.

(*ii*) As in (*i*) with permuting u and v.

1. Introduction

Let F_1 and F_2 be free groups and let $A = \langle u\rangle$ be a nontrivial subgroup of F_1 and $B = \langle v\rangle$ be a nontrivial subgroup of F_2. We let $L = F_1 *_{A=B} F_2$ to be the free product amalgamating A and B, relatively to the natural isomorphism beetween A and B which sends u to v.

If $\{x_1, \cdots, x_n\}$ is a basis of F_1 and $\{y_1, \cdots, y_m\}$ is a basis of F_2, L has as a presentation

$$L = \langle x_1, \cdots, x_n, y_1, \cdots, y_m | u = v\rangle, \tag{1}$$

and in particular L is a one-relator group.

Recall that an element of a free group F is called *primitive* if it is in some basis of F. Clearly if u is primitive in F_1 or if v is primitive in F_2, then L is free. When u is not primitive in F_1 and v in not primitive in F_2, L is called *cyclically pinched one-relator group*. A classical examples of a such group, are surface groups without boundary of genus $n \geq 2$; that is fundamental groups of compact surfaces without boundary of genus $n \geq 2$. Such groups, in the orientable case, have as a presentation

$$G_n = \langle x_1, \cdots, x_n, y_1, \cdots, y_n | [x_1, y_1] \cdots [x_n, y_n] = 1\rangle,$$

which can be rewritten, by putting $u = [x_1, y_1] \cdots [x_{n-1}, y_{n-1}]$ and $v = [x_n, y_n]^{-1}$,

$$G_n = \langle x_1, \cdots, x_n, y_1, \cdots y_n | u = v\rangle.$$

A similar presentation can also be given in the non-orientable case.

A *conjugacy pinched one-relator group* is a group G having a presentation of the form

$$G = \langle x_1, \cdots, x_n | t^{-1} u t = v\rangle, \tag{2}$$

where u and v are nontrivial elements of the free group F on $x_1, \cdots, x_n$. Clearly such a group is an HNN-extension of the free group on $x_1, \cdots, x_n$ with cyclic subgroups generated by u and v.

Suppose that F has a basis $\{u, y_1, \cdots, y_{n-1}\}$ such that $v = a^{-1}v'a$ with $v' \in \langle y_1, \cdots, y_{n-1}\rangle$. Then G can be rewritten as

$$G = \langle u, y_1, \cdots, y_{n-1} | s^{-1}us = v'\rangle,$$

where $s = at$. Then G is free and in particular G is a cyclically pinched one-relator group.

There has been a considerable work on cyclically pinched one-relator groups and conjugacy one-relator groups and it has been showen that in general such groups have several alegbraic properties of surface groups. For more details on these groups, we refer the reader to [FRS97].

However, we have not found in the literature, a complete written proof about the necessarily and sufficient condition for which a cyclically or conjugacy pinched one-relator group is free.

The subject of this note is to give a self-contained "combinatorial" proof of the following theorem, which stats that the configurations noticed above, are the only one for which a group having a presentation as in (1) or (2) is free.

Theorem 1.1.

(*a*) *Let L be a group having a presentation*

$$L = \langle x_1, \cdots, x_n, y_1, \cdots, y_m | u = v\rangle,$$

where $u \in F_1 = \langle x_1, \cdots, x_n\rangle$, $u \neq 1$, $v \in F_2 = \langle y_1, \cdots, y_m\rangle$, $v \neq 1$.

Then L is free if and only if either u is primitive in F_1 or v is primitive in F_2.

(*b*) *Let L be a group having a presentation*

$$L = \langle x_1, \cdots, x_n, t | u^t = v\rangle,$$

where both u and v are nontrivial in the free group F on $x_1, \cdots, x_n$.

Then L is free if and only if one of the following cases holds:

(*i*) *F has a basis $\{u, y_1, \cdots, y_{n-1}\}$ such that v is conjugates to $v' \in \langle y_1, \cdots, y_{n-1}\rangle$.*

(*ii*) *As in (i) with permuting u and v.*

To prove Theorem 1.1, we need some properties of one-relator groups. Recall that a group G is called a one-relator group if it has a presentation $G = \langle x_1, \cdots, x_n | r\rangle$ where r is in the free group on $x_1, \cdots, x_n$. We have the following famous Freiheitssatz of Magnus.

Theorem 1.2. [LS77, Theorem 5.1, Ch IV] *Let $G = \langle x_1, \cdots, x_n | r\rangle$ where r is cyclically reduced. If T is subset of $\{x_1, \cdots, x_n\}$ wich omits a generator occuring in r, then the subgroup generated by T is freely generated by T.* □

We will also use the following two lemmas.

Lemma 1.3. [LS77, Proposition 5.10, Ch II] *If $G = \langle x_1, \cdots, x_n | r\rangle$ is a free group, then either $r = 1$ or r is a primitive element in the free group with basis $\{x_1, \cdots, x_n\}$.* □

Lemma 1.4. [LS77, Theorem 1.8, Ch IV] *Let F be a free group, and let φ be a homomorphism from F onto the free product $*A_i$. Then there is a factorization of F as a free product, $F = *F_i$, such that $\varphi(F_i) = A_i$.* □

We will use the following precise version of Kurosh theorem whose proof is a consequence of Bass-Serre theory. For more details, the reader is refered to [Ser80] and in particular to the proof of Theorem 14 of section 5.5 of that reference.

Lemma 1.5. *Let $L = L_1 * L_2$ be a free product and let F be a subgroup of L. Then F has a factorization*

$$F = *_{i\in I} F \cap L_1^{g_i} * \cdots *_{j\in J} F \cap L_2^{h_j} * D,\ g_i, h_j \in L,$$

with the following properties:

(1) *If $f \in F \cap L_i^g$ for some i and $g \in L$, then there exists $a \in F$ such that $f^a \in F \cap L_1^{g_i}$ for some i, or $f^a \in F \cap L_2^{h_j}$ for some j.*

(2) *D is a free group such that $D \cap L_i^g = 1$ for any i and any $g \in L$.* □

2. Proof of Theorem 1.1.

We prove (a). Let $\{x_1, \cdots, x_n\}$ be a basis of F_1 and let $\{y_1, \cdots, y_m\}$ be a basis of F_2. The group L has the presentation

$$L = \langle x_1, \cdots, x_n, y_1, \cdots, y_m | u = v\rangle. \tag{1}$$

Let

$$F_u = \langle \hat{x}_1, \cdots, \hat{x}_n | \hat{u} = 1\rangle, \quad F_v = \langle \hat{y}_1, \cdots, \hat{y}_n | \hat{v} = 1\rangle, \tag{2}$$

where $\hat{u}$ (resp. $\hat{v}$) is the word obtained from u (resp. v) by replacing each occurence of x_i (resp. y_j) by $\hat{x}_i$ (resp. $\hat{y}_j$).

We suppose that L is free and we show that either u is primitive in F_1 or v is primitive in F_2. By using Lemma 1.3, it is sufficient to prove that either F_u is free or F_v is free.

By the presentations appearing in (1) and (2), we have a homomorphism φ from L onto the free product $F_u * F_v$, such that $\varphi(x_i) = \hat{x}_i$ and $\varphi(y_j) = \hat{y}_j$ for every i and j. By Lemma 1.3, L has a factorization $L = L_1 * L_2$, such that $\varphi(L_1) = F_u$ and $\varphi(L_2) = F_v$. Now we prove the following claim.

Claim 1. One of the following cases holds:
(i) for any $g \in L$, $u^g \notin L_1$,
(ii) for any $g \in L$, $v^g \notin L_2$.

Proof. Suppose towards a contradiction that neither (i) nor (ii) holds. Therefore, there exists $g, g' \in L$ such that $u^g \in L_1$ and $v^{g'} \in L_2$.

Thus there exists $x \in L_1$, $y \in L_2$ such that

$$v = x^{g^{-1}} = y^{g'^{-1}} = u.$$

Since $v \neq 1$, we find $x \neq 1$, $y \neq 1$ and therefore we get a contradiction with well-known conjuguation properties of free products (see [LS77, Theorem 1.4, Ch IV] for more details). □

We treat the case when (i) holds, and we show that in that case F_u is free. The other case is symmetric and can be treated similarly. So we suppose that (i) holds.

Let X (resp. Y) be a basis of L_1 (resp. L_2). Since $u = v$ in L, $\ker(\varphi)$ is equal to the normal closure of u. Write u as a free reduced word in the basis $X \cup Y$. Let u' be a cyclically reduced (in the basis $X \cup Y$) conjugate of u. Then $G = L/\ker(\varphi)$ has a presentation

$$G = \langle X, Y | u'\rangle.$$

Since u' is a conjugate of u we find, by (i), that $u' \notin L_1$. Hence some element of Y occurs in u'. By Theorem 1.2, X is basis of a free group in G; in other words $\varphi(X)$ is basis of a free group in $F_u * F_v$. Since $\varphi(L_1) = \langle \varphi(X) \rangle = F_u$, we find F_u is free as required. This ends the proof of (a). □

We prove now (b). Let F be a free group of finite rank and u and v be nontrivials elements of F. Let

$$L = \langle F, t | u^t = v \rangle.$$

We suppose that L is free and we show that F satisfies either (i) or (ii). Clearly $rk(F) \geq 2$, otherwise L contains a free abelian group of rank 2, which is clearly a contradiction.

The subgroup of L generated by F and F^t is the free product amalgamating v and u^t. Thus, we get by (1) that either u is primitive in F or v is primitive in F. In the rest of the proof we assume that u is primitive in F, the other case is symmetric and can be treated similarly.

Thus F has a basis $\{x_1, \cdots, x_n, u\}$, $n \geq 1$, and

$$L = \langle x_1, \cdots, x_n, u, t | u^t = v \rangle. \tag{3}$$

Let

$$F_{u,v} = \langle \hat{x}_1, \cdots, \hat{x}_n, \hat{u} | \hat{u} = 1, \hat{v} = 1 \rangle, \quad T = \langle \hat{t} | \rangle, \tag{4}$$

where $\hat{v}$ is the word obtained from v by replacing each occurence of x_i by $\hat{x}_i$ and each occurence of u by $\hat{u}$.

By the presentations appearing in (3) and (4), we have a homomorphism φ from L onto the free product $F_{u,v} * T$, such that $\varphi(x_i) = \hat{x}_i$, $\varphi(u) = \hat{u}$, $\varphi(t) = \hat{t}$. By Lemma 1.3, L has a factorization $L = L_1 * L_2$, such that

$$\varphi(L_1) = F_{u,v}, \quad \varphi(L_2) = T. \tag{5}$$

We choose the factorization $L = L_1 * L_2$ to be such that $rk(L_1)$ is maximal among all factorizations of L satisfying (5).

Let X (resp. Y) be a basis of L_1 (resp. L_2). Write v as a reduced word on $X \cup Y$ and let v' be a cyclically reduced conjugate of v.

Claim 2. $v' \notin L_2$.

Proof. Suppose towards a contradiction that $v' \in L_2$. Clearly $\ker(\varphi)$ is the normal closure of v' and $G = \langle X, Y | v' = 1 \rangle$ is isomorphic to $F_{u,v} * T$. As in the proof of (a), by Theorem 1.1 , φ is injective on L_1 and $\varphi(L_1) = F_{u,v}$ is free. It follows that $rk(L_1) = rk(F_{u,v})$. Since $F_{u,v}$ is generated by n elements its rank is bounded by n. Using Lemma 1.3, we have $n - 1 \leq rk(F_{u,v})$. We conclude

$$n - 1 \leq rk(F_{u,v}) \leq n. \tag{6}$$

Using Lemma 1.3, we have

$$rk(L) = n + 1. \tag{7}$$

Combining (6) and (7), we find $rk(L_2) \in \{1, 2\}$.

Suppose first that $rk(L_2) = 1$ and let d generates L_2. Then $v' = d^m$ for some $m \in \mathbb{Z}$ and thus $\varphi(v') = 1 = \hat{t}^m$, which is clearly a contradiction.

Suppose now that $rk(L_2) = 2$. Since $L/\ker(\varphi)$ is free, we find by Lemma 1.3, that v' is primitive in L and thus it is also primitive in L_2. Hence L_2 has a basis $\{v', d\}$.

Let $d' \in L_2$ such that $\varphi(d') = \hat{t}$ and let $m \in \mathbb{Z}$ such that $\varphi(d) = \hat{t}^m$. We claim that $m = \pm 1$. By writting

$$d' = v'^{n_1} d^{m_2} \cdots v'^{n_p} d^{m_p},$$

we find

$$\varphi(d') = \hat{t}^{m(m_1+\cdots+m_p)} = \hat{t},$$

and thus $m = \pm 1$ as claimed.

By setting $L_1' = \langle L_1, v' \rangle = L_1 * \langle v' \rangle$, $L_2 = \langle d \rangle$, we find

$$L = L_1' * L_2', \quad \varphi(L_1') = F_{u,v}, \quad \varphi(L_2') = T,$$

and $rk(L_1') > rk(L_1)$; a contradiction with our choice of the factorization of L. This ends the proof of the claim. □

Claim 3. $v' \in L_1$.

Proof. Suppose towards a contradiction that $v' \notin L_1$. As in the proof of Claim 1, we have $G = \langle X, Y | v' = 1 \rangle$ is isomorphic to $F_{u,v} * T$, and by Theorem 1.1 we find that $\varphi(L_1)$ is free and $rk(L_1) = rk(F_{u,v})$. By Claim 1, $v' \notin L_2$. Again by Theorem 1.1 we find that $\varphi(L_2)$ is free and $rk(L_2) = rk(T)$. Now by Grushko Theorem,

$$rk(F_{u,v} * T) = rk(F_{u,v}) + rk(T),$$

and thus $rk(F_{u,v} * T) = rk(L_1) + rk(L_2) = rk(L)$, but this contradicts Lemma 1.4. □

Claim 4. For any $g \in L$, $F \cap L_2^g = 1$.

Proof. Suppose towards a contradiction that $F \cap L_2^g \neq 1$, for some $g \in L$. Let $1 \neq h \in F \cap L_2^g$, and $1 \neq d \in L_2$ such that $h = d^g$. Then

$$\varphi(h) \in F_{u,v}, \quad \varphi(h) = \varphi(g)^{-1} \hat{t}^m \varphi(g), \text{ for some } m \in \mathbb{Z}, m \neq 0.$$

Thus we find two elements $1 \neq a \in F_{u,v}$, $1 \neq b \in T$ which are conjugate; a contradiction. □

Claim 5. There is a factorization of F, $F = F_1 * F_2$, such that $u \in F_1$ and $v = b^a$ with $b \in F_2$ and $a \in F$.

Proof.

By Claim 3, $v \in L_1^{g'}$ for some $g' \in L$. Since $u^t = v$, we find also that $u \in L_1^g$ for some $g \in L$.

Therefore, by combining Claim 4 and Lemma 1.5 , F has a factorization

$$F = F \cap L_1^{g_1} * \cdots * F \cap L_1^{g_m} * H,$$

such that $u \in (F \cap L_1^{g_p})^a$ for some p and $a \in F$ and $v \in (F \cap L_1^{g_q})^b$ for some q and $b \in F$. Without loss of generality, we assume that $a = 1$ and $p = 1$.

Set $K_i = F \cap L_1^{g_i}$. We claim that $q \neq p = 1$. Suppose towards a contradiction that

$$v \in K_1^b,$$

and in particular we find, $u \in (L_1^{g_1}), v \in (L_1^{g_1})^b$ with $b \in F$. Let $x, y \in L_1$ such that

$$u = x^{g_1} \text{ and } v = y^{g_1 b}.$$

Then $v = y^{g_1 b} = x^{g_1 t}$. Hence

$$y = g_1 b t^{-1} g_1^{-1} x g_1 t b^{-1} g_1^{-1},$$

and since L_1 is malnormal we get

$$g_1 b t^{-1} g_1^{-1} \in L_1. \tag{8}$$

Set $\alpha = \varphi(g)$ and $b' = \varphi(b)$. By the definition of φ we have $b' \in F_{u,v}$.

We find in $F_{u,v} * T$ that

$$\alpha b' \hat{t}^{-1} \alpha^{-1} \in F_{u,v}. \tag{9}$$

Let $\pi : F_{u,v} * T \to F_{u,v} \times T$ to be the natural homomorphism. In $F_{u,v} \times T$, we have

$$\pi(\alpha) b' \pi(\alpha)^{-1} \in F_{u,v}, \tag{10}$$

and combining this with (8) and (9) we get in the group $F_{u,v} \times T$

$$\hat{t}^{-1} \pi(\alpha) b' \pi(\alpha)^{-1} \in F_{u,v},$$

which is clearly a contradiction.

Thus there is $i \neq 1$ such that $v \in K_i^b$ for some $b \in F$. By setting $F_1 = K_1$ and $F_2 = K_2 * \cdots * K_m * H$, we get the required conclusion. This ends the proof of our claim. □

To finish the proof, it is sufficient to observe that since u is primitive in F it is primitive in F_1 and the result follows. □

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