Note on a locally-free existentially closed model of the universal theory of free groups

Abderezak OULD HOUCINE
Institut Gamille Jordan
Université Claude Bernard Lyon-1
Bâtiment Braconnier, 21 Avenue Claude Bernard,
69622 Villeurbanne Cedex, France.
ould@math.univ-lyon1.fr

Abstract

Let Γ be the universal theory of non-abelian free groups. We prove the following: *there exists $\mathcal{M}$ a countable 2-locally-free non-free e.c model of* Γ.

1 Introduction

Given a group G we say that G is *locally-free* if every finitely generated subgroup of G is free. We say that G is *coherent* if every finitely generated subgroup of G is finitely presented. We say that G is *locally-cyclic* if every finitely generated subgroup of G is in some cyclic subgroup of G.

Proposition 1.1 Γ *has the following properties:*
(**1**) Γ *is closed under free product, thus it has the joint embedding property.*
(**2**) Γ *does not have the amalgamation property.*
(**3**) *Every model of* Γ *is coherent.*

Proof

(1). This result is well-know. It is sufficient to see that if G_1 and G_2 are finitely generated models of Γ, then by a know-result they are fully residually free and their free product is also fully residually free.

(2). Let $\mathtt{F}$ be the free group with basis $\{a, b\}$. Let $\mathtt{F}'$ another copy of $\mathtt{F}$ with basis $\{c, d\}$. Let A be the subgroup of $\mathtt{F}$ generated by $[a, b]$, and let B be the subgroup of $\mathtt{F}'$ generated by c^2. Let us show that there is no model of Γ which containing $\mathtt{F}$ and $\mathtt{F}'$ and amalgamating A and B. Suppose that the opposite is true. Then the following sentence :

$$\exists x \exists y \exists z([x, y] \neq 1 \wedge [x, y] = z^2)$$

must be true in free group. But this contradict a result of Schützenberger. (see for example [4], pages 51-52).

Here another proof: Suppose that Γ has the amalgamation property. In [3], M.Burger and S.Mozes has constructed a finitely presented simple group K which can be viewed as the free product of two copy of a finitely generated free group which amalgamating a finitely generated free group G : $\mathtt{F} *_G \mathtt{F}$. Now let M be a model of Γ which amalgamating G over $\mathtt{F}$ and $\mathtt{F}$. Then the subgroup generated by $\mathtt{F}$ and $\mathtt{F}$ is isomorphic to K. Hence K is a model of Γ. Let $W(x_1, ..., x_n)$ be a presentation of K and $v(x_1, ..., x_n)$ be a nontrivial element in K. Then the existential formula $\psi = \exists \bar{x}(W(\bar{x}) \wedge v(\bar{x}) \neq 1)$ is true in K and it is true in some finitely generated free group. Hence there exists a free group which contains finitely generated simple group, and this is a contradiction.

(2). By a result of Kharlampovich-Myasnikov and Sela, every finitely generated model of Γ is finitely presented. Hence every model of Γ is coherent.

□

Definition 1.2 *Let T be a theory and $\mathcal{M}$ be a model of T. We say that $\mathcal{M}$ is* **finitely expansive** *(relatively to T) if there is an existential sentence ψ consistent with T such that $\mathcal{M}$ is embeddable in every model of T which satisfies ψ.*

We have the following fact:

Theorem 1.3 *Let G be a countable model of Γ. Then G is finitely expansive (relatively to Γ) if and only if G is free.*

Proof Let G be a finitely expansive model of Γ. Then there is an existential sentence ψ consistent with T such that $\mathcal{M}$ is embeddable in every model of T which satisfies ψ. But in this case ψ must be true in some free group of finite rank. Hance G is embeddable in free group, thus G is free.

Now let $\mathtt{F}$ be a countable free group. Since $\mathtt{F}$ is embeddable in $\mathtt{F}_2$, it is sufficient to show that $\mathtt{F}_2$ is finitely expansive. Let $\psi(x, y)$ be the following formula ($[x, y] \neq 1$). It is well-known (and it is not difficult to see this), that if a, b are in a model M of Γ such that $[a, b] \neq 1$ then the subgroup generated by a, b is free with basis $\{a, b\}$. Therefore the formula $\exists x \exists y([x, y] \neq 1)$ satisfies the desired properties. □

2 Existentially closed groups of Γ

Notice that since Γ has JEP then every two e.c. models of Γ have the same $\forall\exists$ theory. It is also not difficult to see that if ψ is an $\forall\exists$ sentence which is true in some non-abelian free group, then ψ is true in every e.c. model of Γ.

Definition 2.1 *Let $\psi(\overline{x})$ be an existential formula. We say that ψ is* ***atomic-complete*** *if for every atomic formula $\phi(\overline{x})$ we have:*

$$\Gamma \vdash \psi(\overline{x}) \Rightarrow \phi(\overline{x}) \text{ or } \Gamma \vdash \psi(\overline{x}) \Rightarrow \neg\phi(\overline{x})$$

Theorem 2.2 *There exists a countable locally-free e.c model of* Γ .

Proof

Let $S_n = \{\psi(x_1, ..., x_n) \mid \psi \text{ is an atomic}-\text{complete } formula\}$. Let us show that there is no existential formula $\phi(x_1, ..., x_n)$ consistent with Γ such that : $\Gamma \vdash \phi(x_1, ..., x_n) \Rightarrow \neg\psi(x_1, ..., x_n)$ for every formula $\psi \in S_n$.

Suppose that the opposite is true. Now $\exists x_1 ... \exists x_n \phi(x_1, ..., x_n)$ is true in some free countable group F and then there exists a tuple $d_1, ..., d_n$ in F such that $F \models \phi(d_1, ..., d_n)$. Let $Y = \{y_i | i \in \lambda\}$ be a basis of F where $\lambda \leq \omega$. Let F_2 be the free group with basis $\{a, b\}$. Now F is isomorphic to the subgroup, of F_2, generated by $\{b^{-i}ab^i | i \in \lambda\}$, and $y_i \to b^{-i}ab^i$ is an embedding. Now let $w_1, ...w_n$ be a words on Y such that $d_i = w_i$, in F. Now we see that w_i involve a finite number of element of Y say $y_1, ..., y_m$. Therefore we can write $d_i = w_i(y_1, ..., y_m)$. Let ψ_0 be the following formula:

$$\psi_0(x_1, ..., x_n) = \exists a \exists b([a, b] \neq 1 \bigwedge_{i=1}^{i=n} x_i = w_i(b^{-1}ab, ..., b^{-m}ab^m))$$

We see that $\exists x_1 ... \exists x_n \phi(x_1, ..., x_n) \wedge \psi_0(x_1, ..., x_n)$ is consistent with Γ. It is not difficult to see that ψ_0 is an atomic-complete formula. Therefore by our hypothesis $\Gamma \vdash \phi(x_1, ..., x_n) \Rightarrow \neg\psi_0(x_1, ..., x_n)$, which is a contradiction. Therefore the set $\neg S_n = \{\neg\psi(x_1, ..., x_n) \mid \psi \in S_n\}$ is an universal type, such that there is no existential formula $\phi(x_1, ..., x_n)$ consistent with Γ such that : $\Gamma \vdash \phi(x_1, ..., x_n) \Rightarrow \neg\psi(x_1, ..., x_n)$ for every formula $\psi \in S_n$. Hence by the omitting type theorem, there exists an e.c. model M of Γ which omit every $\neg S_n$, for every n. Therefore every tuple $b_1, ..., b_n$ in M satisfies some atomic complete formula. Therefore every finitely generated group of M is free. □

Remark

It is not difficult to see that in the above proof, by choosing an adequate atomic-complete sentences, in fact we have : every finitely generated subgroup of G is included in free group generated by two elements which is also in G.

Theorem 2.3 *Let G be a locally free group. Then :*
(1) *The centralizer $C_G(x)$ for every x is locally cyclic.*
(2) *The centralizer $C_G([a, b])$ is infinite cyclic, for $[a, b] \neq 1$.*

Proof

(1) The proof is left to the reader.
(2) Since $C_G([a, b])$ is locally cyclic and by the result of Schützenberger, previously cited, we must have $C_G([a, b]) = \langle [a, b] \rangle$. □

References

[1] G. Higman, E. Scott, *Existentially closed groups*, Clarendon Press, Oxford, 1988.

[2] W. Hodges, *Model theory*, Cambridge University Press, 1993.

[3] M. Burger, S. Mozes, *Group acting on trees: from local to global structure*, preprint, ETHZ, 1998.

[4] R.G. Lyndon, P.E. Schupp, *Combinatorial group theory*, Springer-Verlag, Berlin, Heidelberg, New York, 1977.